STUDY ON REGULAR RINGS - A BIORDERED SET APPROACH

Thesis submitted to
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
for the award of the degree of

DOCTOR OF PHILOSOPHY

under the Faculty of Science
by

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May 2018

STUDY ON REGULAR RINGS - A BIORDERED SET APPRAOCH

Ph.D. thesis in the field of Algebra

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Certificate

Certified that the work presented in this thesis entitled "**STUDY ON REGULAR RINGS - A BIORDERED SET APPROACH** "is based on the authentic record of research carried out by Ms. Akhila R under my guidance in the Department of Mathematics, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted for the award of any degree.

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Certified that all the relevant corrections and modifications suggested by the audience during the Pre-synopsis seminar and recommended by the Doctoral Committee of the candidate has been incorporated in the thesis entitled "**STUDY ON REGULAR RINGS - A BIORDERED SET APPROACH**. "

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Declaration

I, AKHILA R, hereby declare that the work presented in this thesis entitled "**STUDY ON REGULAR RINGS - A BIORDERED SET APPROACH** "is based on the original research work carried out by me under the supervision and guidance of Dr. P. G. Romeo, Professor, Department of Mathematics, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted previously for the award of any degree.

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To
My loving
Parents, Teachers
and
my beloved
Husband

Acknowledgments

"The Skies proclaim the work of his hands."
First and foremost, I would like to thank God Almighty for giving me the strength, knowledge, ability and opportunity to undertake this research study and to preserve and complete it satisfactorily. Without his blessing, this achievement would not have been possible.

"Every achiever is inspired by a great mentor."
In my journey towards this degree, I have found a teacher, a friend, and inspiration, a role model and a pillar of support in my Research Guide, Dr. P. G. Romeo. He has been there providing his heartfelt support and guidance at all times and has given me invaluable guidance, inspiration and suggestions in my quest for knowledge. He has given me all the freedom to pursue my research, while silently and non obtrusively ensuring that I stay on course and do not deviate from the core of my research. Without his able guidance, this thesis would not have been possible and I shall eternally be grateful to him for his assistance.

At the very outset, I express my deepest sense for gratitude to Dr. V. B. Kiran Kumar, Head of the Department, Department of Mathematics, CUSAT. My cordial thanks to you for your warm encouragement and thoughtful guidance that has helped in the completion of the thesis.

I take pride in acknowledging the insightful guidance of Dr. A. Vijayakumar, Faculty, Department of Mathematics, CUSAT for sparing his valuable time whenever I approached him and showing me the way ahead.

I would also like to express my gratitude to my faculty, Dr. R. S. Chakravarthy, Dr. M. N. N. Namboodhiri, Dr. Ambily A. A., Dr. Noufal A., Dr. Naveena Chandran, Dr. Prasad, and Ms. Simi Thomas, Department of Mathematics, CUSAT, who have been so helpful and *cooperative in giving their support at all times to help me achieve my goal.*

It would be inappropriate if I omit to mention the staff and librarian in assisting me as per their abilities, in whatever manner possible and for ensuring that good times keep flowing.

I have great pleasure in acknowledging my gratitude to my fellow research scholars Dr. Seethu Varghese, Ms. Savitha K.S., Dr. Jaya, Dr. Dhanya Shajin, Mr. Tijo James, Mr. Balesh K., Dr. Pravas K., Dr. Pamy Sebastian, Ms. Elisabeth Reshma M.T, Dr. Chithra M.R., Mr. Didimos K.V., Dr. Jayaprasad P.N., Dr. Raji George, Dr. Gireesan K.K., Mr. Satheesh Kumar, Ms. Savitha K.N., Ms. Anu Varghese, Dr. Manjunath A. S., Mr. Ajan K. A, Mr. Rahul Rajan, Ms. Aparna Pradeep, Ms. Linet Roslin Anthony, Ms. Susan Mathew Panackal, Ms. Rasila V. A., Ms. Binitha Benny, Ms. Divya V., Ms. Pinky, Ms. Vinitha T. at CUSAT for their interest in my work. I would like to thank my friends in my research area Ms. Sreejamol P. R., Ms. Lejo. J. Manavalan, Ms. Smisha M. A., Ms. Siji Michael, Mr. Bintu Shyam, Ms. Smitha Davies in ensuring that the fire keeps burning and begin their at times when I required motivation and propelling me on the course of this thesis and also for assisting me in all means in the tenure of my research. Their support, encouragement and credible ideas have been a major contribution in the completion of this thesis.

My gratitude also goes to the 'KSCSTE' programme of the Government of Kerala for awarding a research fellowship during the initial stage of my work.

I would like to express my deep sense of gratitude to my teacher Late Dr. Deepthy C. P. for her contribution in shaping my career. If she had not given me thought of wisdom, I wouldn't have studied to this level and with this dedication.

My special gratitude to my sister, a friend and a teacher Mrs. Ranjitha P. who has taught me since my childhood for her valuable advices and her efforts to shape my career and my personality in all aspects.

My acknowledgment would be incomplete without thanking the biggest source of my strength, my family and my friends. The blessing of my parents Mr. G. Ravikumar and Mrs. Praveena P. and the love and care of my brother Adithya R Naik and of course my prime source of ideas, my husband Mr. Mahesh R. and my little princess, my daughter Advaya, who never let things get dull or boring, have all made a tremendous contribution in helping me reach this stage in my life. I thank them for putting up with me in difficult moments where I felt stumped and for goading me on to follow my dream of getting this degree. This would not have been possible without their unwavering and unselfish love and support given to me at all times.

Akhila R

INTRODUCTION

In this thesis 'STUDY ON REGULAR RINGS- A BIORDERED SET APPROACH', we discuss various aspects regarding the idempotents of a regular ring. The set of idempotents of a regular ring together with two quasiorders is characterised as a biordered set and it is also shown that the order ideals generated by these quasiorders called biorder ideals forms complemented modular lattices. Further the a coordinatisation theorem analogous to von Neumann's coordinatization theorem for complemented modular lattices is provided:

The concept of von Neumann regular rings was introduced by John von Neumann in a paper 'On Regular Rings' in 1936. A ring R is called a regular ring if for every $a \in R$ there exists $b \in R$ such that $aba = a$. He used such rings as an algebraic tool for studying certain lattices of projections on algebras of operators on a Hilbert space. A lattice L is said to be coordinatized by a regular ring R, if the lattice L is isomorphic to the lattice of all principal right(left) ideals of a regular ring R. von Neumann proved that every complemented modular lattice with order greater than or equal to 4 is coordinatisable.
It is obvious that the multiplicative reduct of a regular ring is a regular semigroup and thus the study of regular semigroups play a significant role in the study of regular rings. In order to study the structure of a regular semigroup, Nambooripad in 1973 introduced the concept of a biordered set. He characterized the set of idempotents $E(S)$ of a semigroup S as biordered set [25].

Here we extend the biordered set approach from regular semigroups to regular rings by explicitly describing the structure of the multiplicative idempotents E_R of a regular ring R as a bounded and complemented biordered set. The principal ideals generated by the left[right] quasiorder $\omega^l[\omega^r]$ and their intersection ω in the biordered set E_R are called the biorder ideals of R and it is shown that these biorder ideals form a complemented modular lattice $\Omega_l(\Omega_r)$. Subject to certain conditions on the biordered set E_R the lattice Ω_l will have properties like perspectivity, independence and order. We also consider the set of idempotents with respect to the addition $\oplus$ defined by $a \oplus b = a + b - ab$ of the ring R and it is observed that every multiplicative idempotent in R is also an additive idempotent. This set of idempotents is denoted by $E_R^{\oplus}$ and as biordered sets $E_R^{\oplus}$ possesses certain interesting properties as that of E_R.

The converse problem of obtaining a biordered set from a complemented modular lattice was discussed by Pastjin in case of strongly regular baer semigroups (cf.[28]). He defined the normal mappings on a complemented modular lattice L using complementary pairs.These normal mappings is a semigroup $P(L)$ and the set of idempotents $E_{P(L)}$ of $P(L)$ is the biordered set of the complemented modular lattice. Here we extend successfully Pastjin's approach of constructing regular biordered sets of complemented modular lattice to regular rings. It is observed that the set of idempotents of a regular ring E_R is a bounded and complemented biordered set and we identify the conditions for the existence of a biordered subset $E^0_{P(L)}$ so that the lattice L admits a homogeneous basis.

The first chapter is a preliminary where we recall all the basic concepts and definitions regarding partially ordered sets, lattices, semigroups, biordered sets and regular rings which are used in the sequel. The notations and terminologies used are in par with the references [2], [7], [12], [16], [25], [23].

In the second chapter, we consider the set of all multiplicative idempotents E_R of a regular ring and discuss its properties. Here we extend the concept of biordered sets to include the class of all idempotents of a regular ring. As examples, the biordered set of the matrix rings $M_2(\mathbb{Z}_2)$, $M_2(\mathbb{Z}_3)$ and $M_2(\mathbb{Z}_4)$ are given. We generalize this by considering the matrix ring $M_2(\mathbb{Z}_p)$ where p is any prime and describe its biordered set. Further, we also study the additive idempotents in the regular ring R by defining a binary operation $\oplus$ defined by $a \oplus b = a + b - ab$ so that the set of all multiplicative idempotents are idempotents with respect to this addition. Thus the set of idempotents with respect to this addition become a biordered set of special interest.

The third chapter is a study on the biorder ideal of a regular ring. Here we consider the principal ideals obtained from the quasiorders ω^r and ω^l and their intersection ω of the biordered set E_R of a regular ring $(R, +, \cdot)$ which we call the biorder ideals. We define the join and meet of two biorder ideals and show that they are closed under these two operations and hence form the complemented modular lattice Ω_l. Considering the case when these biorder ideals coincide that is $\omega^r = \omega^l = \omega$ it is shown that the set of ω ideals form a complemented distributive lattice. Later, some properties of this complemented modular lattice like perspectivity, independence and order are studied. The perspectivity of two elements of this lattice Ω_l is given in terms of E-sequence, thus showing that

- Two biorder ideals $\omega^l(e)$ and $\omega^l(f)$ are perspective if and only if the length of their E-chain of idempotents, $d_l(e, f)$ is less than or equal to 3.

The condition $e_i \ \omega \ (1-e_j)$ for $i \neq j$ for idempotents $e_1, e_2, \ldots, e_n$ asserts that the biorder ideals generated by these idempotents are independent in the lattice Ω_l with $\omega^l(e_1) \vee \omega^l(e_2) \vee \ldots \omega^l(e_n) = \omega^l(e_1 + e_2 + \ldots + e_n)$. Moreover a necessary and sufficient condition for the independence of elements in the lattice Ω_l is given. Combining all these results regarding perspectivity and independence in the complemented modular lattice Ω_l we arrive at the following:

- Let R be a regular ring with $e_i \, \omega \, (1 - e_j)$ for $i \neq j$, $d_l(e_i, e_j) = 3$ and $e_1 + e_2 + \ldots + e_n = 1$ then the complemented modular lattice Ω_l is of order n.

In the fourth chapter, we study the biordered set $E_{P(L)}$ obtained from the complemented modular lattice L[28] and see that this biordered set $E_{P(L)}$ is bounded and complemented. Further, we describe the biordered subset $E^0_{P(L)}$ of $E_{P(L)}$ satisfying certain conditions, so that the complemented modular lattice admits a homogeneous basis. Finally, analogous to von Neumann's coordinatization we prove a coordinatization theorem for complemented modular lattice by using the biordered set of idempotents $E_{P(L)}$.

Contents

Chapter 1

Preliminaries

In this chapter we recall all basic concepts and results which are used in the sequel.

1.1 Partially Ordered sets and Lattices

Let P be a non-empty set, and let σ be a binary relation on P. The relation σ is called a partial ordering of P if it is reflexive, transitive and antisymmetric. We usually write $x \leq y$ for $x \ \sigma \ y$. The pair $(P, \leq)$ is called a partially ordered set(poset). A non-empty set P together with a binary relation σ is called a quasi-ordered set if σ is reflexive and transitive.

Posets can be depicted as graphs with vertices representing the elements and edges extending upwards to indicate the ordering. These graphs are called Hasse diagrams.

Let $(P, \leq)$ be a poset, $X \subseteq P$ and $a \in P$. When $a \leq x$ for all $x \in X$, we call a a lower bound of X. The element a is called the greatest lower bound or infimum of X if

$$\text{for every } p \in P, p \leq x \text{ and } a \leq x \Longrightarrow p \leq a \text{ for all } x \in X.$$

Analogous definitions for upper bound and least upper bound can be given .

Definition 1.1.1. A subset I of a partially ordered set $(P, \leq)$ is an ideal(order ideal) if the following conditions hold:

1. I is non-empty
2. for every $x \in I$, $y \leq x$ implies that y is in I and
3. for every x, y in I, there is some element z in I, such that $x \leq z$ and $y \leq z$.

The smallest ideal that contains a given element p is called a principal ideal and p is said to be a principal element of the ideal. The principal ideal $\downarrow p$ for a principal p is thus given by $\downarrow p = \{x \in P | x \leq p\}$.

Definition 1.1.2. (cf.[9] page 179) If P is a partially ordered set and $\Phi : P \longrightarrow P$ is an isotone(order preserving) mapping, then Φ will be called normal if

1. $im\Phi$ is a principal ideal of P and
2. whenever $x\Phi = y$, then there exists some $z \leq x$ such that Φ maps the principal ideal $P(z)$ isomorphically onto the principal ideal $P(y)$.

Definition 1.1.3. (cf. [9] page 179) The partially ordered set P will be called regular if for every $e \in P$, $P(e) = im\Phi$ for some normal mapping $\Phi : P \longrightarrow P$ with $\Phi^2 = \Phi$.

If P is a regular partially ordered set, then it is easy to see that the set $S(P)[S^*(P)]$ of normal mappings of P into itself, considered as left [right] operators form a regular semigroup under the composition of mappings.

Definition 1.1.4. A lattice is a partially ordered set in which each pair of elements has a least upper bound and a greatest lower bound.

Let a, b be elements of a lattice L. Then we denote their greatest lower bound (meet) by $a \wedge b$ and the least upper bound (join) by $a \vee b$. It can be easily seen that $a \vee b$ and $a \wedge b$ are unique. The operations thus seen above $\vee$ and $\wedge$ are idempotent, commutative and associative. That is, they satisfy the following:

L1 Idempotency:

$$a \vee a = a; \; a \wedge a = a.$$

L2 Commutativity:

$$a \wedge b = b \wedge a, \; a \vee b = b \vee a.$$

L3 Associativity:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c), \; (a \vee b) \vee c = a \vee (b \vee c).$$

These properties of the operations are also called the *idempotent identities, commutative identities,* and *associative identities*, respectively.
There is another pair of rules that connect $\vee$ and $\wedge$.

L4 Absorption identities:

$$a \wedge (a \vee b) = a, \; a \vee (a \wedge b) = a.$$

An alternate definition treating lattices as algebras is the following:

Definition 1.1.5. An algebra $\langle L; \wedge, \vee \rangle$ is called a lattice if and only if L is a non empty set, $\wedge$ and $\vee$ are binary operations on L, both

$\wedge$ and $\vee$ are idempotent, commutative and associative, and they satisfy the two absorption identities.

The notations $a \vee b$ and $a \wedge b$ are analogous to the notations for the intersections and union of sets. Some properties of union and intersection carry over to lattices but some do not. For instance, the distributive law need not hold in all lattices

Definition 1.1.6. A lattice L is called a distributive lattice if any of the following identities hold:

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Next we define the property of modularity, which is a weak form of distributivity.

Definition 1.1.7. A lattice L is called modular (Dedekind lattice) if the modular law holds in it:

$$a \leq c \Longrightarrow (a \vee b) \wedge c = a \vee (b \wedge c).$$

The two typical examples of non-distributive lattices with 5 elements are N_5 and M_3. It can be seen that M_3 is modular but N_5 is not. N_5 is the smallest non-modular lattice.([2], Theorem 2.8)

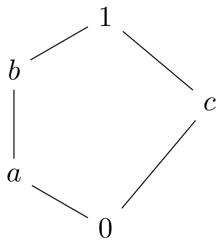


Figure 1.1: N_5

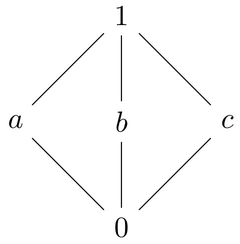


Figure 1.2: M_3

Theorem 1.1.1. *(Dedekind, 1900) Let L be a lattice. The following are equivalent:*

1. *L is modular.*
2. *L satisfies $((x \wedge z) \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.*
3. *L has no sublattice isomorphic to N_5.*

The following theorem is analogous to the above theorem for distributivity ([2] Theorem 2.10)

Theorem 1.1.2. *(Birkhoff) Let L be a lattice. The following are equivalent:*

1. *L is distributive*
2. *L satisfies $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$.*
3. *L has no sublattice isomorphic to either N_5 or M_3.*

A lattice is bounded if it has both a maximum element and a minimum element, we use the symbols 0 and 1 to denote the minimum element and maximum element of a lattice. The notion of complementation in the sense of set theory can be generalized to an arbitrary bounded lattice.

Definition 1.1.8. A bounded lattice L is said to be complemented if for each element a of L, there exists at least one element b such that $a \vee b = 1$ and $a \wedge b = 0$. The element b is referred to as a complement of a.

It is quite possible for an element of a complemented lattice to have many different complements. The lattices M_3 and N_5 illustrate two ways an element can have multiple complements. Now we have the definition of a relative complement.

Definition 1.1.9. An element x is called a complement of a in b if $a \vee x = b$ and $a \wedge x = 0$.

Next we note a simple fact regarding the relative complement in a lattice L. (see [23] Theorem 1.4)

Theorem 1.1.3. *If x is a complement of a in b and y is a complement of b in c, then $x \vee y$ is a complement of a in c.*

Now we proceed to define the notion of independence in lattice elements.

Definition 1.1.10. The elements $x_1, x_2, \cdots x_n$ of a lattice are called independent if

$$(x_1 \vee \cdots \vee x_{i-1} \vee x_{i+1} \vee \cdots \vee x_n) \wedge x_i = 0$$

for every i.

The following proposition characterizes the independence of elements in a modular lattice([33], Proposition 2).

Proposition 1.1.1. For elements $x_i\colon i = 1, 2, \ldots, n$ in a lattice L, if $(x_1 \vee \cdots \vee x_i) \wedge x_{i+1} = 0$ for every i, then $x_1, x_2, \cdots x_n$ are independent.

Next we define the notion of perspectivity, that is closely related to the idea of complement of an element.

Definition 1.1.11. Two elements a and b of a lattice L are said to be perspective (in symbols $a \sim b$) if there exists x in L such that

$$a \vee x = b \vee x, a \wedge x = b \wedge x = 0$$

such an element x is called an axis of perspectivity.

Following is the definition of a basis for a lattice (see[23]).

Definition 1.1.12. Let L be a complemented, modular lattice with zero 0 and unit 1. By a basis of L is meant a system $(a_i\colon i = 1, 2, \ldots n)$ of n elements of L such that

$$(a_i; i = 1, 2, \ldots n) \text{ are independent, } a_1 \vee a_2 \vee \ldots \vee a_n = 1$$

A basis is homogeneous if its elements are pairwise perspective.

$$a_i \sim a_j \ (i, j = 1, 2, \ldots n)$$

The number of elements in a basis is called the order of the basis.

Definition 1.1.13. A complemented modular lattice L is said to have order m in case it has a homogeneous basis of order m.

1.2 Semigroups

In the following we briefly recall some definitions and basic results in semigroup theory. For details of the topic we refer to any book on semigroup theory(cf. [3], [15], [8]).

A semigroup is a pair $(S, \cdot)$, where S is a non-empty set and $\cdot$ is an associative binary operation on S.

A subset T of S is a sub-semigroup of S if T is a semigroup with respect to the restriction of the binary operation of S to T. If T is a subsemigroup of S, then S is called the extension of T.

If S is any semigroup, we can form the semigroup denoted by S^{op} as follows: The set underlying S^{op} is same as the set S and the binary operation of S^{op} (denoted by $\circ$) is defined by

$$x \circ y = yx \text{ for every } y \in S.$$

It is clear that $\circ$ is a binary operation and is called the left-right dual of the binary operation of S is associative and hence S^{op} is a semigroup. We call the semigroup S^{op} as the left-right dual of S.
It should be noted that if P is any statement about a semigroup, then P^{op} is the statement obtained by replacing every occurrence of the binary operation in P, by its left-right dual. If P is true in S then P^{op} must be true in S^{op}. The relation between the statements P and P^{op} is called the left-right duality in semigroups.

If S is any semigroup and $A \subseteq S$, an element $x \in S$ is called a left identity of A if $xa = a$ for every $a \in A$. An element x in S is called a right identity of A in S if it is a a left identity of A in S^{op}. An element x in S which is both a left and a right identity of A in S is called a two-sided identity of A in S.
An element x is a left (right, two-sided) identity of S if the equation $xa = ax = a$ holds with $A = S$. Note that a subset of a semigroup may have more than one left(right) identities. However, an identity of S, if it exists is unique. Given any semigroup S we can always adjoin a new left(right) identity as follows:
Let $T = S \cup \{e\}$ where e is not in S. Extend the multiplication in S to T by

$$ex = x \ (xe = x), \ e^2 = e \text{ for every } x \in S.$$

Clearly this makes T a semigroup and e a left(right) identity of T having S as a subsemigroup. Similarly, a new identity can be adjoined to S by extending the multiplication in S to T by

$$ex = xe = x \text{ for every } x \in S, \;\; e^2 = e.$$

Definition 1.2.1. A semigroup S with identity is called a monoid.

It is clear that any semigroup can be extended to a monoid by adjoining a new identity to S. Given any semigroup S we denote by S^1 the monoid defined as follows:

$$S^1 = \begin{cases} S \text{ if } S \text{ is a monoid} \\ T \text{ if } S \text{ has no identity} \end{cases}$$

An element x in a semigroup S is called a left,[right, two-sided] zero of a subset $A \subseteq S$ if $xa = x[ax = x, ax = x = xa]$ for all $a \in A$. When $A = S$, we say that x is a left,[right, two-sided] zero of S. Left and right zeros of S need not be unique. But a two-sided zero (or just zero for short) of S, when it exists, is unique and will be denoted by 0. As in the case of identities it is possible to adjoin a new left, right or two-sided zero to S. Thus if 0 does not represent an element of S, then $T = S \cup \{0\}$ becomes a semigroup with zero 0 having S as a subsemigroup if we extend the multiplication in S to T by:

$$0x = x0 = 0, \text{ for every } x \in S \text{ and } 00 = 0$$

Again as defined above we define S^0 by

$$S^0 = \begin{cases} S \text{ if } S \text{ has a zero} \\ T \text{ if } S \text{ has no zero} \end{cases}$$

where T is the semigroup obtained by adjoining a zero 0 to S

Definition 1.2.2. An element e in a semigroup S is called an idempotent if $ee = e^2 = e$.

It is clear that the left identities, right identities, identities, left zero, right zero and zero of a semigroup S are idempotents in S. If every element of a semigroup S is idempotent, we shall say S is itself idempotent, or that S is a band.

In the following we give a list of examples of semigroups, for details refer ([8], [15]).

