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Product autoregressive models for non-negative variables

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ABSTRACT

When variables in time series context are non-negative, such as for volatility, survival time or wave heights, a multiplicative autoregressive model of the type $X_t = X_{t-1}^{\alpha} V_t$, $0 \leq \alpha < 1$, $t = 1, 2, \ldots$ may give the preferred dependent structure. In this paper, we study the properties of such models and propose methods for parameter estimation. Explicit solutions of the model are obtained in the case of gamma marginal distribution.

1. Introduction

Linear Autoregressive (AR) models have played a significant role in modeling the dependency structure in the study of Gaussian and non-Gaussian time series. When the time series of interest is a sequence of non-negative random variables such as volatility, survival time or wave heights, the product form of the models is preferable compared to their linear counterparts. Another context where modeling of non-negative random variables plays a major role is in the study of financial time series, where one has to model the evolution of conditional variances. As an alternative one can adopt some of the AR(1) models for non-negative r.v.'s such as Exponential, Gamma, etc. available in the context of non-Gaussian time series (cf: Gaver and Lewis, 1980). When we restrict the variables to be non-negative, the innovation distribution in most of the linear AR(1) models has singular components and that leads to complications while dealing with inference problems. In fact this is one of the drawbacks of the additive models that motivated Engle (2002) to introduce Multiplicative Error Models (MEMs) to analyze the sequence of non-negative r.v.'s. In this paper, we study a class of models defined by

$$X_t = X_{t-1}^{\alpha} V_t, \quad 0 \leq \alpha < 1,\ t = 1, 2, \ldots \tag{1.1}$$

where $\{V_t\}$ is a sequence of independent and identically distributed (i.i.d.) non-negative r.v.'s. We assume that the r.v.'s X_0 and V_1 are independent. The model (1.1) initially introduced by Mckenzie (1982) is referred to as the Product Autoregressive

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model of order 1 (PAR(1)). For an explicit analysis of the model it is important to know the stationary marginal distribution of $\{X_t\}$. This in turn requires us to identify the distribution of $\{V_t\}$ for a specified marginal distribution of $\{X_t\}$, a problem common in the study of non-Gaussian time series models. In fact Mckenzie (1982) developed the model (1.1) to generate a sequence $\{X_t\}$ of gamma r.v.'s through the properties of linear gamma AR(1) (GAR(1)) model of Gaver and Lewis (1980). However, the form of the distribution of V_t was not known explicitly. Mckenzie's interest was to establish a characterizing property of the gamma sequence, namely, $\{X_t\}$ and $\{\log X_t\}$ have the same autocorrelation structure.

The model (1.1) may be viewed as a special case of the MEM of Engle (2002) in which the innovations $\{V_t\}$ are assumed to be i.i.d. non-negative r.v.'s with unit mean. In MEM, a specific form of the innovation distribution is assumed for the analysis, and no attention is given to the stationary marginal distribution of $\{X_t\}$. However, in the context of financial time series, to develop stochastic volatility (SV) models or stochastic conditional duration (SCD) models, it is important to specify the marginal distributions. In view of this, we propose the model defined by (1.1) to generate sequences of volatilities (non-negative r.v.'s) having specified marginal distributions. The literature on financial time series models with latent structures assumes that the volatilities in SV models and the conditional means in SCD models are generated by (1.1) with log-normal marginal distributions (cf. Taylor, 1994, Bauwens and Veredas, 2004). For detailed survey on these models, one may refer Pacurar (2008) or Tsay (2005). In this paper, we are proposing the gamma distribution as an alternative to the log-normal distribution to model the volatilities in the SV and SCD setup.

Moreover, the stationary gamma Markov sequences have their own role in modeling point processes and dam models, (cf. Gaver and Lewis (1980)). In particular, Balakrishna and Lawrance (in press) discussed the PAR(1) models with gamma marginal distribution by approximating the innovation densities and fitted this model to the sea wave height data collected from Bay of Bengal. The approximation was done by comparing the first two moments of the r.v.'s on both sides in (1.1). In the present work, we obtain an explicit form of the innovation random variable V_t which provides gamma marginal distribution for $\{X_t\}$. It is interesting to note that the innovation r.v., for the gamma PAR(1) model is absolutely continuous unlike in the case of linear GAR(1) model, where the innovation has a singular component. Hence the product form of the gamma AR can be more useful to study real life applications.

Rest of the paper is organized as follows. In Section 2, we study some of the useful properties of the sequence generated by the model (1.1). Explicit form of the innovation distribution for the gamma PAR(1) model is obtained in Section 3. A method of simulating the gamma PAR(1) sequence is described in Section 4. Problem of parameter estimation by the method of conditional least squares is discussed in Section 5. Some concluding remarks are given in Section 6.

2. Properties of PAR(1) models

For the detailed analysis of model (1.1), one needs to study the distributional aspects of the variables involved in it. As pointed out by Mckenzie (1982), it is hard to obtain explicit distribution of V_t for a specified stationary marginal distribution of $\{X_t\}$. We derive the form of the innovation distribution for X_t using method of transforms. The log-transform of (1.1) leads to

$$\log X_t = \alpha \log X_{t-1} + \log V_t, \quad 0 \le \alpha < 1, \tag{2.1}$$

which is an AR(1) model in $\log X_t$. In terms of the moment generating function (mgf), we may express (2.1) as

$$\phi_{\log V}(s) = \phi_{\log X}(s)/\phi_{\log X}(\alpha s), \tag{2.2}$$

where $\phi_U(s) = E(\exp(sU))$ is the mgf of U. Thus the model (1.1) defines a stationary sequence $\{X_t\}$ if the right hand side of (2.2) is a proper mgf for every $\alpha \in (0, 1)$. This happens if $\log X_t$ is a self-decomposable r.v. In fact the mgf of $\log X_t$ may be expressed as the Mellin Transform (MT), $M_X(s)$ of X_t, defined by $M_X(s) = E(X_t^s)$, $s \ge 0$, whenever the expectation exists. Thus we can use the Mellin transform to identify the innovation distribution for PAR(1) models. Now Eq. (2.2) can be written in terms of MT as

$$M_V(s) = M_X(s)/M_X(\alpha s). \tag{2.3}$$

If V_t admits a density function $f_V(\cdot)$, then the one step transition pdf of $\{X_t\}$ can be expressed as

$$f(x_t|x_{t-1}) = \frac{1}{x_{t-1}^{\alpha}} f_V(x_t/x_{t-1}^{\alpha}). \tag{2.4}$$

Assuming the finiteness of second moments of the stationary marginal distribution, the autocorrelation function (acf) of PAR(1) sequence $\{X_t\}$ is given by (cf; Mckenzie, 1982):

$$\rho_X(j) = \mathrm{Corr}(X_t, X_{t-j}) = \frac{E(X_t)\left\{E(X_{t-j}^{\alpha^j+1}) - E(X_{t-j}^{\alpha^j})E(X_{t-j})\right\}}{E(X_{t-j}^{\alpha^j})\mathrm{Var}(X_t)}. \tag{2.5}$$

The acf of the squared sequence is also important when we analyze the non-linear time series models. For the PAR(1) model, such acf is given by

$$\rho_{X^2}(j) = \mathrm{Corr}(X_t^2, X_{t-j}^2) = \frac{E(X_t^2)\left\{E(X_{t-j}^{2\alpha^j+2}) - E(X_{t-j}^{2\alpha^j})E(X_{t-j}^2)\right\}}{E(X_{t-j}^{2\alpha^j})\mathrm{Var}(X_t^2)} \tag{2.6}$$

whenever $E(X_t^4) < \infty$.

The above acfs depend only on the moments of stationary marginal distribution.

Result 2.1. Let $\{X_t, t = 0, 1, 2, \ldots\}$ be a stationary sequence of non-negative r.v.s defined by (1.1) with X_0 independent of V_1. Assume that X_0 follows the stationary distribution of the sequence. Then $\{X_t, t = 0, 1, 2, \ldots\}$ is strictly stationary and ergodic.

Proof. The stationary property follows from Mckenzie (1982) when $0 \le \alpha < 1$.