Example 1.2.1. The semigroup of all partial transformations: Let $\mathscr{P}\mathscr{T}_X$ denote the set of all partial transformations(single valued relations) on the set X. A partial mapping of X into itself (usually called a partial transformation of X) is a mapping $\alpha\colon A \longrightarrow X$ whose domain A is a subset of X. When $\alpha\colon A \longrightarrow X$ and $\beta\colon B \longrightarrow X$ are partial transformations, the domain of $\alpha\beta$ is $D = \{x \in A\colon\ x\alpha \in B\}$; then $x(\alpha\beta) = (x\alpha)\beta$ for all $x \in D$. Since composition of single valued relations are single valued, $\mathscr{P}\mathscr{T}_X$ s a semigroup.

Example 1.2.2. The semigroup T_X: The set T_X of all transformations on X (all maps of X into X) is the full transformation semigroup, the operation is composition of mappings; T_X is clearly a subsemigroup of $\mathscr{P}\mathscr{T}_X$.

Example 1.2.3. Semilattices: A semilattice is a commutative semigroup of idempotents (that is, a semigroup S in which every element is an idempotent). Define a partial order by $a \leq b \iff ab = a$ for $a, b \in S$. Then S is a lower semilattice, in which the infimum(g.l.b) $a \wedge b$ of a and b is their product ab.

Ideals and Greens Relations

A subset I of a semigroup S is called a left ideal [right ideal] if for all $x \in I$ and $a \in S, ax \in I$ $[xa \in I]$. I is called a two sided ideal (or simply an ideal) if I is both a left as well as a right ideal. Or equivalently, I is a left ideal if $SI \subseteq I$, a right ideal if $IS \subseteq I$ and an ideal if it is both a left and a right ideal.

Given any subset $A \subseteq S$, the set of ideal that contain A is non empty since S itself is a member of this set. The intersection $L(A)$ of all left ideals on S containing A is the smallest left ideal of S containing A and $L(A)$ is called the left ideal generated by A. Similarly the intersection $R(A)$ $[J(A)]$ of all right [two-sided] ideals of S containing A is the right [two-sided] ideal generated by A. It is easy to show that

$$L(A) = SA \cup A = S^1 A;\ R(A) = A \cup AS = AS^1;\ J(A) = S^1 AS^1$$

When $A = \{a\}$, as usual, we write $L(a)$ for $S^1 a$ is called the principal left ideal generated by a. Similarly aS^1 denotes the principal right ideal and $S^1 a S^1$ denote the principal ideal generated by a.

Study of the structure of the set of ideals (both one-sided and two-sided) via certain equivalence relations induced by them is an important technique for analyzing the structure of a semigroup. These relations were first introduced and studied by Green in 1951. The Greens relations on a semigroup S are defined by

$$\mathscr{L} = \left\{(x, y) : S \times S \colon S^1 x = S^1 y\right\}$$

$$\mathscr{R} = \left\{(x, y) : S \times S \colon xS^1 = yS^1\right\}$$

$$\mathscr{J} = \left\{(x, y) : S \times S \colon S^1 x S^1 = S^1 y S^1\right\}$$

$$\mathscr{D} = \mathscr{L} \vee \mathscr{R}$$

$$\mathscr{H} = \mathscr{L} \wedge \mathscr{R}$$

$\mathscr{D}$ is the smallest equivalence relation that contains both $\mathscr{L}$ and $\mathscr{R}$. It can be shown that the relations $\mathscr{L}$ and $\mathscr{R}$ commute. That is

$$\mathscr{L} o \mathscr{R} = \mathscr{R} o \mathscr{L}$$

and consequently,

$$\mathscr{D} = \mathscr{L} o \mathscr{R}.$$

For $a \in S$, the $\mathscr{L}$-class, $\mathscr{R}$-class, $\mathscr{J}$-class, $\mathscr{H}$-class and the $\mathscr{D}$-class containing a will be denoted respectively by L_a R_a, J_a, H_a and D_a. Since $\mathscr{L}, \mathscr{R}$ and $\mathscr{J}$ are defined in terms of principal ideals, the inclusion order among these ideals induces a partial order on the quotient sets $S/\mathscr{L}, S/\mathscr{R}$, and $S/\mathscr{J}$ by

$$L_a \leq L_b \Longleftrightarrow S^1 a \subseteq S^1 b$$

$$R_a \leq R_b \Longleftrightarrow aS^1 \subseteq bS^1$$

$$J_a \leq J_b \Longleftrightarrow S^1 a S^1 \subseteq S^1 b S^1$$

The following proposition gives an alternate characterization of these relations $\mathscr{L}$ and $\mathscr{R}$ in terms of the "mutual divisibilty" aspect. (see[16], Prop(2.1.1)).

Proposition 1.2.1. Let a, b be elements of a semigroup S. Then $a\mathscr{L}b$ if and only if there exists x, y in S^1 such that $xa = b, yb = a$ and $a\mathscr{R}b$ if and only if there exists u, v in S^1 such that $au = b, bv = a$.

Definition 1.2.3. Let S be a semigroup. A relation R on the set S is called left compatible (with the operation on S) if

$$(\forall s, t, a \in S) \ \ (s, t) \in R \Rightarrow (as, at) \in R,$$

and right compatible if

$$(\forall s, t, a \in S) \ \ (s, t) \in R \Rightarrow (sa, ta) \in R.$$

It is called *compatible* if

$$(\forall s, t, s', t' \in S) \ [(s, t) \in R \text{ and } (s', t') \in R] \Rightarrow (ss', tt') \in R.$$

A left [right] compatible equivalence is called a left [right] congruence and a compatible equivalence relation is called a congruence.

Thus it can be seen that $\mathscr{L}$ is a right congruence and $\mathscr{R}$ is a left congruence. If $a \in R_e$, then $a = ex$ for some $x \in S^1$ and so $ea = e(ex) = e^2x = ex = a$. Similarly, we can see that $be = b$ for all $b \in L_e$. Thus we have the following proposition([15] Prop(2.3.3)):

Proposition 1.2.2. Every idempotent e in a semigroup S is a left identity for R_e and a right identity for L_e.

Every $\mathscr{D}$-class in a semigroup S is a union of $\mathscr{L}$-classes and a union of $\mathscr{R}$-classes. The intersection of an $\mathscr{L}$-class and an $\mathscr{R}$-class is either empty or is an $\mathscr{H}$-class. However, by the definition of $\mathscr{D}$

$$a\mathscr{D}b \Longleftrightarrow R_a \cap L_b \neq \emptyset \Longleftrightarrow L_a \cap R_b \neq \emptyset.$$

Hence a $\mathscr{D}$-class can be visualized as an egg-box picture, in which each row represents an $\mathscr{R}$-class, each column represents an $\mathscr{L}$-class, and each cell represents an $\mathscr{H}$-class.
The main property of these Greens relations is that multiplication by suitable elements induces bijections between $\mathscr{R}, \mathscr{L}$ and $\mathscr{H}$-class.

Lemma 1.2.1 (Green's Lemma)**.** Let $a, b \in S$ and $u, v \in S^1$, such that $ua = b, vb = a$, that is $a\mathscr{L}b$. Then the mappings $\bar{u} : R_a \longrightarrow R_b$ given by $x \longrightarrow ux$ and $\bar{v} : R_b \longrightarrow R_u$ given by $y \longrightarrow vy$ are mutually inverse $\mathscr{L}$-class preserving bijections.

By Green's Lemma, $a\mathscr{L}b$ implies $|R_a| = |R_b|$ and $|H_a| = |H_b|$. Dually, $a\mathscr{R}b$ implies $|L_a| = |L_b|$ and $|H_a| = |H_b|$. Thus any two $\mathscr{H}$-classes contained in the same $\mathscr{D}$-class have the same number of elements and similarly for $\mathscr{L}$- and $\mathscr{R}$-classes.

Regular and Inverse Semigroups

An important concept in the theory of semigroups is that of regularity. The concept of regularity in a semigroup was adapted from an analogous condition on rings, which was defined by J. von Neumann [23] in 1936.

Definition 1.2.4. An element a of a semigroup S is called regular if there exists an element $a' \in S$ such that $aa'a = a$. S is called a regular semigroup, if all the elements of S are regular.

The following result describes the regularity in a $\mathscr{D}$-class ([16](Prop. 3.2.1))

Proposition 1.2.3. If a is a regular element of a semigroup S, then every element of D_a is regular.

Since idempotents e are regular ($eee = e$), it follows that every $\mathscr{D}$ class containing e is regular. Conversely, every regular $\mathscr{D}$-class must contain at least one idempotent.

Proposition 1.2.4. In a regular $\mathscr{D}$-class, each $\mathscr{L}$-class and each $\mathscr{R}$-class contains an idempotent.

Therefore, the following result is quite straightforward.

Proposition 1.2.5. In a regular semigroup, every principal left ideal and every principal right ideal is generated by an idempotent.

An idea of great importance in semigroup theory is that of an inverse of an element. This idea was introduced by Vagner in 1952 and Preston in 1954. Its relationship to Green's relation was explored by Clifford and Miller in 1956.

Definition 1.2.5. Let a be an element of a semigroup S. Then

a' is called an inverse of a if

$$aa'a = a \text{ and } a'aa' = a'.$$

Notice that if an element a has an inverse, then it is necessarily regular. Conversely, every regular element has an inverse since if there exists x such that $axa = a$, then define $a' = xax$ and it is seen that

$$aa'a = a \text{ and } a'aa' = a'.$$

Obviously an element a in a semigroup may have more than one inverses. In a semigroup, the number and location of the inverses of an element a can be determined by the locations of the idempotents in the $\mathscr{D}$-class of a. The following theorem asserts the above statement.

Theorem 1.2.1. *[16] Let a be an element of a regular $\mathscr{D}$-class D in a semigroup S. If a' is an inverse of a, then $a' \in D$ and the two $\mathscr{H}$-classes $R_a \cap L_{a'}$ and $L_a \cap R_{a'}$ contain, respectively, the idempotents aa' and $a'a$. Conversely, if a is an element of S and e, f are idempotents in S with $(e, f) \in D$ then a is regular and there exists an inverse a' of a such that $aa' = e$ and $a'a = f$.*

The following egg-box picture explains this result.

	L_a		$L_{a'}$	
R_a	a		aa'	
$R_{a'}$	$a'a$		a'	

The following theorem can be used to locate the products of elements in a $\mathscr{D}$-class.

Theorem 1.2.2. *[12] Let S be a semigroup and $a, b \in S$ then $ab \in R_a \cap L_b$ if and only if $R_b \cap L_a$ contains an idempotent.*

The eggbox picture is given below

	L_a		L_b	
R_a	a		ab	
R_b	e		b	

1.3 Biordered Set

In many algebraic systems like semigroups, rings etc., the set of idempotents are important in analyzing the structure of the system. For a semigroup S, the idea of using the set of idempotents $E(S)$ in studying the structure has a long history. For example, in the case of inverse and orthodox semigroups, the set of idempotents form a sub-semigroup of known type. In 1966, W. D. Munn constructed the inverse semigroup $T(E)$ now known as the Munn semigroup from an arbitrary semilattice E for which $E(T(E)) \cong E$. This implies that the structure of an inverse semigroup S is determined by its semilattice of idempotents. Note that a semigroup S is orthodox, if $E(S)$ is a band. T. E. Hall(1968) amd Yamada(1970) observed that when S is a regular orthodox semigroup, the structure of S can be described in terms of $E(S)$.

However, for any arbitrary regular semigroup S the set of idempotents $E(S)$ is not a sub-semigroup and hence it is not clear, how one can extend Munn's theory to this class of semigroup and there are different approaches to the use of the set of idempotents in the study of regular semigroups.

K.S.S Nambooripad introduced the concept of biordered sets as an order structure to represent the set of idempotents of a regular semigroup.

Let E be the set of idempotents of a regular semigroup. Nambooripad identified two quasiorders ω^r and ω^l and a set of partial transformations in the set of idempotents of a semigroup S satisfying certain axioms [see the definition below] as a biordered set. A partial algebra is a set X together with a partial binary operation. A partial binary operation on X is a partial mapping from $X \times X$ to X. Let E be a partial algebra and D_E denote the domain of the partial binary operation on E. On E, we define

$$\omega^r = \{(e,f) : fe = e\} \quad \omega^l = \{(e,f) : ef = e\}$$
$$\omega^r(e) = \{f : ef = f\} \quad \omega^l(e) = \{f : fe = f\}$$

$$\begin{aligned}
\mathcal{R} &= \omega^r \cap (\omega^r)^{-1} \\
&= \{(e,f) \colon ef = f \text{ and } fe = e\} \\
\mathcal{L} &= \omega^l \cap (\omega^l)^{-1} \\
&= \{(e,f) \colon ef = e \text{ and } fe = f\} \\
\omega &= \omega^r \cap \omega^l \\
&= \{(e,f) \colon ef = e \text{ and } fe = e\}.
\end{aligned}$$

Then a biordered set is defined as follows:

Definition 1.3.1. Let E be a partial algebra and D_E denote the domain of the partial binary operation on E. Let $\omega^r, \omega^l, \mathcal{R}, \mathcal{L}$ and ω be defined on E as above. Then E is a biordered set if the following axioms and their duals hold:

(B1) ω^r and ω^l are quasi orders on E and

$$D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}.$$

(B2) $f \in \omega^r(e) \Rightarrow f \,\mathcal{R}\, fe \,\omega\, e.$

(B3) $g \,\omega^l\, f \,\text{and}\, f, g \in \omega^r(e) \Rightarrow ge \,\omega^l\, fe.$

(B4) $g \,\omega^r\, f \,\omega^r\, e \Rightarrow gf = (ge)f$

(B5) $g \,\omega^l\, f \,\text{and}\, f, g \in \omega^r(e) \Rightarrow (fg)e = (fe)(ge).$

Let $M(e, f)$ denote the quasi ordered set $(\omega^l(e) \cap \omega^r(f), <)$ where '$<$' is defined by $g < h \Leftrightarrow eg \,\omega^r\, eh$, and $gf \,\omega^l\, hf$. Then the set

$$S(e, f) = \{h \in M(e, f) : g < h \text{ for all } g \in M(e, f)\}$$

is called the sandwich set of e and f.

(B6) $f, g \in \omega^r(e) \Rightarrow S(f, g)e = S(fe, ge)$

We shall often write $E = \langle E, \omega^l, \omega^r \rangle$ to mean that E is a biordered set with quasi-orders ω^l, ω^r. The relation ω defined is a partial order and

$$\omega \cap (\omega)^{-1} \subset \omega^r \cap (\omega^l)^{-1} = 1_E.$$

The partial binary operation defined on E by $ef = e$ or $ef = f$ or $fe = e$ or $fe = f$ is called the basic product on E.

A biordered set E is called a regular biordered set if $S(e, f) \neq \emptyset$ for all $e, f \in E$.

Example 1.3.1. Let S be a semigroup. On $E(S) = \{e \in S : e^2 = e\}$ define

$$e\omega^r f \Longleftrightarrow fe = e$$

$$e\omega^l f \Longleftrightarrow ef = e$$

where the products ef and fe are the products in the semigroup S. Let

$$D_{E(S)} = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$$

If $(e, f) \in D_{E(S)}$ then $(e, f) \in \omega^r \cup \omega^l$ or $(f, e) \in \omega^r \cup \omega^l$. In the first case either $ef = e$ or $fe = e$. If $fe = e, (ef)^2 = e(fe)f = ef$ and so $ef \in E(S)$. Thus $ef \in E(S)$ whenever $(e, f) \in \omega^r \cup \omega^l$. Similarly it can be seen that $ef \in E(S)$ whenever $(f, e) \in \omega^r \cup \omega^l$. Thus by restricting the product in S to $D_{E(S)}$ we obtain a partial algebra on $E(S)$.

Definition 1.3.2. Let E and E' be biordered sets and $\theta\colon E \longrightarrow E'$ be a mapping. Then θ is called a bimorphism if it satisfies the following axiom:

$$(e, f) \in D_E \Longrightarrow (e\theta, f\theta) \in D_{E'}$$

and

$$(ef)\theta = e\theta f\theta.$$

Furthermore, θ is called a regular bimorphism if

$$S(e, f)\theta \subseteq S'(e\theta, f\theta)$$

and

$$S(e, f) \neq \emptyset \Longleftrightarrow S'(e\theta, f\theta) \neq \emptyset$$

for all $e, f \in E$ where $S'(e\theta, f\theta)$ denotes the sandwich set of E'. Call θ a biorder isomorphism if θ is bijective and both θ and θ^{-1} are bimorphisms.

We call F a biordered subset of a biordered set E if $F \subset E$ and F is a partial sub-algebra of E in the sense that $D_F = D_E \cap (F \times F)$ and F satisfies the biordered set axioms with respect to the restrictions of ω^r and ω^l to F.

Definition 1.3.3. Let e and f be idempotents in a semigroup S. By an E-sequence from e to f, we mean a finite sequence $e_0 = e, e_1, e_2, \ldots, e_{n-1}, e_n = f$ of idempotents such that $e_{i-1}(\mathcal{L} \cup \mathcal{R})e_i$ for $i = 1, 2, \ldots, n$; n is called the length of the E-sequence. If there exists an E-sequence from e to f, $d(e, f)$ is the length of the shortest E-sequence from e to f; and $d(e, e) = 1$. If there is no E-sequence from e to f, we define $d(e, f) = 0$. For idempotents e and f, we define $d_l(e, f)$ to be the length of the shortest E-sequence from e to f, which start with the $\mathcal{L}$ relation and $d_r(e, f)$ to be the length of the shortest E-sequence from e to f which start with the $\mathcal{R}$ relation.

The following theorem shows that if S is a regular semigroup, then $E(S)$ is a regular biordered set.

Theorem 1.3.1. *([25], Theorem 1.1) Let S be a semigroup such that $E(S) \neq \phi$.*

1. *The partial algebra $E(S)$ is a biordered set.*
2. *For $e, f \in E(S)$ define*

$$S_1(e, f) = \{h \in M(e, f) \colon ehf = ef\}$$

   Then $S_1(e, f) \subset S(e, f)$.

3. *If $e, f \in E(S)$ then ef is a regular element of S if and only if $S_1(e, f) = S(e, f) \neq \phi$.*
4. *If S is regular, then $E(S)$ is a regular biordered set.*

Nambooripad [1979] showed that any biordered set satisfying regularity condition is the set of idempotents of some regular semigroup. Thus we have the following result from ([25], Corollary 4.15).

Result 1. Every regular biordered set is isomorphic to the biordered set of some regular semigroup.

David Easdown (1985) proved the converse of this result viz., that all biordered sets arise as the biordered set of semigroup. This shows that the biorder axioms of Nambooripad [1979] are both necessary and sufficient in order that the resulting structure represents the set of idempotents of a semigroup.

1.4 Regular Rings

The concept of von Neumann regular rings was introduced by John von Neumann in 1936 as an algebraic tool for studying certain lattices [23]. There he described the regular rings which coordinatize complemented modular lattices. A lattice L is said to be coordinatised by a regular ring R if it is isomorphic to the lattice of principal right ideals of R. As von Neumann showed, almost all complemented modular lattices could be coordinatized by a regular ring.

A ring is a set R together with two binary operations $'+'$ and $'\cdot'$ with the following properties.

1. The set $(R, +)$ is an abelian group.
2. The set $(R, \cdot)$ is a semigroup.
3. The operation $\cdot$ is distributive over $+$.

A ring $(R, +, .)$ is called a ring with identity if the semigroup $(R, .)$ has an identity 1. A ring $(R, +, \cdot)$ is regular if for every $a \in R$ there exists an element a' such that $aa'a = a$, that is, the ring is regular if its multiplicative semigroup is regular.
Throughout the thesis, we deal with rings that are von Neumann regular.

Example 1.4.1. 1. A field is a regular ring, for if F is a field then for $a \neq 0$ in F there exists an a^{-1} in F with $aa^{-1} = 1$ so that $aa^{-1}a = a$.

2. Let V be a vector space over the field F and let R be the ring of all linear transformations of V to itself. Then R is a regular ring. Let t be a linear transformation of V to itself with range A and kernel B. Let A' and B' be complements of A and B in V and let t_0 be the restriction of t to B'. Then t_0 is a bijection onto A and so has the inverse $t_0^{-1} \colon A \longrightarrow B'$. Let t' be a linear extension of t_0^{-1} to V. Then $tt't = t_0 t_0^{-1} t = t$.

3. For any field F the ring of all $n \times n$ matrices over F is isomorphic to the ring of linear transformations of the vector space F^n and so is regular. More generally it is proved that a matrix ring over a regular ring R is also regular see([23]).

Definition 1.4.1. [23] If R is a ring, and if $\mathscr{A} \subseteq R$, then $\mathscr{A}$ is a right ideal in case $x + y \in \mathscr{A}, xz \in \mathscr{A}$ and $\mathscr{A}$ is a left ideal if $x + y \in \mathscr{A}, zx \in \mathscr{A}$ when $x, y \in \mathscr{A}, z \in R$. Finally $\mathscr{A}$ is a called an ideal in case $\mathscr{A}$ is both a left ideal and a right ideal.
Denote by $\mathcal{R}_R$ the class of right ideals and by $\mathcal{L}_R$ the class of all left ideals.

Definition 1.4.2. A principal right ideal is one of the from $\langle a_r \rangle = \{ar : r \in R\}$. Similarly, we can define a principal left ideal. The class of all principal right [left] ideals will be denoted by $\bar{\mathcal{R}}_R$ [$\bar{\mathcal{L}}_R$].

Proposition 1.4.1. ([23], Corollary 2) If $X \subseteq \mathcal{R}_R$ is any class of right ideals, there exists both a smallest right ideal (join or least upper bound of X) containing every element of X and a greatest right ideal (intersection or greatest lower bound of X) contained in every element of X. Thus $\mathcal{R}_R$ is a lattice with $\subset$ and the operations thus defined. The zero element of $\mathcal{R}_R$ is $\langle 0 \rangle_r = 0$ and the unit element is $\langle 1 \rangle_r = R$.

Definition 1.4.3. [24] Let $A, B \in \mathcal{R}_R$. Define $A \vee B = \text{g.l.b}(A, B)$ and $A \wedge B = \text{l.u.b}(A, B)$, $A, B \in \mathcal{R}_R$. Then $(\mathcal{R}_R, \vee, \wedge)$ is a lattice with universal minimum 0 and maximum R. Two right ideals A and B are inverses, if $A \vee B = R$ and $A \wedge B = 0$. (Similarly for left ideals)

Clearly, $A \vee B$ is the set of all $x + y$ such that $x \in A$ and $y \in B$.

In [23] John von Neumann describes the structure of principal ideals of a regular ring.

Lemma 1.4.1. Let $\mathcal{R}$ be a ring, $e \in \mathcal{R}$, then

- e is idempotent if and only if $(1 - e)$ is idempotent.
- $\langle e \rangle_r$ is the set of all x such that $x = ex$.
- $\langle e \rangle_r$ and $\langle 1 - e \rangle_r$ are mutual inverses.
- If $\langle e \rangle_r = \langle f \rangle_r$ and if $\langle 1 - e \rangle_r = \langle 1 - f \rangle_r$ where e and f are idempotents, then $e = f$.