The result is proved once we establish that all the invariant events of $\mathbb{F}_t = \sigma\{X_t, t \ge 1\}$, the minimal sigma field generated by $\{X_t, t \ge 1\}$ have probability 0 or 1. It is also known that for a stationary sequence, every invariant event is a tail event (cf. Breiman, 1968, pp 119). Thus to prove the ergodicity of $\{X_t, t \ge 1\}$ it is enough to show that all its tail events are trivial.

Repeatedly using (1.1) we can write $X_t = X_1^{\alpha^t} \cdot V_t \cdot V_{t-1}^{\alpha} \cdot V_{t-2}^{\alpha^2}, \ldots, V_2^{\alpha^{t-1}}, t = 2, 3, \ldots$ and hence it follows that $\sigma\{X_1, X_2, \ldots, X_t\} \subset \sigma\{X_1, V_2, \ldots, V_t\} = \mathfrak{M}_t$ for $t = 1, 2, \ldots$, where $\mathfrak{M}_t$ is the sigma field induces by a set of independent r.v.'s $X_1, V_2, V_3, \ldots$. This implies that all tail events of $\{X_t, t \ge 1\}$ are contained in the tail sigma field of $X_1, V_2, V_3, \ldots$. The tail events of the latter sigma field are all trivial by Kolmogorov's 0–1 law. This in turn implies that the tail events of $\{X_t, t \ge 1\}$ are also trivial. Thus the result is established. □

3. Innovation distribution of the gamma PAR(1) model

Suppose that X_t has a gamma distribution (Gamma(θ, λ)) with pdf

$$f(x) = e^{-\lambda x}\lambda^{\theta}x^{\theta-1}/\Gamma(\theta), \quad x \ge 0,\ \lambda > 0,\ \theta > 0 \tag{3.1}$$

and the Mellin transform

$$M_X(s) = \lambda^{-s}\Gamma(s+\theta)/\Gamma(\theta).$$

If we want $\{X_t\}$ defined by (1.1) to be a stationary sequence with Gamma(θ, λ) marginal distribution, then the MT of V_t becomes

$$M_V(s) = \lambda^{-(1-\alpha)s}\Gamma(s+\theta)/\Gamma(\alpha s+\theta). \tag{3.2}$$

It is not straightforward to find the distribution of V_t by inverting this MT. Mckenzie (1982) has obtained the distribution of V_t for an Exponential PAR(1) model ($\theta = 1, \lambda = 1$ in the above discussion), and shown that it is distributed as $S^{-\alpha}$, where S is a positive stable random variable with Laplace transform $\phi_S(s) = \exp(-s^{\alpha})$. The resulting pdf of the innovation for an exponential PAR(1) model is given by

$$g_E(x;\alpha) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(k\alpha)}{\Gamma(k)}\sin(k\pi\alpha)\cdot(-x)^{k-1}, \quad x > 0. \tag{3.3}$$

Let us denote the r.v. corresponding to the pdf (3.3) by V_E, where E stands for unit exponential r.v. In the following result, we obtain an explicit form of the innovation distribution for the gamma PAR(1) models.

Result 3.1. If the PAR(1) sequence defined by the model (1.1) has a stationary gamma marginal distribution with pdf (3.1) then the distribution of its innovation r.v. V_t is specified by

$$V_t = \lambda^{-(1-\alpha)}[B(\alpha,\theta)]^{\alpha}V_{\theta}, \tag{3.4}$$

where $B(\alpha, \theta)$ and V_θ are mutually independent i.i.d. r.v.'s with $B(\alpha, \theta)$ being beta$(\alpha\theta,\ (1-\alpha)\theta)$ having pdf:

$$f_B(x) = \frac{\Gamma(\theta)}{\Gamma(\alpha\theta)\Gamma((1-\alpha)\theta)}x^{\alpha\theta-1}(1-x)^{(1-\alpha)\theta-1}, \quad 0 \le x \le 1 \tag{3.5}$$

and the pdf of V_θ is given by

$$g(x;\alpha,\theta) = \frac{\Gamma(\alpha\theta+1)}{\Gamma(\theta+1)}x^{\theta}g_E(x;\alpha), \quad x > 0, \tag{3.6}$$

where $g_E(\cdot)$ is the density function (3.3).

Proof. We prove this result using Mellin transforms. The Mellin transform of the innovation r.v. corresponding to the gamma PAR(1) model is given by (3.2). Here we show that the Mellin transform of $V_t = \lambda^{-(1-\alpha)} \cdot B^{\alpha} V_{\theta}$ is same as (3.2). Consider

$$\begin{aligned} M_V(s) &= \lambda^{-(1-\alpha)} E(V_t^s) = \lambda^{-(1-\alpha)} E(B^{\alpha s}) \cdot E(V_{\theta}^s) \\ &= \lambda^{-(1-\alpha)} \cdot \frac{\Gamma(\theta)\Gamma(\alpha(s+\theta))}{\Gamma(\alpha\theta) \cdot \Gamma(\alpha s+\theta)} \cdot \frac{\Gamma(\alpha\theta+1) \cdot \Gamma(s+\theta+1)}{\Gamma(\theta+1) \cdot \Gamma((s+\theta)\alpha+1)}. \end{aligned}$$

On simplification, the right hand side reduces to that of (3.2). Hence the result is established. □

The density function of V_t for the above Mellin transform may be expressed as

$$f_V(v) = \frac{v^{\theta}}{\Gamma((1-\alpha)\theta)} \int_0^1 u^{-2}(1-u^{1/\alpha})^{(1-\alpha)\theta-1} g_E(v/u; \alpha) du. \tag{3.7}$$

The transition density function of X_t, at x_t given $X_{t-1} = x_{t-1}$, is given by (cf: (2.4))

$$f(x_t|x_{t-1}) = (\lambda x_{t-1})^{-(\theta+1)\alpha} (\lambda x_t)^{\theta} \cdot \frac{\lambda}{\Gamma((1-\alpha)\theta)} \int_0^1 \frac{(1-u^{1/\alpha})^{(1-\alpha)\theta-1}}{u^2} g_E(\lambda^{1-\alpha} x_t/u x_{t-1}^{\alpha}; \alpha) du. \tag{3.8}$$

We can get a Weibull PAR(1) sequence by taking a power transformation of the variables in an Exponential PAR(1) model. The properties of such models are discussed in Balakrishna and Lawrance (in press). For the above gamma PAR(1) model the acfs of $\{X_t\}$ and $\{X_t^2\}$ obtained via (2.5) and (2.6) are respectively given by

$$\rho_X(j) = \alpha^j, \quad j = 0, 1, 2, \ldots \quad \text{and} \quad \rho_{X^2}(j) = \frac{1+2\theta+2\alpha^j}{2\theta+3}\alpha^j, \quad j = 0, 1, 2, \ldots.$$

Both acfs decay geometrically when j increases. Note that the acf of $\{X_t\}$ is free from the parameters θ of the stationary distribution while that of $\{X_t^2\}$ depends on its shape parameter. It is clear from the above expressions that as θ increases the acf of $\{X_t^2\}$ approaches that of $\{X_t\}$.

Remark 3.1. In general, the innovation pdf does not have a closed form expression. However, for $\alpha = 1/2$ we can get a closed form for the pdf of V_{θ} and is expressed as

$$g\left(x; \frac{1}{2}, \theta\right) = \begin{cases} \dfrac{\lambda^{(\theta+1)/2} x^{\theta} e^{-\lambda x^2/2}}{2^{\theta}\sqrt{(\theta+1)/2}}, & x > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

This is the pdf of $\sqrt{Y}$ where Y is a gamma $\left(\frac{\theta+1}{2}, \frac{\lambda}{4}\right)$ r.v. Hence when $\alpha = 1/2$, the distribution of the innovation V_t is given by that of $\sqrt{\text{beta}(\theta/2, \theta/2)}\sqrt{Y}$.

Remark 3.2. We have also identified the innovation distributions for the PAR(1) models when the stationary marginal distributions of the sequences are Uniform, Pareto, Power function, Weibull, etc. Further, problem of parameter estimation also studied. However, the detailed analysis of gamma PAR(1) model is discussed in this paper.