Theorem 1.4.1. *Two right ideals a and b are inverses if and only if there exists an idempotent e such that $a = \langle e \rangle_r$ and $b = \langle 1 - e \rangle_r$. This property characterizes uniquely the idempotent e.*

Next we give some equivalent conditions for a ring R to be regular.

Theorem 1.4.2. *The following statements are equivalent*

1. *Every principal right ideal $\langle a \rangle_r$ has an inverse right ideal.*
2. *For every a there exists an idempotent e such that $\langle a \rangle_r = \langle e \rangle_r$.*
3. *For every a there exists an element x such that $axa = a$.*
4. *For every a there exists an idempotent f such that $\langle a \rangle_l = \langle f \rangle_l$.*
5. *Every principal left ideal $\langle a \rangle_l$ has an inverse left ideal.*

Definition 1.4.4. The ring R is said to be regular in case R possesses any one of the equivalent properties of the above theorem.

Next we give the definition of an annihilator of an ideal.

Definition 1.4.5. For every right ideal $\mathscr{A} \subseteq R$, we define

$$\mathscr{A}^l = \{y : yz = 0 \text{ for every } z \in \mathscr{A}\}\,;$$

for every left ideal $\mathscr{B} \subseteq R$ we define

$$\mathscr{B}^r = \{y : zy = 0 \text{ for every } z \in \mathscr{B}\}\,;$$

$\mathscr{A}^l$ is a left ideal, and $\mathscr{B}^r$ is a right ideal. $\mathscr{A}^l$ is called the left annihilator of the right ideal $\mathscr{A}$ and $\mathscr{B}^r$ is called the right annihilator of the left ideal $\mathscr{B}$.

von Neumann showed that if for every principal right ideal $\mathscr{A} \subset R$, there exists a right ideal $\mathscr{B}$ in R which is inverse to $\mathscr{A}$ then $\bar{\mathcal{R}}_R$ is a complemented lattice. Thus we can state the following theorem.

Theorem 1.4.3. *Let R be a regular ring and $\bar{\mathcal{R}}_R$, the set of all principal right ideals of R then the set $\bar{\mathcal{R}}_R$ is a complemented modular lattice, partially ordered by the relation $\subseteq$, the meet being $\cap$ and the join $\cup$; its zero is $\langle 0\rangle$ and its unit is $\langle 1\rangle_r$.*

Chapter 2

Biordered Sets and Rings

The concept of biordered sets was introduced by K. S. S.Nambooripad in (cf. [25]) to describe the structure of the set of idempotents of a regular semigroup. This biordered set has a significant role in the study of structure theory of regular semigroups. Here we extend the concept of biordered sets to include the class of regular rings. We describe the set of multiplicative idempotents of a regular ring and discuss its properties. Further we also define an addition on the regular ring so that the set of idempotents with respect to this addition will also become a biordered set of interest.

2.1 Multiplicative Idempotents of Regular Rings

The study of idempotents play an important role in describing the structure of a regular semigroup. Since the multiplicative part $(R, \cdot)$ of a regular ring $R = (R, +, \cdot)$ is a regular semigroup, the idempotents of $(R, \cdot)$ is a regular biordered set and we denote it by E_R. Further R being a ring (with unity), the biordered set E_R possess some more interesting properties which we discuss in this section.

Throughout this section, R denotes a regular ring with unity and E_R, the set of all multiplicative idempotents in the ring R. Clearly 0 and 1 belongs to E_R. In the following we prove a series of lemmas that describes the properties of E_R.

Lemma 2.1.1. Let R be a regular ring with unity, if $e \in E_R$ then $(1-e) \in E_R$ and $e(1-e) = (1-e)e = 0$.

Proof. For $e \in E_R$,

$$(1-e)^2 = (1-e)(1-e) = 1 - e - e + e^2 = 1 - e.$$

That is, $(1-e)$ is an idempotent and

$$e(1-e) = e - e^2 = e - e = 0$$

and

$$(1-e)e = e - e^2 = e - e = 0.$$

□

We have already seen that the set of idempotents of a regular semigroup is a regular biordered set(see 1.3) and since for a regular ring R, the semigroup $(R, \cdot)$ is regular E_R is a regular biordered set. Also it is easy to see that $0 \ \omega \ e$ and $e \ \omega \ 1$ for every $e \in E_R$. Thus 0 is the least element and 1 is the greatest element in the partially ordered set (E_R, ω).

Lemma 2.1.2. Let e and f be idempotents in the regular ring R, then $ef = 0$ if and only if $e \ \omega^l \ (1-f)[f \ \omega^r \ (1-e)]$.

Proof. Suppose $ef = 0$. Then

$$e(1-f) = e - ef = e.$$

Thus $e \ \omega^l \ (1-f)$. Conversely, if $e \ \omega^l \ (1-f)$ then $e(1-f) = e$ that is $e - ef = e$, implies $ef = 0$.

Also, if $ef = 0$, then $(1-e)f = f - ef = f$ implies that $f \, \omega^r \, (1-e)$ and conversely, $f \, \omega^r \, (1-e)$ implies $ef = 0$. □

Lemma 2.1.3. Let R be a regular ring and e, $f \in E_R$, then the only idempotent in $M(e, f)$ is 0 if and only if $e \, \omega^l \, (1-f)$.

Proof. Suppose $e \, \omega^l \, (1-f)$. Then by above lemma $ef = 0$. Let $g \in M(e, f)$. Then by definition,

$$ge = g \text{ and } fg = g.$$

Hence

$$g = g^2 = g \cdot g = (ge)(fg) = g(ef)g = g0 = 0$$

Therefore, $M(e, f) = \{0\}$. Conversely, suppose $M(e, f) = \{0\}$. Since R is regular, the element $ef \in R$ has an inverse $a \in R$ so that

$$(ef)a(ef) = ef$$

$$a(ef)a = a$$

Let $g = fae$. Then $g^2 = faefae = fae = g$ and $ge = g = fg$, so $g \in M(e, f)$ hence $g = 0$, by hypothesis. Hence

$$ef = (ef)a(ef) = e(fae)f = egf = 0$$

Thus the proof. □

Proposition 2.1.1. Let e and f be idempotents in a regular ring R. Then the following holds:

1. $e \, \omega^l \, f$ if and only if $(1-f) \, \omega^r \, (1-e)$
2. $e \, \omega^r \, f$ if and only if $(1-f) \, \omega^l \, (1-e)$

Proof. Let $e \, \omega^l \, f$. Then,

$$(1-e)(1-f) = 1 - e - f + ef = 1 - e - f + e = 1 - f.$$

Thus $(1-e)(1-f) = (1-f)$ implies, $(1-f)\ \omega^r\ (1-e)$.
Conversely, suppose $(1-e)(1-f) = (1-f)$ then

$$(1-e)(1-f) = 1-e-f+ef = 1-f$$

That is,

$$e - ef = 0 \text{ so that } ef = e,$$

hence $e\ \omega^l\ f$.
Proof of (2) follows similarly. □

Lemma 2.1.4. Let e,$f \in E_R$ with $ef = fe = 0$. Then $e+f$ is an idempotent and e, $f \in \omega\ (e+f)$.

Proof. Given e, $f \in E_R$ with $ef = fe = 0$, then

$$(e+f)^2 = e^2 + ef + fe + f^2 = e+f,$$

and

$$e(e+f) = e^2 + ef = e + ef = e, \text{ and } (e+f)e = e^2 + fe = e + fe = e.$$

Thus $e\ \omega^l\ (e+f)$ and $e\ \omega^r\ (e+f)$. Therfore, $e\ \omega\ (e+f)$.
Also,

$$f(e+f) = fe + f^2 = fe + f = f \text{ and } (e+f)f = ef + f^2 = ef + f = f.$$

Thus $f\ \omega^r\ (e+f)$. Therefore, $f\ \omega^l\ (e+f)$ and $f\ \omega\ (e+f)$. □

Let e, f be in E_R then the element $e+f$ in E_R can be represented in terms of the sandwich set. Recall that if e and f are idempotents of a regular semigroup S then $S(e,f) = S_1(e,f)$ where

$$S_1(e,f) = \{h \in E_R \colon fhe = h \text{ and } ehf = ef\}.$$

Lemma 2.1.5. Let e and f be in E_R with $ef = fe = 0$. Then

$1-(e+f)$ is the unique element belonging to both the sets $S_1(1-e, 1-f) \cap S_1(1-f, 1-e)$.

Proof. Since $ef = fe = 0$ the element $e+f$ is in E_R and so $k = 1-(e+f)$ is also in E_R. Also

$$(1-f)(1-e) = 1-e-f+fe = 1-e-f = 1-(e+f)$$

and

$$(1-e)(1-f) = 1-f-e+ef = 1-f-e = 1-(e+f)$$

so that

$$k = 1-(e+f) = (1-e)(1-f) = (1-f)(1-e)$$

Hence

$$(1-f)k(1-e) = (1-f)((1-f)(1-e))(1-e) = (1-f)(1-e) = k$$

and

$$(1-e)k(1-f) = (1-e)((1-e)(1-f))(1-f) = (1-e)(1-f)$$

so that $k \in S_1(1-e, 1-f)$. Similarly, it is easy to show that $k \in S_1(1-f, 1-e)$ also.

Now we show that k is the unique element belonging to both these sandwich sets, let g be an idempotent in the ring R belonging to both these sets. Then

$$(1-e)g(1-f) = (1-e)(1-f)$$

since $g \in S_1(1-e, 1-f)$ and

$$(1-e)g(1-f) = g$$

since $g \in S_1(1-f, 1-e)$. Hence

$$g = (1-e)g(1-f) = (1-e)(1-f) = k$$

This proves the result. $\square$

The following properties of the idempotents E_R of the ring R are easy to observe:

1. $1-(1-e) = e$
2. $f \ \omega^l \ e$ if and only if $(1-e) \ \omega^r \ (1-f)$
3. $f \ \omega^l \ (1-e)$ if and only if $M(f, e) = \{0\}$

Further it is obvious that the map $\tau\colon E_R \longrightarrow E_R$ defined by

$$\tau(e) = 1-e$$

is a complementation and so the biordered set E_R of a regular ring is a bounded and complemented biordered set.

On the other hand, given a biordered set E it can be enlarged to a set $\bar{E}$ by including two elements (symbols), 0 and 1 such that $0 \ \omega \ e$ and $e \ \omega \ 1$ and for every $e \in E$ there is an element $e^c \in \bar{E}$ with $0^c = 1$, $(e^c)^c = e$ and $ee^c = e^c e = 0$. Then $\bar{E}$ satisfies the following conditions:

- $e \ \omega^l \ f$ if and only if $f^c \ \omega^r \ e^c$ for $e, f \in \bar{E}$
- $f \ \omega^l \ e^c$ if and only if $M(f, e) = \{0\}$

Then call $\bar{E}$ together with these properties as a bounded and complemented biordered set.

Example 2.1.1. Biordered Set of the Matrix Ring $M_2(\mathbb{Z}_2)$
Consider the matrix ring $R_1 = M_2(\mathbb{Z}_2)$. This ring has 16 elements of which there are 8 idempotents. The idempotents of R_1 denoted as E_{R_1}

are listed below.

$$0=\begin{bmatrix}0&0\\0&0\end{bmatrix}, e_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}, e_2=\begin{bmatrix}0&0\\0&1\end{bmatrix}, e_3=\begin{bmatrix}1&1\\0&0\end{bmatrix},$$

$$e_4=\begin{bmatrix}0&0\\1&1\end{bmatrix}, e_5=\begin{bmatrix}1&0\\1&0\end{bmatrix}, e_6=\begin{bmatrix}0&1\\0&1\end{bmatrix}, 1=\begin{bmatrix}1&0\\0&1\end{bmatrix},$$

It can be easily seen that for each $e \in E_{R_1}$, $1-e \in E_{R_1}$. The ω^r and ω^l class of these idempotents are:

$$\omega^l(0)=0 \text{ and } \omega^l(1)=E_{R_1}\ \omega^r(0)=0 \text{ and } \omega^r(1)=E_{R_1}$$

$$\begin{array}{ll}
\omega^l(e_1)=\{0,e_1,e_5\} & \omega^r(e_1)=\{0,e_1,e_3\}\\
\omega^l(e_2)=\{0,e_2,e_6\} & \omega^r(e_2)=\{0,e_2,e_4\}\\
\omega^l(e_3)=\{0,e_3,e_4\} & \omega^r(e_3)=\{0,e_3,e_1\}\\
\omega^l(e_4)=\{0,e_4,e_3\} & \omega^r(e_4)=\{0,e_4,e_2\}\\
\omega^l(e_5)=\{0,e_5,e_1\} & \omega^r(e_5)=\{0,e_5,e_6\}\\
\omega^l(e_6)=\{0,e_6,e_2\} & \omega^r(e_6)=\{0,e_6,e_5\}
\end{array}$$

Clearly $\omega^l(e_1)=\omega^l(e_5), \omega^l(e_2)=\omega^l(e_6), \omega^l(e_3)=\omega^l(e_4)$, similarly, $\omega^r(e_1)=\omega^r(e_3), \omega^r(e_2)=\omega^r(e_4), \omega^r(e_5)=\omega^r(e_6)$. Thus

$$e_1\mathcal{L}e_5,\ e_2\mathcal{L}e_6,\ e_4\mathcal{L}e_3$$

and

$$e_1\mathcal{R}e_3,\ e_2\mathcal{R}e_4,\ e_5\mathcal{R}e_6.$$

It can viewed that in this ring R_1,

$$\omega^l=(\omega^l)^{-1}=\mathcal{L} \text{ and } \omega^r=(\omega^r)^{-1}=\mathcal{R}.$$

Also it is seen that the cardinality of the $\omega^l(\omega^r)$ class is 3. The egg-box picture of the idempotents is the following:

$$\begin{array}{c|c|c|c|c}
\boxed{1} & & & & \\
\hline
 & e_6 & e_5 & & \\
\hline
 & e_2 & & e_4 & \\
\hline
 & & e_1 & e_3 & \\
\hline
 & & & & \boxed{0}
\end{array}$$

The M-set $M(e,f)=\omega^l(e)\cap\omega^r(f)$ for the ring R_1 is as follows:
$M(e_1,e_2)=M(e_2,e_1)=M(e_1,e_4)=M(e_6,e_1)=M(e_2,e_3)=M(e_5,e_2)=M(e_4,e_5)=M(e_4,e_6)=M(e_5,e_4)=M(e_3,e_5)=M(e_3,e_6)=M(e_6,e_3)=\{0\}$
$M(e_1,e_3)=M(e_5,e_3)=M(e_5,e_1)=\{0,e_1\}$
$M(e_4,e_1)=M(e_3,e_1)=M(e_4,e_3)=\{0,e_3\}$
$M(e_2,e_4)=M(e_6,e_4)=M(e_6,e_2)=\{0,e_2\}$
$M(e_5,e_6)=M(e_1,e_5)=M(e_1,e_6)=\{0,e_5\}$
$M(e_3,e_2)=M(e_4,e_3)=M(e_3,e_4)=\{0,e_4\}$
$M(e_6,e_5)=M(e_2,e_5)=M(e_2,e_6)=\{0,e_6\}$
The sandwich set of two idempotents e and f is the maximum element in the set $M(e,f)=(\omega^l(e)\cap\omega^r(f),<)$ where $g,h\in M(e,f)$, $g<h\iff eg\;\omega^r\;eh$ and $gf\;\omega^l\;hf$.
Hence the sandwich set of the idempotents in $M_2(\mathbb{Z}_2)$ are:
$S(e_1,e_2)=S(e_2,e_1)=S(e_1,e_4)=S(e_6,e_1)=S(e_2,e_3)=S(e_5,e_2)=S(e_4,e_5)=S(e_4,e_6)=S(e_5,e_4)=S(e_3,e_5)=S(e_3,e_6)=S(e_6,e_3)=\{0\}$
$S(e_1,e_3)=S(e_5,e_1)=S(e_5,e_3)=\{e_1\}$
$S(e_2,e_4)=S(e_6,e_2)=S(e_6,e_4)=\{e_2\}$
$S(e_3,e_1)=S(e_4,e_1)=S(e_4,e_3)=\{e_3\}$
$S(e_1,e_5)=S(e_1,e_6)=S(e_5,e_6)=\{e_5\}$
$S(e_3,e_2)=S(e_4,e_2)=S(e_3,e_4)=\{e_4\}$
$S(e_2,e_5)=S(e_2,e_6)=S(e_6,e_5)=\{e_6\}$

From the above computations, it can be seen that the set $M(e_i,e_j)$

can contain at most 2 elements. That is $|M(e_i, e_j)| \leq 2$.
The elements in the sandwich sets can be obtained from the egg-box picture of the semigroup. It is observed that the elements in the sandwich set $S(e_i, e_j)$ is that idempotent element in $L_{e_i} \cap R_{e_j}$ and 0 in the case the class $L_{e_i} \cap R_{e_j}$ has no idempotents.

Example 2.1.2. Biordered set of the ring $M_2(\mathbb{Z}_3)$
Consider the matrix ring $R_1 = M_2(\mathbb{Z}_3)$. This ring has 81 elements out of which there are 14 idempotents. The idempotents of this ring E_{R_2} are listed below.

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, e_8 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, e_{20} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, e_{28} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, e_{29} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$e_{31} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, e_{34} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, e_{37} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, e_{46} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix},$$

$$e_{69} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, e_{81} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

The ω^r and ω^l class of these idempotents are:

$$\omega^l(0) = 0 \text{ and } \omega^l(1) = E_{R_2} \ \omega^r(0) = 0 \text{ and } \omega^r(1) = E_{R_2}$$

$$\begin{aligned}
\omega^l(e_2) &= \{0, e_2, e_{11}, e_{20}\} & \omega^r(e_2) &= \{0, e_2, e_5, e_8\} \\
\omega^l(e_5) &= \{0, e_5, e_{37}, e_{81}\} & \omega^r(e_5) &= \{0, e_2, e_5, e_8\} \\
\omega^l(e_8) &= \{0, e_8, e_{46}, e_{69}\} & \omega^r(e_8) &= \{0, e_2, e_5, e_8\} \\
\omega^l(e_{11}) &= \{0, e_2, e_{11}, e_{20}\} & \omega^r(e_{11}) &= \{0, e_{11}, e_{31}, e_{81}\}
\end{aligned}$$

$$\begin{array}{ll}
\omega^l(e_{20}) = \{0, e_2, e_{11}, e_{20}\} & \omega^r(e_{20}) = \{0, e_{20}, e_{34}, e_{69}\} \\
\omega^l(e_{28}) = \{0, e_{28}, e_{31}, e_{34}\} & \omega^r(e_{28}) = \{0, e_{28}, e_{37}, e_{46}\} \\
\omega^l(e_{31}) = \{0, e_{28}, e_{31}, e_{34}\} & \omega^r(e_{31}) = \{0, e_{11}, e_{31}, e_{81}\} \\
\omega^l(e_{34}) = \{0, e_{28}, e_{31}, e_{34}\} & \omega^r(e_{34}) = \{0, e_{20}, e_{34}, e_{69}\} \\
\omega^l(e_{37}) = \{0, e_5, e_{37}, e_{81}\} & \omega^r(e_{37}) = \{0, e_{28}, e_{37}, e_{46}\} \\
\omega^l(e_{46}) = \{0, e_8, e_{46}, e_{69}\} & \omega^r(e_{46}) = \{0, e_{28}, e_{37}, e_{46}\} \\
\omega^l(e_{69}) = \{0, e_8, e_{46}, e_{69}\} & \omega^r(e_{69}) = \{0, e_{20}, e_{34}, e_{69}\} \\
\omega^l(e_{81}) = \{0, e_5, e_{37}, e_{81}\} & \omega^r(e_{81}) = \{0, e_{11}, e_{31}, e_{46}\}
\end{array}$$

It is easily observed that

$$\begin{array}{rr}
\omega^l(e_2) = \omega^l(e_{11}) = \omega^l(e_{20}), & \omega^r(e_2) = \omega^r(e_5) = \omega^r(e_8) \\
\omega^l(e_5) = \omega^l(e_{37}) = \omega^l(e_{81}), & \omega^r(e_{11}) = \omega^r(e_{31}) = \omega^r(e_{81}) \\
\omega^l(e_8) = \omega^l(e_{46}) = \omega^l(e_{69}), & \omega^r(e_{20}) = \omega^r(e_{34}) = \omega^r(e_{69}) \\
\omega^l(e_{28}) = \omega^l(e_{31}) = \omega^l(e_{34}), & \omega^r(e_{28}) = \omega^r(e_{37}) = \omega^r(e_{46})
\end{array}$$

Also it can be seen that every $\omega^l(\omega^r)$ class has equal number of elements each and the cardinality of the $\omega^l(\omega^r)$ class is 4. The egg-box picture of these idempotents can be drawn as follows:

1					
		e_{11}	e_{81}	e_{31}	
	e_8	e_2	e_5		
	e_{69}	e_{20}		e_{34}	
	e_{46}		e_{37}	e_{28}	
					0

From the above computations, here also it can be seen that the set $M(e_i, e_j)$ contains at most 2 elements. That is $|M(e_i, e_j)| \leq 2$.
The elements in the sandwich sets is obtained from the egg-box picture as given below.

$$S(e_i, e_j) = \begin{cases} L_{e_i} \cap R_{e_j}, \text{ whenever } L_{e_i} \cap R_{e_j} \in E_R \\ 0 \text{ otherwise} \end{cases}$$

Example 2.1.3. Biordered set of the ring $M_2(\mathbb{Z}_4)$
Consider the matrix ring $R_3 = M_2(\mathbb{Z}_4)$. This matrix ring has 256 elements, of which there are 26 idempotents. The idempotents are as follows:

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathrm{e}_6 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \mathrm{e}_{10} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

$$e_{14} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, e_{18} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, e_{34} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, e_{42} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix},$$

$$e_{50} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}, e_{65} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, e_{66} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, e_{69} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$e_{73} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, e_{77} = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}, e_{81} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, e_{97} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix},$$

$$e_{105} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}, e_{113} = \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix}, e_{156} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}, e_{168} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix},$$

$$e_{176} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}, e_{188} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}, e_{219} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, e_{231} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix},$$

$$e_{239} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}, e_{251} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

It can be clearly seen that for each $e \in E_{R_3}$, $(1 - e) \in E_{R_3}$. The ω^r and ω^l class of these idempotents are:

$$\omega^l(0) = 0 \text{ , and } \omega^l(1) = E_{R_3}, \; \omega^r(0) = 0 \text{ , and } \omega^r(1) = E_{R_3}$$

$\omega^l(e_2) = \{0, e_2, e_{18}, e_{34}, e_{50}\}$ $\omega^r(e_2) = \{0, e_2, e_6, e_{10}, e_{14}\}$

$\omega^l(e_6) = \{0, e_6, e_{81}, e_{176}, e_{251}\}$ $\omega^r(e_6) = \{0, e_2, e_6, e_{10}, e_{14}\}$

$\omega^l(e_{10}) = \{0, e_{10}, e_{42}, e_{156}, e_{188}\}$ $\omega^r(e_{10}) = \{0, e_2, e_6, e_{10}, e_{14}\}$

$\omega^l(e_{14}) = \{0, e_{14}, e_{113}, e_{168}, e_{219}\}$ $\omega^r(e_{14}) = \{0, e_2, e_6, e_{10}, e_{14}\}$

$\omega^l(e_{18}) = \{0, e_2, e_{18}, e_{34}, e_{50}\}$ $\omega^r(e_{18}) = \{0, e_{18}, e_{69}, e_{188}, e_{239}\}$

$\omega^l(e_{34}) = \{0, e_2, e_{18}, e_{34}, e_{50}\}$ $\omega^r(e_{34}) = \{0, e_{34}, e_{42}, e_{168}, e_{176}\}$

$\omega^l(e_{42}) = \{0, e_{10}, e_{42}, e_{156}, e_{188}\}$ $\omega^r(e_{42}) = \{0, e_{34}, e_{42}, e_{168}, e_{176}\}$

$\omega^l(e_{50}) = \{0, e_2, e_{18}, e_{34}, e_{50}\}$ $\omega^r(e_{50}) = \{0, e_{50}, e_{77}, e_{156}, e_{231}\}$

$\omega^l(e_{65}) = \{0, e_{65}, e_{69}, e_{73}, e_{77}\}$ $\omega^r(e_{65}) = \{0, e_{65}, e_{81}, e_{97}, e_{113}\}$

$\omega^l(e_{69}) = \{0, e_{65}, e_{69}, e_{73}, e_{77}\}$ $\omega^r(e_{69}) = \{0, e_{18}, e_{69}, e_{188}, e_{239}\}$

$\omega^l(e_{73}) = \{0, e_{65}, e_{69}, e_{73}, e_{77}\}$ $\omega^r(e_{73}) = \{0, e_{73}, e_{105}, e_{219}, e_{251}\}$

$\omega^l(e_{77}) = \{0, e_{65}, e_{69}, e_{73}, e_{77}\}$ $\omega^r(e_{77}) = \{0, e_{50}, e_{77}, e_{156}, e_{231}\}$

$\omega^l(e_{81}) = \{0, e_6, e_{81}, e_{176}, e_{251}\}$ $\omega^r(e_{81}) = \{0, e_{65}, e_{81}, e_{97}, e_{113}\}$

$\omega^l(e_{97}) = \{0, e_{97}, e_{105}, e_{231}, e_{239}\}$ $\omega^r(e_{97}) = \{0, e_{65}, e_{81}, e_{97}, e_{113}\}$

$\omega^l(e_{105}) = \{0, e_{97}, e_{105}, e_{231}, e_{239}\}$ $\omega^r(e_{105}) = \{0, e_{73}, e_{105}, e_{219}, e_{251}\}$

$\omega^l(e_{113}) = \{0, e_{14}, e_{113}, e_{168}, e_{219}\}$ $\omega^r(e_{113}) = \{0, e_{65}, e_{81}, e_{97}, e_{113}\}$

$\omega^l(e_{156}) = \{0, e_{10}, e_{42}, e_{156}, e_{188}\}$ $\omega^r(e_{156}) = \{0, e_{50}, e_{77}, e_{156}, e_{231}\}$

$\omega^l(e_{168}) = \{0, e_{14}, e_{113}, e_{168}, e_{219}\}$ $\omega^r(e_{168}) = \{0, e_{34}, e_{42}, e_{168}, e_{176}\}$

$\omega^l(e_{176}) = \{0, e_6, e_{81}, e_{176}, e_{251}\}$ $\omega^r(e_{176}) = \{0, e_{34}, e_{42}, e_{168}, e_{176}\}$

$\omega^l(e_{188}) = \{0, e_{10}, e_{42}, e_{156}, e_{188}\}$ $\omega^r(e_{188}) = \{0, e_{18}, e_{69}, e_{188}, e_{239}\}$

$\omega^l(e_{219}) = \{0, e_{14}, e_{113}, e_{168}, e_{219}\}$ $\omega^r(e_{219}) = \{0, e_{73}, e_{105}, e_{219}, e_{251}\}$

$\omega^l(e_{231}) = \{0, e_{97}, e_{105}, e_{231}, e_{239}\}$ $\omega^r(e_{231}) = \{0, e_{50}, e_{77}, e_{156}, e_{231}\}$

$\omega^l(e_{239}) = \{0, e_{97}, e_{105}, e_{231}, e_{239}\}$ $\omega^r(e_{239}) = \{0, e_{18}, e_{69}, e_{188}, e_{239}\}$

$\omega^l(e_{251}) = \{0, e_6, e_{81}, e_{176}, e_{251}\}$ $\omega^r(e_{251}) = \{0, e_{73}, e_{105}, e_{219}, e_{251}\}$

From the above ω^l-classes, it can be easily observed that

$$\begin{aligned}
&\omega^l(e_2) = \omega^l(e_{18}) = \omega^l(e_{34}) = \omega^l(e_{50})\\
&\omega^r(e_2) = \omega^r(e_6) = \omega^r(e_{10}) = \omega^r(e_{14})\\
&\omega^l(e_6) = \omega^l(e_{81}) = \omega^l(e_{176}) = \omega^l(e_{251})\\
&\omega^r(e_{18}) = \omega^r(e_{69}) = \omega^r(e_{188}) = \omega^r(e_{239})\\
&\omega^l(e_{10}) = \omega^l(e_{42}) = \omega^l(e_{156}) = \omega^l(e_{188})\\
&\omega^r(e_{34}) = \omega^r(e_{42}) = \omega^r(e_{168}) = \omega^r(e_{176})\\
&\omega^l(e_{14}) = \omega^l(e_{113}) = \omega^l(e_{168}) = \omega^l(e_{219})\\
&\omega^r(e_{50}) = \omega^r(e_{77}) = \omega^r(e_{156}) = \omega^r(e_{231})\\
&\omega^l(e_{65}) = \omega^l(e_{69}) = \omega^l(e_{73}) = \omega^l(e_{77})\\
&\omega^r(e_{65}) = \omega^r(e_{81}) = \omega^r(e_{97}) = \omega^r(e_{113})\\
&\omega^l(e_{97}) = \omega^l(e_{105}) = \omega^l(e_{231}) = \omega^l(e_{239})\\
&\omega^r(e_{73}) = \omega^r(e_{105}) = \omega^r(e_{219}) = \omega^r(e_{251})
\end{aligned}$$

Also it can be seen that every $\omega^l(\omega^r)$ class has equal number of elements each and the cardinality of the $\omega^l(\omega^r)$ class is 5. The egg-box picture of these idempotents can be drawn as follows:

1							
	e_{97}	e_{113}		e_{81}		e_{65}	
		e_{168}	e_{34}	e_{176}	e_{42}		
	e_{239}		e_{18}		e_{188}	e_{69}	
		e_{14}	e_2	e_6	e_{10}		
	e_{231}		e_{50}		e_{156}	e_{77}	
	e_{105}	e_{219}		e_{251}		e_{73}	
							0

From the above computations, here also it can be seen that the set $M(e_i, e_j)$ contains at most 2 elements. That is $|M(e_i, e_j)| \leq 2$.
The elements in the sandwich sets is obtained from the egg-box picture

as given below.

$$S(e_i, e_j) = \begin{cases} L_{e_i} \cap R_{e_j}, \text{ whenever } L_{e_i} \cap R_{e_j} \in E_R \\ 0 \text{ otherwise} \end{cases}$$

Let R be the ring of all 2×2 matrices over a finite field then we have the following theorem:

Theorem 2.1.1. *Let $R = M_2(\mathbb{Z}_p)$ and E_R denote the set of all idempotents in R. Then for $E_i, E_j \in E_R$,*

1. *All the idempotents in R other than 0 and 1 are in the same $\mathscr{D}$-class.*
2. *$|M(E_i, E_j)| \leq 2$, that is $M(E_i, E_j)$ has at most two elements.*
3. *Each $\omega^l(\omega^r)$- ideal has the same number of elements*
4. *The elements in the sandwich set can be characterized as:*

$$S(E_i, E_j) = \begin{cases} L_{E_i} \cap R_{E_j}, \textit{ whenever } L_{E_i} \cap R_{E_j} \in E_R \\ 0 \textit{ otherwise} \end{cases}$$

Proof.

1. In $M_2(\mathbb{Z}_p)$, all the idempotents other than 0 and 1 have the same rank. Therefore, from Lemma(2.1)([17]), all the idempotents with same rank are $\mathscr{D}$- related. Therefore, they lie in the same same $\mathscr{D}$-class.

2. Now we prove that $|M(E_i, E_j)|$ cannot exceed 2. For that suppose there exists two idempotents E_h and E_k in $M(E_i, E_j)$ other than 0. We prove that $E_h = E_k$.
   In the ring $R = M_2(\mathbb{Z}_p)$, $E_i \ \omega^l \ E_j$ implies $\omega^l(E_i) = \omega^l(E_j)$. Thus we have $\omega^l(E_i) = \omega^l(E_h)$, $\omega^r(E_j) = \omega^r(E_h)$ and $\omega^l(E_i) = \omega^l(E_k)$, $\omega^r(E_j) = \omega^r(E_k)$. Therefore, $\omega^l(E_i) = \omega^l(E_h) = \omega^l(E_k)$ and

$\omega^r(E_j) = \omega^r(E_h) = \omega^r(E_k)$. Hence

$$E_h = E_j E_h E_i = E_k E_j E_h E_i E_k = E_k E_h E_i E_k = E_k E_h E_k = E_k$$

3. We know that in this ring R, $\omega^l = (\omega^l)^{-1} = \mathcal{L}$. We have by Greens lemma, that any two $\mathcal{L}$-classes contained in the same $\mathscr{D}$-class have the same number of elements. Here any two $\omega^l(\omega^r)$-class generated by idempotents other than 0 and 1 are in the same $\mathscr{D}$-class. Therefore, any two $\omega^l(\omega^r)$-class in this $\mathscr{D}$-class has the same number of elements.

4. We show that

$$S(E_i, E_j) = \begin{cases} L_{E_i} \cap R_{E_j}, \text{ whenever } L_{E_i} \cap R_{E_j} \in E_R \\ 0 \text{ otherwise} \end{cases}$$

Since E_i and E_j have the same rank, $E_i \mathscr{D} E_j$. Therefore, there exists $M_{ij} \in E_R$ such that $E_i \mathscr{L} M_{ij} \mathscr{R} E_j$. Therefore, by definition of $\mathscr{L}$ and $\mathscr{R}$ classes, there exists matrices X, X', Y, Y' such that

$$XE_i = M_{ij} \text{ and } E_j X' = M_{ij}$$

$$YM_{ij} = E_i \text{ and } M_{ij} Y' = E_j$$

Suppose $M_{ij} \in E_R$. Then

$$E_i M_{ij} = (YM_{ij})M_{ij} = YM_{ij}^2 = YM_{ij} = E_i$$

$$M_{ij} E_i = (XE_i)E_i = XE_i^2 = XE_i = M_{ij}$$

Similarly, $M_{ij}E_j = E_j$ and $E_j M_{ij} = M_{ij}$. Also,

$$M_{ij}(E_i E_j)M_{ij} = (M_{ij}E_i)E_j M_{ij} = M_{ij}E_j M_{ij} = E_j M_{ij} = M_{ij}$$

$$E_i E_j M_{ij} E_i E_j = E_i(E_j M_{ij})E_i E_j = E_i M_{ij} E_i E_j = E_i M_{ij} E_j = E_i E_j$$

Thus $M_{ij} \in V(E_i, E_j)$. Moreover,

$$M_{ij}E_i = (XE_i)E_i = XE_i^2 = XE_i = M_{ij}$$

and

$$E_jM_{ij} = E_j(E_jX') = E_j^2X' = E_jX' = M_{ij}$$

Therefore, $M_{ij} \in S(E_i, E_j)$.
Suppose $M_{ij} \notin E_R$. But $M_{ij}E_i = M_{ij} = E_jM_{ij}$. Since the ring R is regular, $S(E_i, E_j) \neq \emptyset$. But $0 \in M(E_i, E_j)$. Therefore, $S(E_i, E_j) = \{0\}$.
Conversely, if $H \in S(E_i, E_j)$ then $H \ \omega^l \ E_i$ and $H \ \omega^r \ E_j$. But in this ring $M_2(\mathbb{Z}_p)$, $H \ \omega^l \ E_i$ implies $H\mathcal{L}E_i$ and $H \ \omega^r \ E_j$ implies $H\mathcal{R}E_j$. Hence $H \in L_{E_i} \cap R_{E_j}$ and if $H = 0$, then $S(E_i, E_j) = \{0\}$.
□

2.2 Additive Idempotents in Regular Rings

In general the only idempotents in a regular ring which are idempotents with respect to addition are 0 and 1. Hence the biordered set theory for the set of additive idempotents in a regular ring collapses to trivial. However, if we define an addition $\oplus$ on the ring $(R, +, \cdot)$ by

$$a \oplus b = a + b - ab \text{ for every } a, b \in R$$

it is easily seen that $\oplus$ is an associative binary operation on R and the additive reduct $(R, \oplus)$ is a semigroup.
Let e be an idempotent in R. Now consider e as an element in $(R, \oplus)$. Then

$$e \oplus e = e + e - e.e = e.$$

That is, e is also an idempotent in $(R, \oplus)$. Thus every multiplicative idempotent in the ring $(R, +, \cdot)$ is an additive idempotent with respect to $\oplus$ in R. We denote the idempotent set in $(R, \oplus)$ by $E_R^{\oplus}$. As sets the multiplicative biordered set E_R coincides with $E_R^{\oplus}$, further it is seen that the biorder structure on $E_R^{\oplus}$ is determined by the biorder structure on E_R.

Lemma 2.2.1. Let R be a ring and e, f be idempotents in R. Then

1. $e\,\omega^l\,f$ in $E_R^{\oplus} \Longleftrightarrow f\,\omega^r\,e$ in E_R,

2. $e\,\omega^r\,f$ in $E_R^{\oplus} \Longleftrightarrow f\,\omega^l\,e$ in E_R.

Proof. Let $e\,\omega^l\,f$ in $E_R^{\oplus}$ then $e \oplus f = e$. Therefore by definition,

$$e \oplus f = e + f - ef = e$$

That is, $ef = f$ therefore, $f\ \omega^r\ e$. Conversely, let $f\ \omega^r\ e$ in E_R, then $ef = f$. Therefore,

$$e \oplus f = e + f - ef = e + f - f = e.$$

Thus $e \oplus f = e$. That is $e\,\omega^l\,f$ in $E_R^{\oplus}$. Similarly, assume that $e\,\omega^r\,f$ in $E_R^{\oplus}$ then $f \oplus e = e$. Therefore by definition,

$$f \oplus e = f + e - fe = e$$

That is $fe = f$, therefore, $f\,\omega^l\,e$. Conversely, let $f\,\omega^l\,e$ in E_R, then $fe = f$. Therefore,

$$f \oplus e = f + e - fe = f + e - f = e.$$

Thus $e\,\omega^r\,f$ in $E_R^{\oplus}$. □

Let

$$D_{E_R^{\oplus}} = \left(\omega^r \cup \omega^l\right) \cup \left(\omega^r \cup \omega^l\right)^{-1}$$

For $(e, f) \in D_{E_R^{\oplus}}$ either $(e, f) \in \omega^r \cup \omega^l$ or $(f, e) \in \omega^r \cup \omega^l$. In the first case, either $f \oplus e = e$ or $e \oplus f = e$. If $f \oplus e = e$ then $(e \oplus f)^2 = (e \oplus f) \oplus (e \oplus f) = e \oplus (f \oplus e) \oplus f = e \oplus e \oplus f = e \oplus f$ and so $e \oplus f \in E_R^{\oplus}$. Thus $e \oplus f \in E_R^{\oplus}$ whenever $(e, f) \in \omega^r \cup \omega^l$. Similarly, it can be seen that $e \oplus f \in E_R^{\oplus}$ whenever $(f, e) \in \omega^r \cup \omega^l$ also. Thus, by restricting the operations in the ring R to $(D_{E_R}, \oplus)$ we obtain a partial algebra on $E_R^{\oplus}$. Let E_R denote the biordered set with the relation ω^r replaced by $(\omega^l)^{-1}$ and ω^l by $(\omega^r)^{-1}$. Thus we can say that as biordered sets, E_R is same as $E_R^{\oplus}$.

For $e, f \in E_R^{\oplus}$ the set $\tilde{M}(e, f)$ of e and f in that order is defined by

$$\tilde{M}(e, f) = \{g : e\omega^r g \,\text{and}\, f\omega^l g\}.$$

For $g,\ h \in \tilde{M}(e, f)$ we define $g \prec h$ if and only if $h < g$ in $M(f, e)$. The sandwich set of e and f (in that order) in $E_R^{\oplus}$ is defined as follows

$$\tilde{S}(e, f) = \left\{ g \in \tilde{M}(e, f) \ \text{such that} \ g \prec h \ \text{for all} \ h \in \tilde{M}(e, f) \right\}.$$

If E and F are biordered sets, a bimorphism $\phi : E \to F$ is called a biorder isomorphism if ϕ is bijective. That is,

$$e\omega^r f \ \text{if and only if} \ e\phi\omega^r f\phi \ \text{and} \ (ef)\phi = (e\phi)(f\phi)$$

$$e\omega^l f \ \text{if and only if} \ e\phi\omega^l f\phi \ \text{and} \ (fe)\phi = f\phi e\phi.$$

If E and F are biordered sets a bijective map $\phi : E \to F$ which preserves product and satisfies

$$f\omega^r e \ \text{if and only if} \ e\phi\omega^l f\phi \ \text{and} \ f\omega^l e \ \text{if and only if} \ e\phi\omega^r f\phi$$

then ϕ is called a biorder anti isomorphism. Two biordered sets E and F are said to be anti isomorphic if there exists an anti biorder isomorphism between them.

Theorem 2.2.1. *The biordered sets* E_R *and* $E_R^{\oplus}$ *derived from the ring* $(R, +, \cdot)$ *are anti isomorphic.*

Proof. Consider the map ϕ from E_R to $E_R^{\oplus}$ defined by

$$(e \cdot f)\phi = e\phi \oplus f\phi, \text{ for all } e, f \in E_R$$

clearly ϕ is a bijective homomorphism. For $f\ \omega^r\ e$ in E_R, we have

$$\begin{aligned}(e)\phi \oplus (f)\phi &= (e)\phi + (f)\phi - (ef)\phi \\ &= (e)\phi + (f)\phi - (f)\phi \\ &= (e)\phi\end{aligned}$$

That is, $e\phi\ \omega^l\ f\phi$. Similarly it is seen that if $f\ \omega^l\ e$ in E_R then $(e)\phi\ \omega^r\ (f)\phi$ in $E_R^{\oplus}$. Thus E_R and $E_R^{\oplus}$ are anti isomorphic. □

Example 2.2.1. Consider the ring $(R, +, \cdot)$, where $x \cdot y = x \wedge y$ and $x + y = (x \wedge y') \vee (x' \wedge y)$, clearly $x \cdot x = x$ and $x + x = 0$, that is $(R, +, \cdot)$ is a multiplicative band in which $x = -x$. Thus $(R, +, \cdot)$ is a Boolean ring. Define $\oplus$ in R by

$$x \oplus y = x + y + x \cdot y, \text{ for all } x, y \in R$$

then $E_R = E_R^{\oplus} = R$ and since R is commutative we have $\omega^r = \omega^l = \omega$. Thus the sandwich set of e and f in E_R is $S(e, f) = \{ef\}$ and if $e\omega^l f$ $S(e, f) = \{e\}$ and in $E_R^{\oplus}$, $\tilde{S}(e, f) = e \oplus f = \{f\}$.

In the following example we consider the semigroup ring $\mathbb{Z}_2\,[R_2]$, (see cf.[6] page 47).

Example 2.2.2. Let $R_2 = \{x, y\}$ be the two element right zero band. Consider the ring $\mathbb{Z}_2\,[R_2] = \{0, x, y, x + y\}$ with operations '$+$' and '$\cdot$' defined by

$+$	0	x	y	$x+y$
0	0	x	y	$x+y$
x	x	0	$x+y$	y
y	y	$x+y$	0	x
$x+y$	$x+y$	y	x	0

$\cdot$	0	x	y	$x+y$
0	0	0	0	0
x	0	x	y	$x+y$
y	0	x	y	$x+y$
$x+y$	0	$x+y$	$x+y$	0

clearly $E_R = \{0, x, y\}$ is the set of idempotents with respect to '$\cdot$' and with respect to the addition $\oplus$ defined by

$$x \oplus y = x + y + x \cdot y \text{ for all } x, y \in \mathbb{Z}_2[R_2]$$

the idempotent set $E_R^{\oplus}$ coincides with E_R. The biorder relations in the semigroup ring $\mathbb{Z}_2\,[R_2]$ are $x\mathcal{R}y, 0\ \omega\ x, 0\ \omega\ y$, $\omega^l(x) = \{0, x\}\,, \omega^r(y) = \{0, x, y\}$ and

$$M(x, y) = (\omega^l(x) \cap \omega^r(y), <) = \{0, x\}\,,$$

$S(x, y)$ being the maximum of elements in $M(x, y)$ with respect to $<$, we have

$$S(x, y) = \{x\}\,.$$

The quasi ordered set $\tilde{M}(x, y) = \{y\}$ and additive sandwich set $\tilde{S}(x, y) = \{y\}$.