4. Simulation of the gamma PAR(1) model

Simulation of a sequence from a gamma PAR(1) model requires the simulation from the innovation r.v. V_t described in Result 3.1. This can be done by drawing independent samples from B, V_{θ} and then using (3.4). Note that the pdf of V_{θ} is the weighted version of the innovation pdf (3.3) of an exponential PAR(1) model with a weight function $w(x) = x^{\theta}$. So we adopt Mckenzie's method to simulate V_E, the innovation r.v. corresponding to an exponential PAR(1) model and then obtain the sample from V_{θ} by accept–reject method (cf: Ripley, 1987). The simulation algorithm is described below:

Step 1: Specify the values for the parameters α, λ, θ and generate a random sample of large size from (3.3) using the formula: $V_E = E^{1-\alpha} \sin U \cdot (\sin(\alpha U))^{-\alpha} \cdot (\sin((1-\alpha)U))^{-(1-\alpha)}$ proposed by Mckenzie (1982), where U is a uniform r.v. over $(0, \pi)$ and E is a unit exponential r.v. independent of U. Let $\{V_E(i), i = 1, 2, \ldots, N\}$ denote the resulting sample and let $w_i = (V_E(i))^{\theta}$, $i = 1, 2, \ldots, N$ be the weights.
Step 2: Let $M = \max(w_i) + 1$ and define $p_i = w_i/M$, $i = 1, 2, \ldots, N$.
Step 3: For $t = 1, 2, \ldots$ draw a random number R_t from U(0,1). If $R_t < p_t$ then accept the tth observation as $V_{\theta}(t) = V_E(t)$, otherwise reject it.
Step 4: Continue Step 3 until we get a sample of required size.
Step 5: Generate an independent sample from $B(\alpha, \theta)$ with pdf (3.5) and then obtain a sample from the gamma innovation V_t using the formula (3.4).
Step 6: Finally obtain the gamma PAR(1) sequence using (1.1).

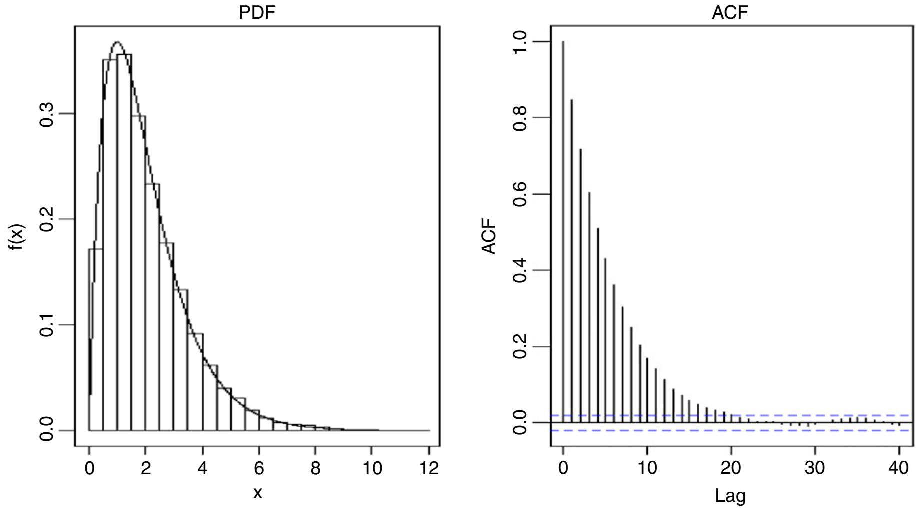

Fig. 1. Histogram of the simulated sample of size 10 000 from a gamma PAR(1) sequence for $\alpha = 0.85, \theta = 2, \lambda = 1$. The line on the histogram is the theoretical density curve of the corresponding gamma distribution. The second graph is the autocorrelation function of the simulated sequence.

The first plot in the following Fig. 1 is a histogram of the realization generated from a gamma PAR(1) model super-imposed by the gamma pdf with corresponding parameters shows good agreement between the simulated and theoretical pdfs. The second plot is the acf of the simulated series which is geometrically decreasing, a characterizing property of gamma PAR(1) sequence.

Remark 4.1. The simulation procedure described in Steps (1)–(4) leads to lot of rejections and hence we need to generate a large number of observations from V_E to get a sample of reasonable size from V_θ. For example, when $(\theta = 1, \alpha = 0.4)$ the rejection was 85% and when $(\theta = 2, \alpha = 0.4)$ it was 95%. For larger values of α the rejection rate is relatively low. If $(\theta = 1, \alpha = 0.95)$ and $(\theta = 2, \alpha = 0.95)$ the rejection rates were 57% and 63%, respectively.