Example 2.2.3. Consider the ring $\mathbb{Z}_2\,[R_2^1]$ where R_2^1 is 1 included to R_2 in the previous example. Thus $\mathbb{Z}_2\,[R_2^1] = \{0, 1, x, y, x + y, 1 + x, 1 + y\}$ with operations $+$ and . defined by

$+$	0	1	x	y	$1+x$	$1+y$	$x+y$
0	0	1	x	y	$1+x$	$1+y$	$x+y$
1	1	0	$1+x$	$1+y$	x	y	$1+x+y$
x	x	$1+x$	0	$x+y$	1	$1+x+y$	y
y	y	$1+y$	$x+y$	0	$1+x+y$	1	x
$1+x$	$1+x$	x	1	$1+x+y$	0	$x+y$	$1+y$
$1+y$	$1+y$	y	$1+x+y$	1	$x+y$	0	$1+x$
$x+y$	$x+y$	$1+x+y$	y	x	$1+y$	$1+x$	0

$.$	0	1	x	y	$1+x$	$1+y$	$x+y$
0	0	0	0	0	0	0	0
1	0	1	x	y	$1+x$	$1+y$	$x+y$
x	0	x	x	y	0	$x+y$	$x+y$
y	0	y	x	y	$x+y$	0	$x+y$
$1+x$	0	$1+x$	0	0	$1+x$	$1+x$	0
$1+y$	0	$1+y$	0	0	$1+y$	$1+y$	0
$x+y$	0	$x+y$	0	0	$x+y$	$x+y$	0

The biordered set is $\{1,0,x,y,1+x,1+y\}$ with biorder relation $x\ \omega\ 1,\ \ y\ \omega\ 1,\ \ x\mathcal{R}y,\ \ (1+x)\ \omega\ 1,\ \ (1+y)\ \omega\ 1,\ \ (1+x)\mathcal{L}(1+y),\ \ 0\ \omega\ x,\ \ 0\ \omega\ y,\ \ 0\ \omega\ (1+x),\ \ 0\ \omega\ (1+y)$. Thus $\omega^l(x)=\{0,x\}\,,\omega^r(y)=\{0,x,y\}$. Hence

$$M(x,y)=\{x,0\} \text{ and } S(x,y)=\{x\}\,.$$

Since, $x\in\omega^r(y),\ y\in\omega^l(y)$ and $x\in\omega^r(1),\ y\in\omega^l(1)$, we have

$$\tilde{M}(x,y)=\{1,y\} \text{ and since } y\ \omega\ 1\ \tilde{S}(x,y)=\{y\}$$

Example 2.2.4. Let $B=\{e,f,ef\}$ be the three element band whose biordered relations are defined by $e\ \omega^r\ f, e\mathcal{R}ef, ef\ \omega\ f$. Now consider the semigroup ring $\mathbb{Z}_2\,[B]=\{0,e,f,ef,e+f,e+ef,f+ef\}\,.$ Then it has the following biordered set $\{0,e,f,ef,f+ef,e\oplus f\}$ with relations defined by $0\ \omega\ e$, $0\ \omega\ f$, $0\ \omega\ ef$, $0\ \omega\ (f+ef)$, $e\ \omega\ (e\oplus f)$, $e\mathcal{R}ef$, $e\ \omega^r\ f$, $ef\ \omega\ f$, $(f+ef)\ \omega\ f$, $(e\oplus f)\mathcal{R}f$. Consider idempotents $e\oplus f$ and f. Then $\omega^l(e\oplus f)=\{0,e\oplus f,e,f+ef\}$ and $\omega^r(f)=$

$\{0, f, e, ef, f + ef, e \oplus f\}$. Hence,

$$M(e \oplus f, f) = \{0, e, f + ef, e \oplus f\}\,.$$

We have $0 \ \omega \ e$, $0 \ \omega \ f + ef$, $0 \ \omega \ e \oplus f$, $f + ef \ \omega \ e \oplus f$, $e \ \omega \ e \oplus f$. Thus

$$S(e \oplus f, f) = \{e \oplus f\}$$

Now, since $e \oplus f \in \omega^r(f)$ and $f \in \omega^l(f)$, the additive sandwich set is given by

$$\tilde{M}(e \oplus f, f) = \{f\} \text{ and } \tilde{S}(e \oplus f, f) = \{f\}\,.$$

Chapter 3

Lattice of Biorder Ideals on Regular Rings

In this chapter, we consider the principal ideals obtained from the biorders ω^r, ω^l and their intersection ω of the biordered set E_R of a regular ring $(R, +, \cdot)$, which we call the biorder ideals. We discuss several properties of the biorder ideals and it is shown that the collection of all biorder ideals Ω_l obtained from the left quasiorder ω^l and the collection of biorder ideals Ω_r obtained from the right quasiorder ω^r are complemented modular lattices. Many of the result included in this chapter has already appeared in the paper entitled *Biorder Ideals and Regular Rings*, Algebra and its Applications, Springer Proceedings in Mathematics and Statistics, ICSAA, Aligarh, 2014 Vol **174**, ISSN 2194-1017, 265-274.

3.1 Biorder Ideals of a Regular Ring

Let R be a regular ring with unity and E_R is the bounded and complemented biordered set discussed in chapter(2.1).

Then for e in E_R define

$$\omega^l(e) = \{f\colon fe = f\}\,;\quad \omega^r(e) = \{f : ef = f\}$$

where ω^r and ω^l are quasiorders defined as in [25]. Then $\omega^l(e)[\,\omega^r(e)])$ is called the left[right] principal ideal in E_R and are called the left[right]biorder ideals. The set $\omega(e) = \{f\colon fe = ef = f\}$ is the two sided biorder ideal generated by e.

Denote by Ω_r the class of all principal ω^r-ideals and by Ω_l the class of all principal ω^l-ideals. For e, $f \in E_R$ with $e\ \omega^l\ f$ then $\omega^l(e) \subseteq \omega^l(f)$. Hence we can define a relation $\leq$ on the set of all principal left biorder ideals in E_R by

$$\omega^l(e) \leq \omega^l(f) \text{ if and only if } e\ \omega^l\ f.$$

Also by the definition of $\leq$, it is obvious that $\leq$ is a partial order on E_R. Similarly we can define a partial order $\leq$ on the set Ω_r of the principal right biorder ideals in E_R by

$$\omega^r(e) \leq \omega^r(f) \text{ if and only if } e\ \omega^r\ f.$$

The following proposition shows the relation between the biorder ideals of idempotents and their inverses in a regular ring.

Proposition 3.1.1. Let e and f be idempotents in the ring R. Then the following hold.

1. $\omega^l(e) = \omega^l(f)$ if and only if $\omega^r(1-e) = \omega^r(1-f)$
2. $\omega^r(e) = \omega^r(f)$ if and only if $\omega^l(1-e) = \omega^l(1-f)$

Proof. Suppose $\omega^l(e) = \omega^l(f)$. Then from the definitions of $e\ \omega^l\ f$

and $f\ \omega^l\ e$ we have,

$$(1-e)(1-f) = 1-e-f+ef = 1-e-f+e = 1-f$$

and

$$(1-f)(1-e) = 1-f-e+fe = 1-f-e+f = 1-e.$$

Hence $(1-f)\ \omega^r\ (1-e)$ and $(1-e)\ \omega^r\ (1-f)$. Thus,

$$\omega^r(1-e) = \omega^r(1-f).$$

Similarly (2) can also be proved. □

Proposition 3.1.2. Let e and f be idempotents in the ring R, if $\omega^r(e) = \omega^r(f), \omega^r(1-e) = \omega^r(1-f)$, then $e = f$.

Proof. Suppose $\omega^r(e) = \omega^r(f)$. By above proposition $\omega^r(1-e) = \omega^r(1-f)$ implies $\omega^l(e) = \omega^l(f)$. Thus we have $\omega(e) = \omega(f)$ implies $e = f$. □

In the next lemma, it is shown that the biorder ideals of idempotents in a regular ring R is closed under the operation join and meet.

Lemma 3.1.1. Let R be a regular ring, let $e, f \in E_R$ and choose $h \in S_1(e, 1-f)$. Then

$$\omega^l(e) \vee \omega^l(f) = \omega^l(h(1-f)+f) \quad \text{and} \quad \omega^l(e) \wedge \omega^l(f) = \omega^l(e(1-h)).$$

Proof. By hypothesis, we have $h \in S_1(e, 1-f)$ so that h is in E_R with

$$he = h = (1-f)h \text{ and } eh(1-f) = e(1-f).$$

Let $k = h(1-f)$, then

$$k^2 = h((1-f)h)(1-f) = h \cdot h(1-f) = h(1-f) = k.$$

Therefore, k is an idempotent in R. Define $g = k + f$ then

$$kf = h(1-f)f = 0, \quad fk = fh(1-f) = 0.$$

Hence $g = k + f$ is an idempotent with

$$eg = e(k+f) = ek + ef = eh(1-f) + ef = e(1-f) + ef = e.$$

Hence $e\,\omega^l\,g$ and

$$\omega^l(e) \subseteq \omega^l(g).$$

Also,

$$fg = f(k+f) = fk + f = f.$$

so that $f\,\omega^l\,g$ and

$$\omega^l(f) \subseteq \omega^l(g).$$

Thus,

$$\omega^l(e) \vee \omega^l(f) \subseteq \omega^l(g).$$

But

$$g = k + f = h(1-f) + f = he - hf + f = he + (1-h)f$$

Thus

$$g \in \omega^l(e) \vee \omega^l(f).$$

Therefore,

$$\omega^l(g) \subseteq \omega^l(e) \vee \omega^l(f)$$

and so

$$\omega^l(e) \vee \omega^l(f) = \omega^l(g).$$

Next we find an idempotent $g' \in E_R$ and prove that

$$\omega^l(e) \wedge \omega^l(f) = \omega^l(g').$$

Let $g' = e(1-h)$. Then

$$(g')^2 = (e(1-h))^2 = e(e-he)(1-h) = e(e-eh) = e(1-h) = g'.$$

and

$$g'e = e(1-h)e = e - ehe = e - eh = e(1-h) = g'$$

implies, $g' \in \omega^l(e)$. Thus

$$\omega^l(g') \subseteq \omega^l(e).$$

Also,

$$g'f = e(1-h)f = ef - ehf$$

and since $eh(1-f) = e(1-f)$, we have $eh - ehf = e - ef$ from which we have $ehf = eh - e + ef$, so we get $g'f = ef - eh + e - ef = e - eh = e(1-h) = g'$ implies $g' \in \omega^l(f)$. Thus

$$\omega^l(g') \subseteq \omega^l(f).$$

Hence

$$\omega^l(g') \subseteq \omega^l(e) \wedge \omega^l(f).$$

To prove the reverse inclusion, let $x \in \omega^l(e) \wedge \omega^l(f)$, so that $xe = x = xf$. Hence

$$xg' = x(1-h) = xf(1-h) = x(1-(1-f)h) = x - x(1-f)h$$

and since $xf = x, x(1-f) = 0$, so that

$$xg' = x - x(1-f)h = x.$$

That is $x \in \omega^l(g')$. Therefore,

$$\omega^l(e) \wedge \omega^l(f) = \omega^l(g').$$

□

In the light of the above lemma, we have the following theorem.

Theorem 3.1.1. *The set of all ω^l-ideals Ω_l is closed with respect to the operation $\vee$ and $\wedge$ defined in Ω_l. Thus $(\Omega_l, \vee, \wedge)$ is a lattice.*

Next we introduce the notion of annihilators in the principal ω^r and ω^l-ideals.

Definition 3.1.1. For every ω^r-ideal we define

$$(\omega^r(e))^L = \{y \colon yz = 0 \text{ for every } z \in \omega^r(e)\}$$

and for every ω^l-ideal,

$$(\omega^l(e))^R = \left\{y \colon zy = 0 \text{ for every } z \in \omega^l(e)\right\}.$$

Then $(\omega^r(e))^L$ is a left ideal and $(\omega^l(e))^R$ is a right ideal.

Using the concept of annihilators of biorder ideals in E_R we can show that the lattice of all principal ω^l-ideals Ω_l is isomorphic with the dual of the lattice of all principal ω^r-ideals Ω_r.

Proposition 3.1.3. For $e \in E_R$, $(\omega^l(e))^R$ is a principal ω^r-ideal and $(\omega^r(e))^L$ is a principal ω^l-ideal. In fact, $(\omega^l(e))^R = \omega^r(1-e)$ and $(\omega^r(e))^L = \omega^l(1-e)$.

Proof. We have by definition,

$$\begin{aligned}
\omega^r(e) &= \{g \colon eg = g\} \\
&= \{g \colon (1-e)g = 0\} \\
&= \{g \colon u(1-e)g = 0; \text{ for every } u \in E_R\} \\
&= \left\{g \colon hg = 0 \text{ for every } h \in \omega^l(1-e)\right\}
\end{aligned}$$

where $h = u(1-e)$. Since $h(1-e) = u(1-e)(1-e) = u(1-e) = h$ we have $h \in \omega^l(1-e)$. Thus $\omega^r(e) = (\omega^l(1-e))^R$. Therefore, replacing e by $1-e$ we get,

$$\omega^r(1-e) = (\omega^l(e))^R.$$

The proof for left annihilators of principal ω^r-ideals is similar. □

Lemma 3.1.2. Let $e, f \in E_R$ and $\omega^r(e)$ and $\omega^r(f)$ be ideals generated by e and f then

1. $\omega^r(e) \subset \omega^r(f) \Rightarrow (\omega^r(e))^L \supset (\omega^r(f))^L$.
2. $\omega^r(e) = (\omega^r(e))^{LR}$ and $(\omega^r(e))^L = (\omega^r(e))^{LRL}$.

Proof. 1. Let $g \in (\omega^r(f))^L$ then $gh = 0$ for every $h \in \omega^r(f)$. If $\omega^r(e) \subset \omega^r(f)$ then $gh = 0$ for every $h \in \omega^r(e)$ thus $g \in (\omega^r(e))^L$ and so

$$(\omega^r(f))^L \subset (\omega^r(e))^L.$$

2. By above Proposition,

$$(\omega^r(e))^{LR} = ((\omega^r(e))^L)^R = (\omega^l(1-e))^R = \omega^r(1-(1-e)) = \omega^r(e)$$

and

$$(\omega^r(e))^{LRL} = (\omega^r(e))^L.$$

□

Lemma 3.1.3. Let R be a regular ring and E_R the set of idempotents in R. Define θ and ρ on Ω_l and Ω_r by

$$\theta(\omega^l(e)) = (\omega^l(e))^R \text{ and } \rho(\omega^r(e)) = (\omega^r(e))^L.$$

Then θ and ρ define a one-one correspondence between Ω_l and Ω_r and hence they are inverse anti-isomorphisms between these sets.

Proof. Let e be in E_R with $\omega^l(e) \in \Omega_l$ then

$$\theta(\omega^l(e)) = (\omega^l(e))^R = \omega^r(1-e)$$

Thus θ maps the lattice Ω_l to Ω_r. Now for idempotents $e, f \in E_R$ suppose $\omega^l(e)$, $\omega^l(f) \in \Omega_l$ such that $\omega^l(e) \subseteq \omega^l(f)$. But from the above lemma $\omega^l(e) \subseteq \omega^l(f)$ implies $(\omega^l(f)^R \subseteq (\omega^l(e))^R$ and $\theta(\omega^l(e)) \subseteq \theta(\omega^l(f))$. Similarly, ρ is an order preserving map from Ω_r to Ω_l. Moreover, for $\omega^l(e) \in \Omega_l$

$$\rho(\theta(\omega^l(e))) = \rho(\omega^r(1-e)) = (\omega^r(1-e))^L = \omega^l(1-(1-e)) = \omega^l(e).$$

Thus for $\omega^l(e) \in \Omega_l$ we have $\rho\theta(\omega^l(e)) = \omega^l(e)$ and similarly for $\omega^r(e) \in \Omega_r$, $\theta\rho(\omega^r(e)) = \omega^r(e)$. Hence θ and ρ are mutually inverse anti-isomorphisms between Ω_l and Ω_r. □

For any idempotent $e \in E_R$, $\omega^l(e) \vee \omega^l(1-e) = \omega^l(he + (1-e)) = \omega^l(e+1-e) = \omega^l(1) = E_R$ and $\omega^l(e) \wedge \omega^l(1-e) = \omega^l(e(1-e)) = \omega^l(0) = \{0\}$, since $h \in S(e,e) = \{e\}$. Thus $\omega^l(e)$ and $\omega^l(1-e)$ are complements of each other in the lattice of all principal left ω-ideals. Similarly, $\omega^r(e)$ and $\omega^r(1-e)$ are complements to each other in the lattice of all principal right ω-ideals of E_R.
Thus we have the following theorem.

Theorem 3.1.2. *Let R be a regular ring and E_R denote the set of idempotents in R. Then Ω_l the set of all principal left biorder ideals in E_R is a complemented modular lattice with respect to the order defined by*

$$\omega^l(e) \leq \omega^l(f) \text{ if and only if } e\ \omega^l\ f$$

and the join and meet are given by

$$\omega^l(e) \vee \omega^l(f) = \omega^l(h(1-f)+f) \text{ and } \omega^l(e) \wedge \omega^l(f) = \omega^l(e(1-h))$$

where $h \in S_1(e, 1-f)$ *its zero being* 0 *and its unit is* $\omega^l(1)$. *Dually, the set* Ω_r *is a complemented modular lattice and the map* $\omega^l(e) \longrightarrow \omega^r(1-e)$ *is an anti-isomorphism of* Ω_l *onto* Ω_r.

Now we consider a special case when $\omega^l = \omega^r = \omega$ and we describe the structure of the ω-ideals.

Lemma 3.1.4. Let e and f be two idempotents in E_R and $\omega(e)$ and $\omega(f)$ denote the ω-ideals generated by e and f respectively. Then $\omega(e) \vee \omega(f)$ and $\omega(e) \wedge \omega(f)$ are both principal ω-ideals.

Proof. Suppose $S_1(e, 1-f) \cap S_1(1-f, e) \neq \emptyset$. Choose an $h \in S_1(e, 1-f) \cap S_1(1-f, e)$, so that h is in E_R with

$$he = h = (1-f)h \text{ and } eh(1-f) = e(1-f),$$

$$h(1-f) = h = eh \text{ and } (1-f)he = (1-f)e$$

Define $g = h + f$. Then

$$g^2 = (h+f)(h+f) = h + hf + fh + f = h + f = g.$$

Hence g is an idempotent satisfying

$$eg = e(h+f) = eh + ef = eh + e - eh + ehf = e + ehf = e,$$

since $ef = e - eh + ehf$ and

$$ge = (h+f)e = he + fe = he + e - he + fhe = e + fhe = e$$

since $fe = e - he + fhe$ Therefore,

$$eg = ge = e, \text{ implies } \omega(e) \subseteq \omega(g)$$

that is,

$$\omega(e) \subseteq \omega(h+f)$$

Also

$$fg = f(h+f) = fh + f = f$$

and

$$gf = (h+f)f = hf + f = f.$$

But

$$fg = gf = f \text{ implies } \omega(f) \subseteq \omega(g)$$

and

$$\omega(e) \vee \omega(f) \subseteq \omega(g).$$

Thus

$$g = h + f = ehe + f \in \omega(e) \vee \omega(f)$$

and so

$$\omega(g) \subseteq \omega(e) \vee \omega(f).$$

Hence

$$\omega(e) \vee \omega(f) = \omega(h+f).$$

Let $g' = e(1-h)$. Then

$$(g')^2 = e(1-h)e(1-h) = e(e-he)(1-h) = e^2(1-h) = e(1-h).$$

with

$$g'e = e(1-h)e = e(e-he) = (e-ehe) = e(1-h) = g'$$

$$eg' = ee(1-h) = e(1-h) = g'$$

thus

$$g' \ \omega \ e \text{ hence } \omega(g') \subseteq \omega(e).$$

Now

$$g'f = e(1-h)f = e(f-hf) = ef - hf$$

and since $eh(1-f) = e(1-f)$ we have $eh - ehf = e - ef$ and

$ehf = eh - e + ef$ so we get $g'f = ef - eh + e - ef = e - eh = e(1-h) = g'$. Also,

$$fg' = fe(1-h) = fe - feh = fe - fh = fe - fhe$$

and since $(1-f)he = (1-f)e$ we have $he - fhe = e - fe$, and $fhe = he - e + fe$ so we get $fg' = fe - he + e - fe = e - he = e(1-h) = g'$. Thus $g' \; \omega \; f$ and so $\omega(g') \subseteq \omega(f)$. To prove the converse, let $x \in \omega(e) \cap \omega(f)$. Then $xe = x = xf$ and $ex = x = fx$. Then

$$xg' = xe(1-h) = xe - xeh = x(1-h) = xf(1-h) = xf - xfh = xf = x$$

and

$$g'x = e(1-h)x = ex - ehx = x - ehx = x - hx = x - hfx = x$$

that is

$$xg' = g'x = x \text{ hence } x \; \omega \; g'$$

Therefore,

$$\omega(x) \subseteq \omega(g')$$

Thus

$$\omega(e) \wedge \omega(f) = \omega(e(1-h)).$$

□

Denote the class of all ω-ideals of E_R by $\Omega(R)$. Then $\Omega(R)$ is a partially ordered set with the usual set containment.

Lemma 3.1.5. Suppose e, f be elements in E_R. Then $S_1(e, 1-f) \cap S_1(1-f, e) \neq \emptyset$ if and only if $e(1-f) = (1-f)e$. Further, in this case $h = e(1-f) = (1-f)e$ is the unique element in $S(e, 1-f) \cap S(1-f, e)$

Proof. Suppose that $S_1(e, 1-f) \cap S_1(1-f, e) \neq \emptyset$ and let $h \in$

$S_1(e, 1-f) \cap S_1(1-f, e)$. Since $h \in S(e, 1-f) = S_1(e, 1-f)$ we have

$$eh(1-f) = e(1-f)\ ;\ (1-f)he = h$$

and since $h \in S(1-f, e) = S_1(1-f, e)$ we have

$$(1-f)he = (1-f)e\ ;\ eh(1-f) = h$$

From these equations it follows that $h = e(1-f) = (1-f)e$ and therefore, $ef = fe$.

Conversely suppose that $h = e(1-f) = (1-f)e$ then

$$(1-f)he = (1-f)((1-f)e)e = (1-f)^2e^2 = (1-f)e = h$$

and

$$eh(1-f) = ee(1-f)(1-f) = e^2(1-f)^2 = e(1-f)$$

hence $h \in S_1(e, 1-f) = S(e, 1-f)$. Similarly, $h \in S(1-f, e)$. Thus $h \in S(e, 1-f) \cap S(1-f, e)$ and $S(e, 1-f) \cap S(1-f, e) \neq \emptyset$. □

Using Lemma(3.1.4) it follows that

$$\omega(e) \vee \omega(f) = \omega(h+f) = \omega(e(1-f)+f) = \omega(e+f-ef)$$

and

$$\omega(e) \wedge \omega(f) = \omega(e(1-h)) = \omega(e(1-e(1-f))) = \omega(e-e+ef) = \omega(ef).$$

Now we proceed to show that the lattice $\Omega(R)$ is a distributive lattice.

Lemma 3.1.6. Let $e, f, g \in E_R$. Then

$$(\omega(e) \vee \omega(f)) \wedge \omega(g) = (\omega(e) \wedge \omega(g)) \vee (\omega(f) \wedge \omega(g))\,.$$

Proof. By Lemma(3.1.4), we have

$$\omega(e) \vee \omega(f) = \omega(e + f - ef).$$

Therefore,

$$(\omega(e) \vee \omega(f)) \wedge \omega(g) = \omega(e + f - ef) \wedge \omega(g).$$

Again by Lemma(3.1.4), we get

$$(\omega(e) \vee \omega(f)) \wedge \omega(g) = \omega((e + f - ef)g) = \omega(eg + fg - efg)$$

and

$$(\omega(e) \wedge \omega(g)) \vee (\omega(f) \wedge \omega(g)) = \omega(eg) \vee \omega(fg)$$

thus

$$(\omega(e) \wedge \omega(g)) \vee (\omega(f) \wedge \omega(g)) = \omega(eg + fg - egfg)$$

Since $gf = fg$ we have

$$(\omega(e) \wedge \omega(g)) \vee (\omega(f) \wedge \omega(g)) = \omega(eg + fg - efg).$$

Thus the distributive law holds.

□

For any idempotent e, $f \in E_R$, $\omega(e)$ and $\omega(f)$ are complements if and only if

$$\omega(e) \vee \omega(f) = \omega(1) \text{ and } \omega(e) \wedge \omega(f) = \omega(0).$$

Since

$$\omega(e) \vee \omega(f) = \omega(e + f - ef) = \omega(1)$$

and

$$\omega(e) \wedge \omega(f) = \omega(ef) = \omega(0)$$

we have

$$e+f-ef=1 \text{ and } ef=0$$

that is

$$e+f=1 \text{ implies } f=1-e.$$

Hence

$$\omega(e)\vee\omega(1-e)=\omega(1)=E_R$$

and

$$\omega(e)\wedge\omega(1-e)=\omega(e(1-e))=\omega(0)=\{0\}\,.$$

That is $\omega(e)$ and $\omega(1-e)$ are complements of each other and $\omega(1-e)$ is the unique complement of $\omega(e)$ in the lattice of all principal ω-ideals in E_R. Thus we have the following theorem.

Theorem 3.1.3. *Let R be a regular ring with unity, then the set of all principal ω-ideals $\Omega(R)$ of E_R is a complemented, distributive lattice with its zero being* 0*, and its unit being* $\omega(1)$.

3.2 Order of the Complemented Modular Lattice

In the following we discuss the properties of the complemented modular lattice $\Omega_l[\Omega_r]$ such as perspectivity, independence, order etc.
The following lemma characterizes when two ω^l ideals are complements to each other.

Lemma 3.2.1. Two biorder ideals $\omega^l(e)$ and $\omega^l(f)$ are complements in Ω_l if and only if there exists an idempotent $h\in S(e,1-f)$ such that $\omega^l(e)=\omega^l(h)$ and $\omega^l(f)=\omega^l(1-h)$.

Proof. Suppose there exists an idempotent $h\in S(e,1-f)$ with $\omega^l(h)=\omega^l(e)$ and $\omega^l(1-h)=\omega^l(f)$, then $\omega^l(e)\vee\omega^l(f)=\omega^l(h)\vee\omega^l(1-h)=\omega^l(1)$ and $\omega^l(e)\wedge\omega^l(f)=\omega^l(h)\wedge\omega^l(1-h)=0$. Hence $\omega^l(e)$ and $\omega^l(f)$ are complements of each other.

Conversely, suppose that $\omega^l(e)$ and $\omega^l(f)$ are complements of each other in Ω_l. Then

$$\omega^l(e) \vee \omega^l(f) = \omega^l(1) \text{ and } \omega^l(e) \wedge \omega^l(f) = \{0\}$$

and there exists $h \in S_1(e, 1-f)$, so that by lemma 3.1.1

$$\omega^l(e) \vee \omega^l(f) = \omega^l(h(1-f)+f) \text{ and } \omega^l(e) \wedge \omega^l(f) = \omega^l(e(1-h))$$

Therefore,

$$\omega^l(h(1-f)+f) = \omega^l(1) \text{ and } \omega^l(e(1-h)) = \{0\}$$

and so by definition, $(h(1-f)+f)1 = h(1-f)+f$ and $1(h(1-f)+f) = 1$ and $e(1-h)0 = 0$ and $0(e(1-h)) = e(1-h)$. Hence, $h(1-f)+f = 1$ and $e(1-h) = 0$ so that $h(1-f) = (1-f)$ and $e(1-h) = 0$, thus $(1-f)\ \omega^r\ h$ and $e\ \omega^l\ h$ so that $\omega^r(1-f) \subseteq \omega^r(h)$ and $\omega^l(e) \subseteq \omega^l(h)$. Since $h \in S_1(e, 1-f)$, we have $h\ \omega^r\ (1-f)$ and $h\ \omega^l\ e$, so that $\omega^l(h) \subseteq \omega^l(e)$ and $\omega^r(h) \subseteq \omega^r(1-f)$ hence $\omega^r(h) = \omega^r(1-f)$ and $\omega^l(h) = \omega^l(e)$. Now, since $\omega^r(h) = \omega^r(1-f)$, we have $\omega^l(1-h) = \omega^l(f)$. Thus $\omega^l(h) = \omega^l(e)$ and $\omega^l(1-h) = \omega^l(f)$. □

Similarly two ω^r-ideals, $\omega^r(e)$ and $\omega^r(f)$ are complements if and only if there exists an idempotent k such that $\omega^r(e) = \omega^r(k)$ and $\omega^r(f) = \omega^r(1-k)$.

Two elements of a lattice are said to be in perspective if they have a common complement.

Now, we describe perspectivity of two members of Ω_l in a regular ring in terms of the E-sequence as follows:

Lemma 3.2.2. Let $\omega^l(e)$ and $\omega^l(f)$ be biorder ideals in Ω_l. Then $\omega^l(e)$ and $\omega^l(f)$ are perspective in Ω_l if and only if $1 \leq d_l(e,f) \leq 3$.

Proof. Suppose that $\omega^l(e)$ and $\omega^l(f)$ are in perspective. Then there exists a common complement $\omega^l(g)$ of $\omega^l(e)$ and $\omega^l(f)$ in Ω_l. Since $\omega^l(e)$ and $\omega^l(g)$ are complements of each other in Ω_l, there exists h in E_R by lemma 3.2.1 with

$$\omega^l(h) = \omega^l(e) \text{ and } \omega^l(1-h) = \omega^l(g)$$

Again, since $\omega^l(f)$ and $\omega^l(g)$ are complements of each other, there exists k in E_R such that

$$\omega^l(k) = \omega^l(f) \text{ and } \omega^l(1-k) = \omega^l(g)$$

But $\omega^l(e) = \omega^l(h)$, so that $e \mathcal{L} h$ and since $\omega^l(k) = \omega^l(f)$, we have $k \mathcal{L} f$. Also, $\omega^l(1-h) = \omega^l(g) = \omega^l(1-k)$ so $(1-h) \mathcal{L} (1-k)$ and hence $h \mathcal{R} k$. Thus $e \mathcal{L} h \mathcal{R} k \mathcal{L} f$, so the E-sequence from e to f is of length 3.
Conversely, suppose $1 \leq d_l(e,f) \leq 3$, then there exist g and h in E_R with $e \mathcal{L} g \mathcal{R} h \mathcal{L} f$. Since $e \mathcal{L} g$, we have $e\, \omega^l \cap (\omega^l)^{-1} g$, so $e\, \omega^l\, g$ and $g\, \omega^l\, e$. Thus $\omega^l(e) \subseteq \omega^l(g)$ and $\omega^l(g) \subseteq \omega^l(e)$. Hence $\omega^l(e) = \omega^l(g)$ and so $\omega^l(1-g)$ is a complement of $\omega^l(g) = \omega^l(e)$. Also, from $g \mathcal{R} h$, we have $(1-g)\mathcal{L}(1-h)$ so that $\omega^l(1-g) = \omega^l(1-h)$ and so $\omega^l(1-g)$ is a complement of $\omega^l(h)$. Moreover, from $h \mathcal{L} f$, we have $\omega^l(h) = \omega^l(f)$. Hence $\omega^l(1-g)$ is a complement of $\omega^l(h) = \omega^l(f)$, thus $\omega^l(1-g)$ is a complement of both $\omega^l(e)$ and $\omega^l(f)$. □

Definition 3.2.1. Let Ω_l be a complemented modular lattice with zero 0 and unit $\omega^l(1)$. A basis of Ω_l is a collection $\{\omega^l(e_i)\colon i = 1, 2, \ldots n\} \in \Omega_L$ such that $\{\omega^l(e_i)\colon i = 1, 2, \ldots, n\}$ are independent, $\omega^l(e_1) \vee \ldots \vee \omega^l(e_n) = \omega^l(1)$. The number of elements in a basis is called the order of the basis. Further, a basis is homogeneous if its elements are pairwise perspective.

Proposition 3.2.1. Let $e, f \in E_R$ and $f\ \omega\ (1-e)$ then

$$\omega^l(e) \vee \omega^l(f) = \omega^l(e+f),\ \omega^l(e) \wedge \omega^l(f) = \{0\}$$

in the lattice Ω_l.

Proof. Since $f\ \omega\ (1-e)$, clearly $e+f \in E_R$ and $e+f \in \omega^l(e) \vee \omega^l(f)$. Thus

$$\omega^l(e+f) \subseteq \omega^l(e) \vee \omega^l(f).$$

Also, since $e(e+f) = e$ and $f(e+f) = f$, $\omega^l(e) \subseteq \omega^l(e+f)$ and $\omega^l(f) \subseteq \omega^l(e+f)$, and so $\omega^l(e) \vee \omega^l(f) \subseteq \omega^l(e+f)$. Hence $\omega^l(e) \vee \omega^l(f) = \omega^l(e+f)$. Let $g \in \omega^l(e) \wedge \omega^l(f)$, then $ge = gf = g$, so $g = ge = gfe = 0$. Thus $\omega^l(e) \wedge \omega^l(f) = \{0\}$ whenever $f\ \omega\ (1-e)$. □

The above result can be extended to a finite number of idempotents. Let $e_1, e_2, e_3 \in E_R$ with $e_i\ \omega\ (1-e_j), i \neq j$, then

$$e_1 + e_2 + e_3 \in \omega^l(e_1) \vee \omega^l(e_2) \vee \omega^l(e_3).$$

Hence,

$$\omega^l(e_1 + e_2 + e_3) \subseteq \omega^l(e_1) \vee \omega^l(e_2) \vee \omega^l(e_3).$$

Since $e_i\ \omega^l\ (e_1 + e_2 + e_3)$, $i = 1, 2, 3$

$$\omega^l(e_i) \subseteq \omega^l(e_1 + e_2 + e_3)$$

and so,

$$\omega^l(e_1) \vee \omega^l(e_2) \vee \omega^l(e_3) \subseteq \omega^l(e_1 + e_2 + e_3).$$

Thus

$$\omega^l(e_1) \vee \omega^l(e_2) \vee \omega^l(e_3) = \omega^l(e_1 + e_2 + e_3).$$

Since $e_i \omega (1-e_j)$ we have $e_i e_j = 0$ for $i \neq j$, $i, j = 1, 2, 3$. Thus $e_1 e_2 = e_2 e_1 = 0$ implies $e_1 + e_2 \in E_R$ and therefore

$$(\omega^l(e_1) \vee \omega^l(e_2)) \wedge \omega^l(e_3) = \omega^l(e_1 + e_2) \wedge \omega^l(e_3).$$

Now let $e_1 + e_2 = k$. Then since $e_i e_j = 0$ for $i \neq j$, we have $ke_3 = 0$ and $e_3 k = 0$. Hence $k\ \omega^l\ (1-e_3)$ and $k\ \omega^r\ (1-e_3)$. Therefore, $k\ \omega\ (1-e_3)$

and by above lemma 3.2.1

$$(\omega^l(e_1) \vee \omega^l(e_2)) \wedge \omega^l(e_3) = \omega^l(e_1 + e_2) \wedge \omega^l(e_3) = \omega^l(0) = 0$$

Thus generalizing the above result for n idempotents, we have the following lemma.

Lemma 3.2.3. Let $e_1, e_2, \ldots e_n \in E_R$ with $e_i \ \omega \ (1 - e_j)$ for $i \neq j$ for any i, j. Then $\omega^l(e_1), \omega^l(e_2), \ldots, \omega^l(e_n)$ are independent elements in the lattice Ω_l with $\omega^l(e_1) \vee \omega^l(e_2) \vee \ldots \omega^l(e_n) = \omega^l(e_1 + e_2 + \ldots + e_n)$.

Lemma 3.2.4. Let $e_1, e_2, \ldots e_n \in E_R$. Then $\omega^l(e_1), \omega^l(e_2), \ldots, \omega^l(e_n)$ are independent elements in the lattice Ω_l if and only if $e_i \ \omega \ (1 - e_j)$ for $i \neq j$, $i, j = 1, 2, \ldots, n$.

Proof. Suppose $e_i \ \omega \ (1 - e_j)$. By above lemma $\omega^l(e_1), \omega^l(e_2), \ldots, \omega^l(e_n)$ are independent. Conversely, suppose $n = 1$, then the statement follows trivially. Suppose $\omega^l(e_1), \omega^l(e_2), \ldots, \omega^l(e_{n+1})$ are independent. Then by definition

$$(\omega^l(e_1) \vee \omega^l(e_2) \vee \ldots \vee \omega^l(e_n)) \wedge \omega^l(e_{n+1}) = \{0\}$$

Now by corollary to Theorem (1.4)(part 1)[23], there is a complement, $\omega^l(e_k)$ of $\omega^l(e_{n+1})$ such that

$$\omega^l(e_k) \geq \omega^l(e_1) \vee \omega^l(e_2) \vee \ldots \vee \omega^l(e_n).$$

By Lemma 3.2.1, there exists an idempotent e such that $\omega^l(e_k) = \omega^l(e)$ and $\omega^l(e_{n+1}) = \omega^l(1 - e)$. Since $\omega^l(e_1), \omega^l(e_2), \ldots, \omega^l(e_n)$ are independent, by induction hypothesis, there exists idempotents $e_1, e_2, \ldots e_n$ such that $e_i e_j = 0$. Now define $e'_i = e e_i (i = 1, 2, \ldots, n)$ and $e_{n+1} = 1 - e$. We show that $e'_i e'_j = 0$ for $i \neq j$. Since $e_i \in \omega^l(e_i)$, we have $e_i \in \omega^l(e_k) = \omega^l(e)$ and so $e_i e = e_i$. Therefore, $(e'_i)^2 = e e_i e e_i = e e_i e_i = e e_i = e'_i$. Therefore, e'_i is idempotent for $i = 1, 2, \ldots, n$ and obviously

e_{n+1} is an idempotent. Now $e'_i \in \omega^l(e_i)$; also $e'_i e_i = (ee_i)e_i = ee_i = e'_i$. Thus $e'_i \in \omega^l(e_i)$ and hence $\omega^l(e'_i) \subseteq \omega^l(e_i)$. Also $e'_i = ee_i \in \omega^l(e_i)$. Now $e_i e'_i = e_i(ee_i) = (e_i e)e_i = e_i e_i = e_i$. Therefore, $e_i \in \omega^l(e'_i)$. Hence $\omega^l(e_i) \subseteq \omega^l(e'_i)$ and so $\omega^l(e_i) = \omega^l(e'_i)$ for $i = 1, 2, \ldots, n$.
Finally for $i, j = 1, 2, \ldots, n$, $i \neq j$,

$$e'_i e'_j = (ee_i)(ee_j) = e(e_i e)e_j = ee_i e_j = e0 = 0$$

$$e_{n+1} e'_i = (1-e)ee'_i = 0$$

and

$$e'_i e_{n+1} = ee_i(1-e) = ee_i - ee_i e = ee_i - ee_i = 0$$

therefore, this result holds for $i = n+1$. By induction this result holds for every n. □

Lemma 3.2.5. Let $e_1, e_2, \ldots, e_n \in E_R$ with $e_i\ \omega\ (1-e_j)$ for $i \neq j$. Then $d_l(e_i, e_j) = 3$ for $i \neq j$.

Proof. Suppose all these $\omega^l(e_i)$'s are perspective to each other. Then for each i and j with $i \neq j$, there exists a common complement $\omega^l(e_{ij})$ of $\omega^l(e_i)$ and $\omega^l(e_j)$ in Ω_l. Since $\omega^l(e_i)$ and $\omega^l(e_{ij})$ are complements of each other in the lattice Ω_l, there exists some $\omega^l(e_{ji})$ in Ω_l such that $e_i \mathcal{L} e_{ij} \mathcal{R} e_{ji} \mathcal{L} e_j$ and so $d_l(e_i, e_j) \leq 3$.
Since $e_i e_j = 0$ for $i \neq j$, e_i and e_j are neither $\mathcal{L}$-related nor $\mathcal{R}$ related. So, $d_l(e_i, e_j) \neq 1$. Again if there is an idempotent $f \in E_R$ with $e_i \mathcal{L} f \mathcal{R} e_j$ then $e_i \mathcal{R} e_i e_j = 0$, by Clifford Miller Theorem, so that $e_i = 0$ which is not true. Therefore, it follows that $d_l(e_i, e_j) \neq 2$. Thus $d_l(e_i, e_j) = 3$. □

In the light of the above Lemmas and Propositions, we have the following theorem.

Theorem 3.2.1. *Let R be regular ring with $e_i\ \omega\ (1-e_j)$ for $i \neq j$, $d_l(e_i, e_j) = 3$ and $e_1 + e_2 + \ldots + e_n = 1$, Then the complemented,* *modular lattice Ω_l is of order n.*

Proof. Since $M(e_i, e_j) = \{0\}$ we have by above Lemma(3.2.3) that $\omega^l(e_1), \ldots, \omega^l(e_n)$ are independent elements in Ω_l with $\omega^l(e_1) \vee \omega^l(e_2) \vee \ldots \vee \omega^l(e_n) = \omega^l(e_1 + e_2 + \ldots + e_n)$ and since $e_1 + e_2 + \ldots + e_n = 1$, $\omega^l(e_1) \vee \omega^l(e_2) \vee \ldots \omega^l(e_n) = \omega^l(1)$. Since $d_l(e_i, e_j) = 3$, by Lemma(3.2.2) we have $\omega^l(e_i)$ and $\omega^l(e_j)$ are perspective to each other. Therefore by the definition of homogeneous basis, Ω_l admits a homogeneous basis of rank n. Thus Ω_l is a complemented, modular lattice of order n. $\square$

Example 3.2.1. Consider the matrix ring $R = M_2(\mathbb{Z}_2)$. Clearly, this ring R is a regular ring with $|M_2(\mathbb{Z}_2)| = 16$. The idempotent set E_R has 8 elements and are listed as follows:

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, e_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$e_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, e_5 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, e_6 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

The biorder ideals generated by the idempotents in this ring is as follows:

$$\omega^l(0) = 0 \text{ and } \omega^l(1) = E_R, \omega^r(0) = 0 \text{ and } \omega^r(1) = E_R$$

$$\omega^l(e_1) = \{0, e_1, e_5\} \quad \omega^r(e_1) = \{0, e_1, e_3\}$$

$$\omega^l(e_2) = \{0, e_2, e_6\} \quad \omega^r(e_2) = \{0, e_2, e_4\}$$

$$\omega^l(e_3) = \{0, e_3, e_4\} \quad \omega^r(e_3) = \{0, e_3, e_1\}$$

$$\omega^l(e_4) = \{0, e_4, e_3\} \quad \omega^r(e_4) = \{0, e_4, e_2\}$$

$$\omega^l(e_5) = \{0, e_5, e_1\} \quad \omega^r(e_5) = \{0, e_5, e_6\}$$

$$\omega^l(e_6) = \{0, e_6, e_2\} \quad \omega^r(e_6) = \{0, e_6, e_5\}$$

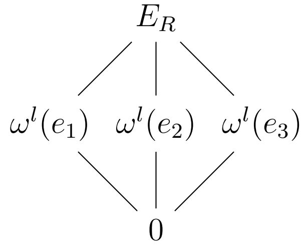
It can be observed that

$$e_1\mathcal{L}e_5,\ e_2\mathcal{L}e_6,\ e_4\mathcal{L}e_3.$$

and

$$e_1\mathcal{R}e_3,\ e_2\mathcal{R}e_4,\ e_5\mathcal{R}e_6.$$

It can be seen that $\omega^l(e_1)$ and $\omega^l(e_2)$ are complements to each other, since there exists an idempotent, e_5 such that $\omega^l(e_1) = \omega^l(e_5)$ and $\omega^l(e_2) = \omega^l(1-e_5)$. Similarly, there exists an idempotent, e_6 such that $\omega^l(e_2) = \omega^l(e_6)$ and $\omega^l(e_3) = \omega^l(1-e_6)$. Therefore, $\omega^l(e_2)$ and $\omega^l(e_3)$ are complements of each other. Also, $\omega^l(e_3) = \omega^l(e_4)$ and $\omega^l(e_1) = \omega^l(1-e_4)$. Therefore, $\omega^l(e_1)$ and $\omega^l(e_3)$ are complements of each other. The complemented modular lattice Ω_l of this ring is as shown below:



Thus it can be seen that $(E_R, 0)$, $(\omega^l(e_1), \omega^l(e_2))$, $(\omega^l(e_1), \omega^l(e_3))$, $(\omega^l(e_2), \omega^l(e_3))$ are the complementary pairs in the lattice Ω_l and the pairs $(\omega^l(e_1), \omega^l(e_2))$, $(\omega^l(e_1), \omega^l(e_3))$, $(\omega^l(e_2), \omega^l(e_3))$ are the perspective elements in this lattice Ω_l.

The egg-box diagram of elements of $M_2(\mathbb{Z}_2)$ is given by

1				
	e_6	e_5		
	e_2		e_4	
		e_1	e_3	
				0

Also, it can be seen from the egg-box picture that $M(e_1, e_2) = M(e_2, e_1) = M(e_2, e_3) = \{0\}$. Thus we get $\big\{0, \omega^l(e_1), \omega^l(e_2)\big\}$ is a basis of this complemented modular lattice Ω_l and $d_l(e_1, e_2) = 3$. Thus this lattice Ω_l has a homogeneous basis of order 2.