In the next section, we study the problem of estimation by the method of conditional least squares.

5. Parameter estimation by conditional least squares

The complex structure of the innovation r.v. makes it difficult to obtain the likelihood based estimation of the parameters. So we employ the method of Conditional Least Squares proposed by Klimko and Nelson (1978) to estimate the parameters of the gamma PAR(1) model. Let $\{X_t\}$ be a stationary Markov sequence. The CLS estimator of the parameter is obtained by minimizing

$$Q_n(\mu) = \sum_{t=1}^{n} [X_t - g(\mu; X_{t-1})]^2 \tag{5.1}$$

with respect to the parameter vector $\mu = (\mu_1, \mu_2, \ldots, \mu_p)'$, where

$$g(\mu; X_{t-1}) = E(X_t | X_{t-1}). \tag{5.2}$$

The CLS estimates are obtained by solving the least squares equations:

$$\frac{\partial Q_n(\mu)}{\partial \mu_i} = 0, \quad i = 1, 2, \ldots, p. \tag{5.3}$$

Klimko and Nelson (1978) proved under certain regularity conditions that the CLS estimators are consistent and asymptotically normal (CAN) as stated in the flowing lemma.

Lemma 5.1. *Let $\{X_t\}$ be a stationary and ergodic Markov sequence with finite third order moments. Under the regularity conditions of Klimko and Nelson (1978), the CLS estimator $\hat{\mu}$ of μ is CAN. That is, as $n \to \infty$*

$$\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{L} N_p(0, V^{-1}WV^{-1}),$$

where V and W are pxp matrices, whose (i, j)th elements are respectively given by

$$V_{ij} = E\left(\frac{\partial g(\mu; X_{t-1})}{\partial \mu_i} \cdot \frac{\partial g(\mu; X_{t-1})}{\partial \mu_j}\right), \quad i, j = 1, 2, \ldots, p$$

and

$$W_{ij} = E\left(u_t^2(\mu)\frac{\partial g(\mu;X_{t-1})}{\partial \mu_i}\cdot\frac{\partial g(\mu;X_{t-1})}{\partial \mu_j}\right),\quad i,j=1,2,\ldots,p \quad \textit{and} \quad u_t = X_t - g(\mu;X_{t-1}).$$

We now show that $\{X_t\}$ defined by (1.1) satisfies these regularity conditions. Result 2.1 shows that the sequence $\{X_t\}$ is stationary and ergodic. Other regularity conditions will follow if we assume that the third order moments of the marginal distribution are finite. Now we obtain the CLS estimators for the gamma PAR(1) model described in the earlier sections. We assume the scale parameter $\lambda = 1$ in order to avoid the identifiability problem and let $\mu = (\alpha,\theta)'$. The conditional expectation in this case is $g(\mu;X_{t-1}) = \frac{\Gamma(1+\theta)}{\Gamma(\alpha+\theta)}X_{t-1}^{\alpha}$ and the corresponding least squares equations lead to the following relations:

$$\sum_{t=1}^{n} X_t X_{t-1}^{\alpha} = \frac{\Gamma(1+\theta)}{\Gamma(\alpha+\theta)}\sum_{t=1}^{n} X_{t-1}^{2\alpha} \tag{5.4}$$

and

$$\frac{\sum_{t=1}^{n} X_t X_{t-1}^{\alpha}}{\sum_{t=1}^{n} X_t X_{t-1}^{\alpha}\ln(X_{t-1})} = \frac{\sum_{t=1}^{n} X_{t-1}^{2\alpha}}{\sum_{t=1}^{n} X_{t-1}^{2\alpha}\ln(X_{t-1})}. \tag{5.5}$$

Solving these equations we can get the LSE estimates of $\mu = (\alpha,\theta)'$.