Example 3.2.2. Consider the matrix ring $R = M_3(\mathbb{Z}_2)$. Clearly, this ring R is a regular ring with $|M_3(\mathbb{Z}_2)| = 512$. The idempotent set E_R has 58 elements and are listed as follows:

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$e_6 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_8 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$e_{17} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_{18} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_{19} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$e_{22} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_{25} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_{46} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$e_{49} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, e_{50} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, e_{54} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$e_{55} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, e_{57} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, e_{66} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$e_{74} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, e_{82} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, e_{122} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$e_{145} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, e_{146} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, e_{147} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$e_{152} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, e_{196} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, e_{210} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

$$e_{217} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, e_{239} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, e_{257} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$e_{258} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e_{260} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e_{261} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$e_{266} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e_{273} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e_{274} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$e_{275} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e_{277} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e_{279} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$e_{281} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e_{289} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, e_{290} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$e_{293} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, e_{296} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, e_{298} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$e_{317} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, e_{321} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, e_{337} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$e_{345} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, e_{361} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, e_{385} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$e_{386} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, e_{388} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, e_{391} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$e_{449} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, e_{458} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, e_{467} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$e_{512} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

It can be seen that in this ring the ω^l ideals satisfy

$$
\begin{aligned}
&(I_1) && \omega^l(e_2) = \omega^l(e_{10}) = \omega^l(e_{66}) = \omega^l(e_{74})\\
&(I_2) && \omega^l(e_4) = \omega^l(e_{25}) = \omega^l(e_{196}) = \omega^l(e_{217})\\
&(I_3) && \omega^l(e_6) = \omega^l(e_{46}) = \omega^l(e_{321}) = \omega^l(e_{361})\\
&(I_4) && \omega^l(e_8) = \omega^l(e_{57}) = \omega^l(e_{449}) = \omega^l(e_{512})\\
&(I_5) && \omega^l(e_{17}) = \omega^l(e_{19}) = \omega^l(e_{145}) = \omega^l(e_{147})\\
&(I_6) && \omega^l(e_{18}) = \omega^l(e_{82}) = \omega^l(e_{146}) = \omega^l(e_{210})\\
&(I_7) && \omega^l(e_{22}) = \omega^l(e_{152}) = \omega^l(e_{337}) = \omega^l(e_{467})\\
&(I_8) && \omega^l(e_{49}) = \omega^l(e_{55}) = \omega^l(e_{385}) = \omega^l(e_{391})\\
&(I_9) && \omega^l(e_{50}) = \omega^l(e_{122}) = \omega^l(e_{386}) = \omega^l(e_{458})\\
&(I_{10}) && \omega^l(e_{54}) = \omega^l(e_{239}) = \omega^l(e_{345}) = \omega^l(e_{388})\\
&(I_{11}) && \omega^l(e_{257}) = \omega^l(e_{261}) = \omega^l(e_{289}) = \omega^l(e_{293})\\
&(I_{12}) && \omega^l(e_{317}) = \omega^l(e_{260}) = \omega^l(e_{296}) = \omega^l(e_{281})\\
&(I_{13}) && \omega^l(e_{258}) = \omega^l(e_{266}) = \omega^l(e_{290}) = \omega^l(e_{298})\\
&(I_{14}) && \omega^l(e_{273}) = \omega^l(e_{275}) = \omega^l(e_{277}) = \omega^l(e_{279})
\end{aligned}
$$

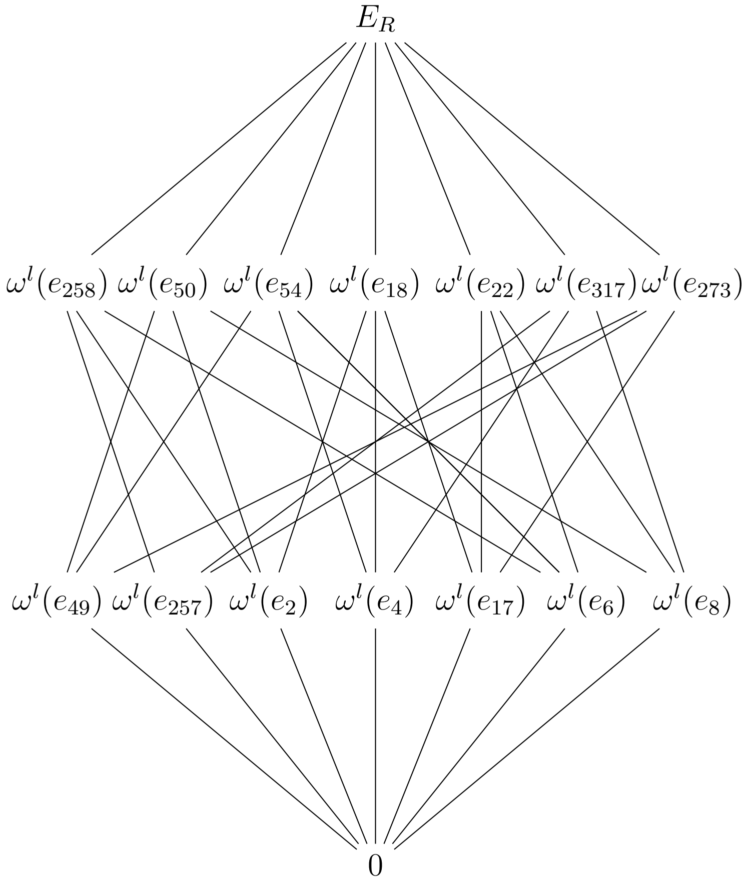
and it can be sen that

$$
\begin{gathered}
I_1, I_2, I_5 \subseteq I_6\\
I_3, I_4, I_5 \subseteq I_7\\
I_2, I_4, I_{11} \subseteq I_{12}\\
I_1, I_4, I_8 \subseteq I_9\\
I_2, I_3, I_8 \subseteq I_{10}\\
I_5, I_8, I_{11} \subseteq I_{14}\\
I_1, I_3, I_{11} \subseteq I_{13}
\end{gathered}
$$

The ω^r ideals satisfy

$$\begin{aligned}
&\omega^r(e_2) = \omega^r(e_4) = \omega^r(e_6) = \omega^r(e_8)\\
&\omega^r(e_{17}) = \omega^r(e_{25}) = \omega^r(e_{49}) = \omega^r(e_{57})\\
&\omega^r(e_{10}) = \omega^r(e_{19}) = \omega^r(e_{46}) = \omega^r(e_{55})\\
&\omega^r(e_{66}) = \omega^r(e_{196}) = \omega^r(e_{261}) = \omega^r(e_{391})\\
&\omega^r(e_{18}) = \omega^r(e_{22}) = \omega^r(e_{50}) = \omega^r(e_{54})\\
&\omega^r(e_{74}) = \omega^r(e_{147}) = \omega^r(e_{293}) = \omega^r(e_{512})\\
&\omega^r(e_{82}) = \omega^r(e_{122}) = \omega^r(e_{277}) = \omega^r(e_{317})\\
&\omega^r(e_{145}) = \omega^r(e_{217}) = \omega^r(e_{289}) = \omega^r(e_{361})\\
&\omega^r(e_{146}) = \omega^r(e_{152}) = \omega^r(e_{290}) = \omega^r(e_{296})\\
&\omega^r(e_{210}) = \omega^r(e_{239}) = \omega^r(e_{279}) = \omega^r(e_{298})\\
&\omega^r(e_{257}) = \omega^r(e_{321}) = \omega^r(e_{385}) = \omega^r(e_{449})\\
&\omega^r(e_{266}) = \omega^r(e_{275}) = \omega^r(e_{458}) = \omega^r(e_{467})\\
&\omega^r(e_{260}) = \omega^r(e_{258}) = \omega^r(e_{386}) = \omega^r(e_{388})\\
&\omega^r(e_{273}) = \omega^r(e_{281}) = \omega^r(e_{337}) = \omega^r(e_{345})
\end{aligned}$$

The complemented modular lattice Ω_l of the biorder ideals is as follows:



It can be easily seen from this diagram that this lattice has a homogeneous basis with 3 elements. For example the set $\{\omega^l(e_2), \omega^l(e_4), \omega^l(e_6)\}$ is a homogeneous basis of this lattice. Thus we can say the lattice is of order 3.

Chapter 4

Biordered Sets and Complemented Modular Lattices

In [28], Pastjin has constructed a biordered set $E_{P(L)}$ from a complemented modular lattice L. In this chapter, we discuss the properties of this biordered set $E_{P(L)}$ and it is shown that the set of idempotents E_R of a regular ring R is isomorphic to $E_{P(L)}$.

4.1 Biordered sets of lattices and homogeneous basis

In the following we briefly recall the construction of the biordered set $E_{P(L)}$ of the complemented modular lattice L. It is shown that this biordered set is bounded and complemented. Some interesting properties of the biordered set $E_{P(L)}$ are also discussed. Finally we describe the biordered subset satisfying certain conditions as $E^0_{P(L)}$ so that the complemented modular lattice admits a homogeneous basis.

Let L be a complemented modular lattice and $(n; v)$ be any pair of

complementary elements of L
Let $(n;v) : L \longrightarrow L$, be the map defined by

$$x \longrightarrow v \wedge (n \vee x) \text{ for all } x \text{ in } L$$

and $(n;v)' : L \longrightarrow L$ defined by

$$x \longrightarrow n \vee (v \wedge x) \text{ for all } x \text{ in } L$$

are idempotent order preserving normal mappings. We let the map $(n;v)$ act on L as right operator and denote by $P(L)$ the subsemigroup of $S^*(L)$ which is generated by these idempotent normal mappings $(n;v)$, $n, v \in L$. Analogously the mapping $(n;v)'$ is an order preserving idempotent mapping of L onto the principal ideal $[1,n]$ of $(L,\vee)$; hence $(n;v)'$ is a normal mapping of $(L,\vee)$ into itself. Letting $(n;v)'$ act on L as left operators, denote by $P(L)'$ the subsemigroup of $S(L)$ which is generated by these idempotent normal mappings $(n;v)', n, v \in L$.

Let

$$E_{P(L)} = \{(n;v) : n, v \in L \; n \vee v = 1, n \wedge v = 0\}$$

and

$$E_{P(L)'} = \{(n;v)' : n, v \in L \; n \vee v = 1, n \wedge v = 0\}$$

we refer to the elements $(n;v)[(n;v)']$ as idempotent generators of $P(L)[P(L)]'$.

Theorem 4.1.1 (cf.[28], Theorem 1). *Let L be a complemented modular lattice. Then*

1. *$P(L)$ is a regular subsemigroup of $S^*(L)$ and*
   $E_{P(L)} = \{(n;v) : n, v \in L \; n \vee v = 1, n \wedge v = 0\}$.

2. *In* $(E_{P(L)}, \omega^l, \omega^r)$ *we have*

$$(n_1; v_1)\ \omega^l\ (n_2; v_2) \Longleftrightarrow v_1 \leq v_2 \text{ in } L$$

   and then

$$(n_2; v_2)(n_1; v_1) = (n_2 \vee (v_2 \wedge n_1); v_1);$$

   we have

$$(n_1; v_1)\ \omega^r\ (n_2; v_2) \Longleftrightarrow n_2 \leq n_1 \text{ in } L$$

   and then

$$(n_1; v_1)(n_2; v_2) = (n_1; v_2 \wedge (n_2 \vee v_1))$$

3. *Let* $(n_1; v_1)$ *and* $(n_2; v_2)$ *be any idempotent of* $P(L)$. *Let* n *be any complement of* $v_1 \vee n_2$ *in* $[n_2, 1]$; *let* v *be any complement of* $v_1 \wedge n_2$ *in* $[0, v_1]$; *then* n *and* v *are complementary in* L *and* $(n; v)$ *is an element in the sandwich set* $S((n_1; v_1), (n_2; v_2))$. *Conversely, any element in the sandwich set* $S((n_1; v_1), (n_2; v_2))$ *can be obtained in this way.*

The above theorem provides a biordered set $E_{P(L)}$ from a complemented modular lattice L.
The zero of $P(L)$ is $(1; 0)$ and the identity is $(0; 1)$, obviously $(1; 0)$ and $(0; 1)$ are in $E_{P(L)}$. For any $(n; v)$ in biordered set $E_{P(L)}$, $(n; v)\ \omega\ (0; 1)$ and $(1; 0)\ \omega\ (n; v)$.

The following lemma is immediate.

Lemma 4.1.1. Let $(n_1; v_1)$, $(n_2; v_2) \in E_{P(L)}$ then

1. $S((n_1; v_1), (n_2; v_2)) = S((n_2; v_2), (n_1; v_1)) = \{(1; 0)\}$ if and only if $v_1 \leq n_2$ and $v_2 \leq n_1$.

2. For $v_1 \leq n_2$ and $v_2 \leq n_1$, $(v_1 \vee v_2; n_2 \wedge n_1)$ is the unique element in $S((v_1; n_1)(v_2; n_2)) = S((v_2; n_2)(v_1; n_1))$.

Proof. 1. Let $(n_1; v_1), (n_2; v_2) \in E_{P(L)}$ with $v_1 \leq n_2$. Now let $(n; v) \in S((n_1; v_1), (n_2; v_2))$. Then by definition of sandwich set as in [[28], Theorem 1] n is a complement of n_2 in $[n_2, 1]$ and v is a complement of v_1 in $[0, v_1]$. Since, $(n; v)\ \omega^l\ (n_1; v_1)$ and $(n; v)\ \omega^r\ (n_2; v_2)$, it follows that $n_2 \leq n$ and $n \vee n_2 = 1$ implies $n = 1$ and $v \leq v_1$ and $v \wedge v_1 = 0$ implies $v = 0$. Thus $S((n_1; v_1), (n_2; v_2)) = \{(1; 0)\}$. Similarly, $S((n_2; v_2), (n_1; v_1)) = \{(1; 0)\}$ if $v_2 \leq n_1$. The converse follows immediately.

2. For $v_1 \leq n_2$ and $v_2 \leq n_1$, by definition,

$$(v_1; n_1)(v_2; n_2) = (v_1 \vee (n_1 \wedge v_2); n_2 \wedge (v_2 \vee n_1)) = (v_1 \vee v_2; n_2 \wedge n_1)$$

and

$$(v_2; n_2)(v_1; n_1) = (v_2 \vee (n_2 \wedge v_1); n_1 \wedge (v_1 \vee n_2)) = (v_2 \vee v_1; n_1 \wedge n_2).$$

Thus $(v_1; n_1)(v_2; n_2) = (v_2; n_2)(v_1; n_1)$. It can be easily seen that $(v_1 \vee v_2)$ is a complement of $v_2 \vee n_1$ in $[v_2, 1]$ and $(n_2 \wedge n_1)$ is a complement of $v_2 \wedge n_1$ in $[0, n_1]$. Thus $(v_1 \vee v_2; n_2 \wedge n_1) \in S((v_1; n_1)(v_2; n_2))$. Similarly, $(v_1 \vee v_2)$ is a complement of $v_1 \vee n_2$ in $[v_1, 1]$ and $(n_2 \wedge n_1)$ is a complement of $v_1 \wedge n_2$ in $[0, n_2]$. Therefore, $(v_1 \vee v_2; n_2 \wedge n_1) \in S((v_2; n_2)(v_1; n_1))$.

Now it remains to prove the uniqueness of this element. Suppose there exists another element say $(a; a') \in S((v_1; n_1)(v_2; n_2)) \cap S((v_2; n_2)(v_1; n_1))$. Then $v_1 \leq a, a' \leq n_1, v_2 \leq a, a' \leq n_2$ and from the definition of sandwich set as in [28] it can be seen that $a \vee n_1 = 1, a' \wedge v_2 = 0, a \vee n_2 = 1, a' \wedge v_1 = 0$ and $a \wedge n_1 \leq v_2, n_1 \leq a' \vee v_2, a \wedge n_2 \leq v_1, n_2 \leq a' \vee v_1$. Thus $a = v_1 \vee v_2$ and $a' = n_1 \wedge n_2$ and $(a; a') = (v_1; n_1)(v_2; n_2)$ is unique.

□

Thus $E_{P(L)}$ has the following properties:

For each $(n;v) \in E_{P(L)}$ there exists an element $(v;n) \in E_{P(L)}$ such that $(n;v)(v;n) = (v;n)(n;v) = (1;0)$. The element $(v;n)$ is called the inverse of $(n;v)$.

- $(n_1;v_1)\ \omega^l\ (n_2;v_2) \Longleftrightarrow (v_2;n_2)\ \omega^r\ (v_1;n_1)$.
- $v_1 \leq n_2 \Longleftrightarrow S((n_1;v_1)(n_2;v_2)) = (1;0)$

Hence the biordered set $E_{P(L)}$ is a bounded and complemented biordered set.

From here onwards we consider the biordered subset of $E_{P(L)}$ satisfying $v_i \leq n_j$ for all $(n_i;v_i)$, $(n_j;v_j)$ and $i \neq j$.

For $(n_i;v_i), (n_j;v_j)$, with $i \neq j$ in the biordered subset we have $(v_i;n_i)(v_j;n_j) = (v_j;n_j)(v_i;n_i) = (v_i \vee v_j; n_j \wedge n_i)$. Now define

$$(n_i;v_i) \oplus (n_j;v_j) = (n_i \wedge n_j; v_i \vee v_j)$$

.

Lemma 4.1.2. For the biordered subset of $E_{P(L)}$ with $v_i \leq n_j$ for $i \neq j$ and let $(p;q) = (n_i;v_i) \oplus (n_j;v_j)$. Then $(p;q)$ satisfies the following properties:

1. $(n_i;v_i), (n_j;v_j) \in \omega((p;q))$
2. If $(r;s) \in E_{P(L)}$ with $(n_i;v_i), (n_j;v_j) \in\omega^l\ ((r;s))$ then $(p;q) \in\omega^l\ ((r;s))$.
3. If $(r;s) \in E_{P(L)}$ with $(n_i;v_i), (n_j;v_j) \in\omega^r\ ((r;s))$, then $(p;q)\ \omega^r\ ((r;s))$.

Proof. 1. Note that $(p;q) \in S((n_i;v_i)(n_j;v_j)) \cap S((n_j;v_j)(n_i;v_i))$. Therefore, $(q;p)\ \omega^l\ (v_i;n_i), (q;p)\ \omega^r\ (v_j;n_j), (q;p)\ \omega^l\ (v_j;n_j)$ and $(q;p)\ \omega^r\ (v_i;n_i)$. Thus $p \leq n_i, p \leq n_j, v_j \leq q, v_i \leq q$ and $(n_i;v_i)\ \omega\ (p;q)$ and $(n_j;v_j)\ \omega\ (p;q)$.

2. Let $(r;s) \in E_{P(L)}$ with $(n_i;v_i)\ \omega^l\ (r;s)$ and $(n_j;v_j)\ \omega^l\ (r;s)$, then $v_i \leq s$ and $v_j \leq s$. Then as seen above in lemma 4.1.1(2),

$$(q;p) = (v_i;n_i)(v_j;n_j).$$

Thus, $v_j \leq s$ implies $(s;r)\ \omega^r\ (v_j;n_j)$, that is $(v_j;n_j)(s;r) = (s;r)$. Similarly, $v_i \leq s$ implies $(s;r)\ \omega^r\ (v_i;n_i)$, that is $(v_i;n_i)(s;r) = (s;r)$.
Therefore,

$$(q;p)(s;r) = (v_i;n_i)(v_j;n_j)(s;r) = (s;r),$$

that is $(s;r)\ \omega^r\ (q;p)$ also $(p;q)\ \omega^l\ (r;s)$.

3. The proof follows similarly as above.

□

The next lemma shows that the addition defined is cancellative.

Lemma 4.1.3. Let $(n_i;v_i),(n_j;v_j),(n_k;v_k) \in E_{P(L)}$ with $v_i \leq n_j, v_j \leq n_i$ and $v_i \leq n_k, v_k \leq n_i$ for $i \neq j \neq k$. Then $(n_i;v_i) \oplus (n_j;v_j) = (n_i;v_i) \oplus (n_k;v_k)$ if and only if $(n_j;v_j) = (n_k;v_k)$.

Proof. If $(n_j;v_j) = (n_k;v_k)$, then in $E_{P(L)}$

$$(n_i;v_i)\oplus(n_j;v_j) = (n_j \wedge n_i; v_i \vee v_j) = (n_k \wedge n_i; v_i \vee v_k) = (n_i;v_i)\oplus(n_k;v_k).$$

Conversely suppose that $(n_i;v_i) \oplus (n_j;v_j) = (n_i;v_i) \oplus (n_k;v_k)$. Then

$$(n_j \wedge n_i; v_i \vee v_j) = (n_i \wedge n_j; v_j \vee v_i) = (n_k \wedge n_i; v_i \vee v_k) = (n_i \wedge n_k; v_k \vee v_i).$$

Also since $v_j \leq n_i$ and $v_i \leq n_j$;

$$\begin{aligned}
(n_j; v_j)(v_k; n_k) &= (n_j; v_j)(v_i; n_i)(v_k; n_k) \\
&= (n_j; v_j)((v_i; n_i)(v_k; n_k)) \\
&= (n_j; v_j)(v_i \vee v_k; n_i \wedge n_k) \\
&= (n_j; v_j)(v_j \vee v_i; n_j \wedge n_i) \\
&= (n_j; v_j)(v_j; n_j)(v_i; n_i) \\
&= (1; 0)(v_i; n_i) \\
&= (1; 0)
\end{aligned}$$

Therefore, $S((n_j; v_j)(v_j; n_j)) = \{(1; 0)\}$ and so $v_j \leq v_k$ and

$$\begin{aligned}
(v_k; n_k)(n_j; v_j) &= (v_k; n_k)(v_i; n_i)(n_j; v_j) \\
&= (v_k; n_k)((v_i; n_i)(n_j; v_j)) \\
&= (v_k \vee v_i; n_i \wedge n_k)(n_j; v_j) \\
&= (v_i \vee v_j; n_j \wedge n_i)(n_j; v_j) \\
&= (v_i; n_i)(v_j; n_j)(n_j; v_j) \\
&= (v_i; n_i)(1; 0) \\
&= (1; 0)
\end{aligned}$$

Therefore, $S((v_k; n_k)(n_j; v_j)) = \{(1; 0)\}$ and so $n_j \leq n_k$. Interchanging $(n_k; v_k)$ and $(n_j; v_j)$, $n_j \leq n_k$. Thus $n_j = n_k$ and $v_j = v_k$. That is, $(n_j; v_j) = (n_k; v_k)$. $\square$

Corollary 4.1.1. *Let $(n_i; v_i), (n_j; v_j) \in E_{P(L)}$ with $v_j \leq n_i$ for $i \neq j$. Then*

$$(n_i; v_i) \oplus (n_j; v_j) = (0; 1) \text{ if and only if } (n_j; v_j) = (v_i; n_i).$$

Proof. By Lemma(4.1.1) we have

$$S((n_i; v_i)(v_i; n_i)) = S((v_i; n_i)(n_i; v_i)) = \{(1; 0)\}.$$

Therefore,

$$S((n_i;v_i)(v_i;n_i)) \cap S((v_i;n_i)(n_i;v_i)) = \{(1;0)\}.$$

That is, $((v_i \vee n_i; n_i \wedge v_i)) = (1;0)$. Thus we get

$$(n_i;v_i) \oplus (v_i;n_i) = (n_i \wedge v_i; v_i \vee n_i) = (0;1).$$

Conversely, suppose $(n_i;v_i) \oplus (n_j;v_j) = (1;0)$. Since $(n_i;v_i) \oplus (v_i;n_i) = (1;0)$, it follows that $(n_j;v_j) = (v_i;n_i)$, by above lemma. □

Lemma 4.1.4. Let $E_{P(L)}$ be the biordered set with $v_i \leq n_j$, $v_j \leq n_i$, $v_i \leq n_k$, $v_k \leq n_i$, $v_j \leq n_k$, $v_k \leq n_j$ for $i \neq j \neq k$. Then for elements $(n_i;v_i), (n_j;v_j), (n_k;v_k)$, $i,j,k = 1,2,\ldots,N$ with $i \neq j \neq k$ in $E_{P(L)}$, the collection $\{v_1, v_2, \ldots, v_N\}$ are independent elements in the lattice L.

Proof. We have the set $\{(n_i;v_i)\colon i = 1,2,\ldots,N\}$ in $E_{P(L)}$ so that the elements $v_1, v_2, \ldots, v_N$ are in the complemented modular lattice L. We show that the collection $\{v_1, v_2, \ldots, v_N\}$ are independent elements in the lattice. Since $v_i \leq n_k$ and $v_j \leq n_k$ for $i \neq j$ in $E_{P(L)}$, we have $(v_i \vee v_j) \leq n_k$ for $i \neq j \neq k$.
Then

$$(v_i \vee v_j) \wedge v_k \leq n_k \wedge v_k = 0$$

Thus for any such pairs $(n_i;v_i)$, v_i's satisfy this property. Hence the collection $\{v_i\colon i = 1,2\ldots,n\}$ are independent. □

For any biordered subset of $E_{P(L)}$ consisting of N elements, with $v_i \leq n_j$, $i,j = 1,2,\ldots,N$, $i \neq j$, the collection $\{v_1, v_2, \ldots, v_N\}$ are independent in the lattice L.
In the following we assume that the biordered set $E_{P(L)}$ has elements $\{(n_i;v_i)\colon i = 1,2,\ldots,N\}$ satisfying the following properties:

1. $v_i \leq n_j$ for $i \neq j$

2. $(n_1; v_1) \oplus (n_2; v_2) \oplus \ldots \oplus (n_N; v_N) = (0; 1)$
3. $d_l((n_i; v_i), (n_j; v_j)) = 3$ for $i \neq j$.

and denote this biordered set as $E^0_{P(L)}$.

In the light of (Theorem 6, [28]) stated below,

Theorem 4.1.2. *Let L be any complemented modular lattice, let $v_1, v_2 \in L$ and let $n_1[n_2]$ be any complement of $v_1[v_2]$ in L. Then $v_1 \sim v_2$ in L if and only if $(n_1; v_1)$ and $(n_2; v_2)$ are connected by an E-sequence in $E_{P(L)}$.*

From the fact that the perspectivity in the complemented modular lattice L is transitive if and only if any two elements $(n_1; v_1)$ and $(n_2; v_2)$ are connected by an E-sequence of length 3 (see[28], page 218), it is easy to see that the complemented modular lattice L with the biordered set $E^0_{P(L)}$ having elements $\{(n_i; v_i), : i = 1, 2, \ldots N\}$ the collection $\{v_1, v_2, \ldots v_N\}$ in L satisfies, $v_1 \vee v_2 \vee \ldots v_N = 1$ and each v_i's are pairwise perspective. That is, the complemented modular lattice admits a homogeneous basis of order N that is L is a lattice of order N(see definition in [23]).

Example 4.1.1. Consider the complemented modular lattice $\Omega_l = \{\omega^l(e_i) \colon e_i \in R\}$ of order n (See Chapter3, Theorems 3.1.2, 3.2.1) where

$$\omega^l(e) \vee \omega^l(f) = \omega^l(h(1-f)+f) \text{ and } \omega^l(e) \wedge \omega^l(f) = \omega^l(e(1-h))$$

The maps
$(\omega^l(1-e); \omega^l(e))$, $(\omega^l(1-e); \omega^l(e))' \colon \Omega_l \longrightarrow \Omega_l$ defined by

$$(\omega^l(1-e); \omega^l(e))(x) \longrightarrow \omega^l(e) \wedge (\omega^l(1-e) \vee x)$$

and

$$(\omega^l(1-e);\omega^l(e))(x) \longrightarrow \omega^l(1-e) \vee (\omega^l(e) \wedge x)$$

are idempotent order preserving normal mappings. We denote by $P(\Omega_l)$ the subsemigroup of $S^*(\Omega_l)$ which is generated by these idempotent normal mappings $(\omega^l(1-e);\omega^l(e))$ defined as in [28]. Then

$$E_{P(\Omega_l)} = \left\{(\omega^l(1-e),\omega^l(e))\colon \omega^l(1-e) \vee \omega^l(e) = 1, \omega^l(1-e) \wedge \omega^l(e) = 0\right\}.$$

$E_{P(\Omega_l)}$ is the biordered set of the semigroup $P(\Omega_l)$ and it can be easily seen that $E_{P(\Omega_l)}$ has elements $(\omega^l(1-e_i);\omega^l(e_i))\colon i = 1, 2, \ldots, N$ satisfying all the properties of $E^0_{P(L)}$.

The next lemma gives a biorder isomorphism between the biordered set of idempotents in the ring R and the biordered set $E_{P(\Omega_l)}$.

Lemma 4.1.5. Every idempotent e in a ring R is associated with a pair $(\omega^l(1-e);\omega^l(e))$ of complementary biorder ideals in E_R. The map $\epsilon\colon E_R \longrightarrow E_{P(\Omega_l)}$ defined by $\epsilon(e) = (\omega^l(1-e);\omega^l(e))$ is a biorder isomorphism.

Proof. For each $e \in E_R$, $(\omega^l(1-e);\omega^l(e))$ is a complementary pair in the lattice Ω_l and the mapping

$$\epsilon\colon e \longrightarrow (\omega^l(1-e);\omega^l(e)) \text{ for all } e \in E_R$$

is a map of E_R into $E_{P(\Omega_l)}$. The map ϵ is clearly injective. It follows from the definition of biordered set [[25], Definition 1] and the equation 1 in Theorem(1) [28] that the map ϵ preserve basic products and hence $\epsilon\colon E_R \longrightarrow E_{P(\Omega_l)}$ is a biorder isomorphism. Also, it can be easily seen that this map ϵ is a regular bimorphism. □

Thus we have $E_{P(\Omega_l)}$ and E_R are biorder isomorphic. Now we show that there exists elements $e_1, e_2, \ldots, e_N$ in E_R satisfying all the conditions of $E^0_{P(\Omega_l)}$.

Consider $E^0_{P(\Omega_l)}$. Then there are elements $((\omega^l(1-e_i);\omega^l(e_i)\colon i = 1,2,\ldots,N)$ such that

1. $\omega^l(1-e_i) \leq \omega^l(e_i)$ for $i \neq j$
2. $(\omega^l(1-e_1);\omega^l(e_1)) \oplus (\omega^l(1-e_2);\omega^l(e_2)) \oplus \ldots \oplus (\omega^l(1-e_N);\omega^l(e_N)) = (0;1)$
3. $d_l((\omega^l(1-e_i);\omega^l(e_i)),(\omega^l(1-e_j);\omega^l(e_j))) = 3$

Since $E_{P(\Omega_l)}$ and E_R are biorder isomorphic, corresponding to each $((\omega^l(1-e_i);\omega^l(e_i)\colon i = 1,2,\ldots,N)$, there exists elements $e_1, e_2, \ldots, e_N$ such that

1. $\omega^l(1-e_i) \leq \omega^l(e_j)$ for $i \neq j$ implies $(\omega^l(1-e_i);\omega^l(e_i))(\omega^l(1-e_j);\omega^l(e_j)) = \{(0;1)\}$. But $\omega^l(1-e_i) \leq \omega^l(e_j)$ implies $1-e_i\ \omega^l\ e_j$ thus $e_i\ \omega^r\ (1-e_j)$ and so $e_ie_j = 0$ for $i \neq j$.

   The second condition implies

2. $(\omega^l(1-e_1);\omega^l(e_1)) \oplus \ldots \oplus (\omega^l(1-e_N);\omega^l(e_N)) = (0;\omega^l(1))$ which implies $\omega^l(e_1) \vee \omega^l(e_2) \vee \ldots \vee \omega^l(e_N) = \omega^l(1)$. But we have from Lemma 3.2.3, $e_ie_j = 0$ for $i \neq j$ implies $\omega^l(e_1) \vee \omega^l(e_2) \vee \ldots \vee \omega^l(e_N) = \omega^l(e_1+e_2+\ldots+e_N) = \omega^l(1)$ and hence $e_1+e_2+\ldots+e_N = 1$.
3. $d_l((\omega^l(1-e_i);\omega^l(e_i)),(\omega^l(1-e_j);\omega^l(e_j))) = 3$ implies there exists elements $(\omega^l(1-e_i);\omega^l(e_i))\mathcal{L}\ (\omega^l(1-e_h);\omega^l(e_h))\mathcal{R}\ (\omega^l(1-e_k);\omega^l(e_k))\mathcal{L}(\omega^l(1-e_j);\omega^l(e_j))$. Therefore, by definition of $\mathcal{L}$ and $\mathcal{R}$, $\omega^l(e_i) = \omega^l(e_h)$ and $\omega^l(1-e_h) = \omega^l(1-e_k)$ and $\omega^l(e_k) = \omega^l(e_j)$. But $\omega^l(1-e_h) = \omega^l(1-e_k)$ implies $\omega^r(e_h) = \omega^r(e_k)$. Thus we get $e_i\mathcal{L}e_h\mathcal{R}e_k\mathcal{L}e_j$ and hence $d_l(e_i,e_j) = 3$.

Since $E^0_{P(\Omega_l)}$ is a biorder subset of $E_{P(\Omega_l)}$, and corresponding to each element in $E_{P(\Omega_l)}$, there exists elements in E_R satisfying all the conditions of $E^0_{P(\Omega_l)}$, as shown above, we have $E^0_{P(\Omega_l)}$ and E_R are biorder isomorphic.

4.2 Von Neumann coordinatisation Theorem and its analogue

A coordinatisation theorem is a statement that expresses a class of geometric objects in algebraic terms. See for example the classical coordinatisation theorem of Arguesian affine planes(cf [1], page 101). This idea was extended to the coordinatisation of modular lattices by regular rings due to von Neumann [23].

Von Neumann's Coordinatisation Theorem:

Theorem 4.2.1. *If a complemented modular lattice L has a spanning finite homogeneous basis with at least four elements, then there exists a von Neuman regular ring R such that L is isomorphic to the lattice of all principal right[left] ideals of R.*

In the previous section, we have shown that if a complemented modular lattice L admits the biordered subset $E^0_{P(L)}$ consisting of N elements, then L has a homogeneous basis of order N. Thus analogous to von-Neumann's coordinatization theorem, we have the following theorem:

Theorem 4.2.2. *Let L be a complemented modular lattice admitting a biordered subset with at least 4 elements, having the following properties:*

1. $v_i \leq n_j$ *for* $i \neq j$

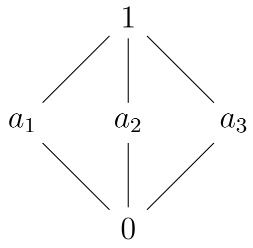
2. $(n_1; v_1) \oplus (n_2; v_2) \oplus \ldots \oplus (n_N; v_N) = (0; 1)$

3. $d_l((n_i; v_i), (n_j; v_j)) = 3$ *for* $i \neq j$,

then there exists a von Neumann regular ring R such that L is isomorphic to the lattice of all principal left ideals of R.

In the following, we provide some examples of complemented modular lattice L with biordered set $E_{P(L)}$ admitting biordered subsets $E^0_{P(L)}$ having 2 elements.

Example 4.2.1. Consider the lattice $M_3 = \{0, 1, a_1, a_2, a_3\}$



The biordered set

$$E(M_3) = \{(a_1; a_2), (a_2; a_3), (a_1; a_3), (a_2; a_1), (a_3; a_2), (a_3; a_1), (0; 1), (1; 0)\}$$

and the biorder relations are as follows:

$$(a_1; a_2)\mathcal{L}(a_3; a_2),\ (a_2; a_1)\mathcal{L}(a_3; a_1),\ (a_1; a_3)\mathcal{L}(a_2; a_3)$$

and

$$(a_1; a_2)\mathcal{R}(a_1; a_3),\ (a_2; a_3)\mathcal{R}(a_2; a_1),\ (a_3; a_1)\mathcal{R}(a_3; a_2)$$

The egg-box picture of this biordered set is as follows:

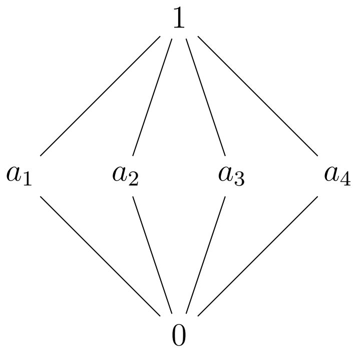
	$(a_1; a_2)$	$(a_1; a_3)$
$(a_2; a_1)$		$(a_2; a_3)$
$(a_3; a_1)$	$(a_3; a_2)$	

This lattice M_3 has a homogeneous basis of order 2, since the biordered subset $E^0_{P(L)}$ has only 2 elements $\{(a_1; a_2), (a_2; a_1)\}$. Recall the matrix ring $M_2(\mathbb{Z}_2)$ as in (Chap.2, Example 2.1.1). The egg-box picture of the idempotents of this ring is the following:

1				
		e_6	e_5	
	e_3		e_1	
	e_4	e_2		
				0

It is easily seen that these two biordered sets $E(M_3)$ and $E(M_2(\mathbb{Z}_2))$ are isomorphic. As seen in Chapter 3, Example 3.2.1 that the ω^l-ideal of the ring $M_2(\mathbb{Z}_2)$ is the complemented modular lattice M_3. Therefore, the lattice M_3 is coordinatised by the ring $M_2(\mathbb{Z}_2)$.

Example 4.2.2. Consider the lattice $M_4 = \{0, 1, a_1, a_2, a_3, a_4\}$



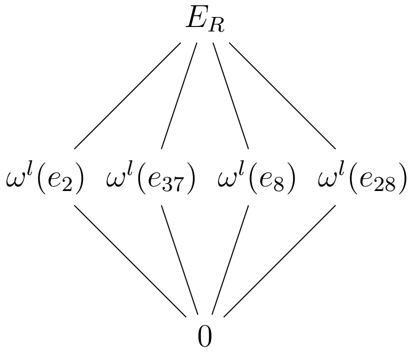
The biordered sets of M_4 is the following:

	$(a_1; a_2)$	$(a_1; a_3)$	$(a_1; a_4)$
$(a_2; a_1)$		$(a_2; a_3)$	$(a_2; a_4)$
$(a_3; a_1)$	$(a_3; a_2)$		$(a_3; a_4)$
$(a_4; a_1)$	$(a_4; a_2)$	$(a_4; a_3)$	

It can be seen from the biordered subset $E^0_{P(L)}$ that the lattice M_4 also has a homogenoeous basis of order 2. Consider the ring $M_2(\mathbb{Z}_3)$ and the biordered set of $M_2(\mathbb{Z}_3)$ is

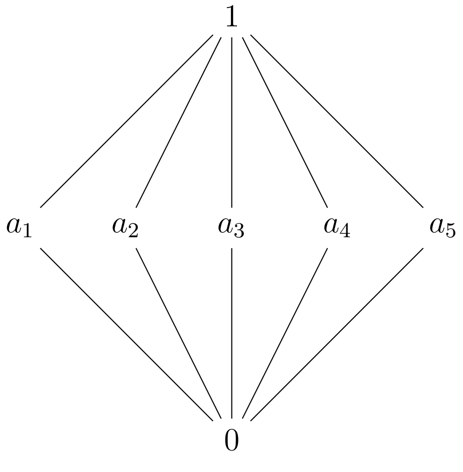
1					
		e_{11}	e_{81}	e_{31}	
	e_{46}		e_{37}	e_{28}	
	e_{69}	e_{20}		e_{34}	
	e_8	e_2	e_5		
					0

Here also it is easy to observe that the biordered sets $E(M_4)$ and $E(M_2(\mathbb{Z}_3))$ are isomorphic. The lattice of biorder ideals of the ring $M_2(\mathbb{Z}_3)$ is the lattice M_4 given below:



Thus $M_2(\mathbb{Z}_3)$ is coordinatised by M_4.

Example 4.2.3. Consider the lattice $M_5 = \{0, 1, a_1, a_2, a_3, a_4, a_5\}$



has a homogenoeous basis of order 2. The biordered sets of M_5 is the following:

	$(a_1;a_2)$	$(a_1;a_3)$	$(a_1;a_4)$	$(a_1;a_5)$
$(a_2;a_1)$		$(a_2;a_3)$	$(a_2;a_4)$	$(a_2;a_5)$
$(a_3;a_1)$	$(a_3;a_2)$		$(a_3;a_4)$	$(a_3;a_4)$
$(a_4;a_1)$	$(a_4;a_2)$	$(a_4;a_3)$		$(a_4;a_5)$
$(a_5;a_1)$	$(a_5;a_2)$	$(a_5;a_3)$	$(a_5;a_4)$	

Consider the ring $M_2(\mathbb{F}_4)$ where $\mathbb{F}_4$ is the field of order 4 defined by $\mathbb{F}_4 = \{0, 1, \beta, \beta+1\}$ where β is a root of x^2+x+1, $x \in \mathbb{Z}_2$. The idempotents of this matrix ring is as follows:

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$e_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, e_5 = \begin{bmatrix} 0 & \beta \\ 0 & 1 \end{bmatrix}, e_6 = \begin{bmatrix} 0 & \beta^2 \\ 0 & 1 \end{bmatrix}, e_7 = \begin{bmatrix} 0 & 0 \\ \beta & 1 \end{bmatrix},$$

$$e_8 = \begin{bmatrix} 0 & 0 \\ \beta^2 & 1 \end{bmatrix}, e_9 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, e_{10} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, e_{11} = \begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix},$$

$$e_{12} = \begin{bmatrix} 1 & \beta^2 \\ 0 & 0 \end{bmatrix}, e_{13} = \begin{bmatrix} 1 & 0 \\ \beta & 0 \end{bmatrix}, e_{14} = \begin{bmatrix} 1 & 0 \\ \beta^2 & 0 \end{bmatrix}, e_{15} = \begin{bmatrix} \beta & 1 \\ 1 & \beta^2 \end{bmatrix},$$

$$e_{16} = \begin{bmatrix} \beta^2 & 1 \\ 1 & \beta \end{bmatrix}, e_{17} = \begin{bmatrix} \beta & \beta \\ \beta^2 & \beta^2 \end{bmatrix}, e_{18} = \begin{bmatrix} \beta^2 & \beta^2 \\ \beta & \beta \end{bmatrix}, e_{19} = \begin{bmatrix} \beta & \beta^2 \\ \beta & \beta^2 \end{bmatrix},$$

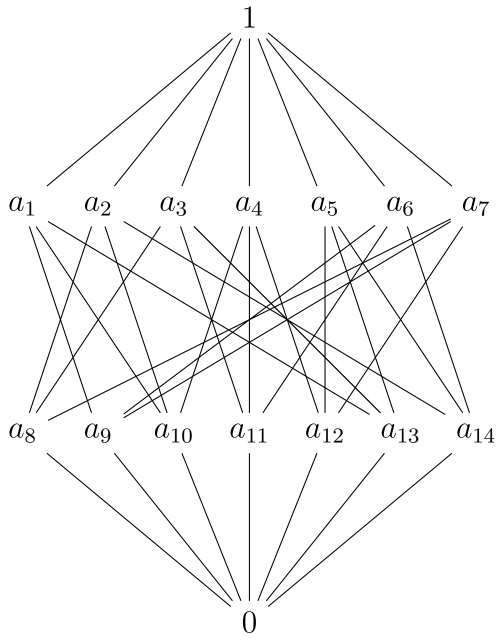
$$e_{20} = \begin{bmatrix} \beta^2 & \beta \\ \beta^2 & \beta \end{bmatrix}, 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the biordered set of $M_2(\mathbb{F}_4)$ is

1						
		e_{14}	e_{18}	e_{15}	e_5	
	e_8		e_4	e_7	e_2	
	e_{19}	e_{10}		e_{20}	e_3	
	e_{16}	e_{13}	e_{17}		e_6	
	e_{11}	e_1	e_9	e_{12}		
						0

Here also it is easily seen that the biordered sets that $E(M_5)$ and $E(M_2(\mathbb{F}_4))$ are isomorphic and the lattice of biorder ideals of $E(M_2(\mathbb{F}_4))$ is M_5.

Example 4.2.4. Consider the following lattice.



The complementary pairs of this lattice are:

$$(a_1;a_8),(a_4;a_8),(a_5;a_8),(a_6;a_8),(a_1;a_{11}),(a_2;a_{11}),(a_7;a_{11}),(a_5;a_{11}),$$

$$(a_1;a_{12}),(a_2;a_{12}),(a_3;a_{12}),(a_6;a_{12}),(a_1;a_{14}),(a_3;a_{14}),(a_4;a_{14}),(a_7;a_{14}),$$

$$(a_2;a_9),(a_3;a_9),(a_4;a_9),(a_5;a_9),(a_2;a_{13}),(a_4;a_{13}),(a_7;a_{13}),(a_6;a_{13})$$

$$(a_3; a_{10}), (a_5; a_{10}), (a_6; a_{10}), (a_7; a_{10}), (a_8; a_1), (a_8; a_4), (a_8; a_5), (a_8; a_6),$$

$$(a_{11}; a_1), (a_{11}; a_2), (a_{11}; a_7), (a_{11}; a_5), (a_{12}; a_1), (a_{12}; a_2), (a_{12}; a_3), (a_{12}; a_6)$$

$$(a_{14}; a_1), (a_{14}; a_3), (a_{14}; a_4), (a_{14}; a_7), (a_9; a_2), (a_9; a_3), (a_9; a_4), (a_9; a_5)$$

$$(a_{13}; a_2), (a_{13}; a_4), (a_{13}; a_7), (a_{13}; a_6), (a_{10}; a_3), (a_{10}; a_5), (a_{10}; a_6), (a_{10}; a_7)$$

and the eggbox picture of the biordered set of this lattice is given in page 93:

Consider the elements $\{a_{10}, a_{11}, a_{13}\}$ in the lattice L. It is easily seen that these elements are independent in this lattice and hence the biordered set $E_{P(L)}$ has a biordered subset

$$E^0_{P(L)} = \{(a_3; a_{10}), (a_1; a_{11}), (a_4; a_{13})\}$$

Thus this lattice has a homogenoeus basis with 3 elements and so the lattice L is of order 3. Also from Example 3.2.2, it is evident that this lattice is coordinatised by the ring $M_3(\mathbb{Z}_2)$.

							$(a_1; a_8)$			$(a_1; a_{11})$	$(a_1; a_{12})$		$(a_1; a_{14})$
								$(a_2; a_9)$		$(a_2; a_{11})$	$(a_2; a_{12})$	$(a_2; a_{13})$	
								$(a_3; a_9)$	$(a_3; a_{10})$		$(a_3; a_{12})$		$(a_3; a_{14})$
							$(a_4; a_8)$	$(a_4; a_9)$				$(a_4; a_{13})$	$(a_4; a_{14})$
							$(a_5; a_8)$	$(a_5; a_9)$	$(a_5; a_{10})$	$(a_5; a_{11})$			
							$(a_6; a_8)$		$(a_6; a_{10})$		$(a_6; a_{12})$	$(a_6; a_{13})$	
									$(a_7; a_{10})$	$(a_7; a_{11})$		$(a_7; a_{13})$	$(a_7; a_{14})$
$(a_8; a_1)$			$(a_8; a_4)$		$(a_8; a_5)$	$(a_8; a_6)$							
	$(a_9; a_2)$	$(a_9; a_3)$	$(a_9; a_4)$	$(a_9; a_5)$									
		$(a_{10}; a_3)$		$(a_{10}; a_5)$	$(a_{10}; a_6)$	$(a_{10}; a_7)$							
$(a_{11}; a_1)$	$(a_{11}; a_2)$			$(a_{11}; a_5)$		$(a_{11}; a_7)$							
$(a_{12}; a_1)$	$(a_{12}; a_2)$	$(a_{12}; a_3)$			$(a_{12}; a_6)$								
	$(a_{13}; a_2)$		$(a_{13}; a_4)$		$(a_{13}; a_6)$	$(a_{13}; a_7)$							
$(a_{14}; a_1)$		$(a_{14}; a_3)$	$(a_{14}; a_4)$			$(a_{14}; a_7)$							

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- *P. G. Romeo and R Akhila*: **Additive Biordered Set Derived from a Ring**, , Southeast Asian Bulletin Of Mathematics, (2018) 42: 111-116.
- *P. G. Romeo and R Akhila*: **Biorder Ideals and Regular Rings**, Algebra and its Applications, Springer Proceedings in Mathematics and Statistics, ICSAA, Aligarh, 2014 Vol **174**, ISSN 2194-1017, 265-274.
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- *P. G. Romeo and R Akhila*: **Biorder Ideals of Regular Rings**, International conference of Algebra and its Applications, Aligarh Muslin University, Aligarh, India 15-17, December 2014.
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- *R. Akhila and P. G. Romeo*: **Rings and Distributive Lattices**, International seminar on Algebra and Coding Theory, 8-10, January 2017

Scope of Further Study

In this thesis the biordered sets (both additive and multiplicative) of a regular ring are described. But the converse problem of constructing a regular ring from the biordered set of idempotents was successfully done only for some very special class of rings. So one can look into this problem for various classes of rings.

It is shown that the biorder ideals of a regular ring is a complemented modular lattice and an analogous theorem to von Neumann's coordinatization theorem is provided. But the actual construction of the ring coordinatizing the lattice using biordered sets (independent of von Neumann's construction of L- numbers) is yet to achieve.

In chapter 4, we provide some examples for biordered sets of lattices of order two and three. But the existence of the biordered set does not guarantee the coordinatization of the lattices of order less than 4. This demands further study in this direction.

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