Since all moments of the gamma distribution are finite the Klimko–Nelson regularity conditions are satisfied for the gamma PAR(1) sequence. Hence the CLS estimator $\hat{\mu} = (\hat{\alpha},\hat{\theta})'$ is CAN for $\mu = (\alpha,\theta)'$. The asymptotic dispersion matrix of $\hat{\mu}$ is given by $V^{-1}WV^{-1}$, where $V = \begin{pmatrix} v_{11} & v_{12}\\ v_{21} & v_{22}\end{pmatrix}$

$$\begin{aligned}
v_{11} &= \frac{\Gamma^2(1+\theta)}{\Gamma^4(\alpha+\theta)\Gamma(\theta)}\left[\begin{matrix}0.25\Gamma^2(\alpha+\theta)\Gamma''_{\alpha}(2\alpha+\theta) + \Gamma(2\alpha+\theta)\left(\Gamma'_{\alpha}(\alpha+\theta)\right)^2\\ -\Gamma(\alpha+\theta)\cdot\Gamma'_{\alpha}(\alpha+\theta)\cdot\Gamma'_{\alpha}(2\alpha+\theta)\end{matrix}\right]\\
v_{12} &= \frac{\Gamma(1+\theta)}{\Gamma^4(\alpha+\theta)\Gamma(\theta)}\left[\begin{matrix}\left(\Gamma(\alpha+\theta)\Gamma'_{\theta}(1+\theta) - \Gamma(1+\theta)\Gamma'_{\theta}(\alpha+\theta)\right)\\ \times\left(\Gamma(\alpha+\theta)\cdot\Gamma'_{\theta}(2\alpha+\theta) - \Gamma(2\alpha+\theta)\cdot\Gamma'_{\alpha}(\alpha+\theta)\right)\end{matrix}\right]\\
&= v_{21}\\
v_{22} &= \frac{\Gamma(2\alpha+\theta)}{\Gamma^4(\alpha+\theta)\cdot\Gamma(\theta)}\left[\Gamma(\alpha+\theta)\cdot\Gamma'_{\theta}(1+\theta) - \Gamma(1+\theta)\cdot\Gamma'_{\theta}(\alpha+\theta)\right]^2
\end{aligned}$$

and

$$W = \frac{\sigma_V^2}{\Gamma(\theta)}\cdot\begin{pmatrix} w_{11} & w_{12}\\ w_{21} & w_{22}\end{pmatrix}$$

where

$$\begin{aligned}
\sigma_V^2 &= \frac{\Gamma(2+\theta)}{\Gamma(2\alpha+\theta)} - \left(\frac{\Gamma(1+\theta)}{\Gamma(\alpha+\theta)}\right)^2, \text{ is the variance of the innovation r.v.,}\\
w_{11} &= \frac{\Gamma^2(1+\theta)}{\Gamma^4(\alpha+\theta)}\left[\begin{matrix}(1/16)\Gamma^2(\alpha+\theta)\Gamma''_{\alpha}(4\alpha+\theta) + \Gamma(4\alpha+\theta)\left(\Gamma'_{\alpha}(\alpha+\theta)\right)^2\\ -(1/2)\Gamma(\alpha+\theta)\cdot\Gamma'_{\alpha}(\alpha+\theta)\cdot\Gamma'_{\alpha}(4\alpha+\theta)\end{matrix}\right]\\
w_{12} &= \frac{\Gamma(1+\theta)}{\Gamma^4(\alpha+\theta)}\left[\begin{matrix}\left(\Gamma(\alpha+\theta)\Gamma'_{\theta}(1+\theta) - \Gamma(1+\theta)\Gamma'_{\theta}(\alpha+\theta)\right)\\ \times\left((1/4)\cdot\Gamma(\alpha+\theta)\cdot\Gamma'_{\alpha}(4\alpha+\theta) - \Gamma(4\alpha+\theta)\cdot\Gamma'_{\alpha}(\alpha+\theta)\right)\end{matrix}\right]\\
&= w_{21}\\
w_{22} &= \frac{\Gamma(4\alpha+\theta)}{\Gamma^4(\alpha+\theta)}\left[\Gamma(\alpha+\theta)\cdot\Gamma'_{\theta}(1+\theta) - \Gamma(1+\theta)\cdot\Gamma'_{\theta}(\alpha+\theta)\right]^2.
\end{aligned}$$

In the above expressions, we have used the following notations:

$$\Gamma(x) = \int_0^{\infty} e^{-u}u^{x-1}du,\qquad \Gamma'_y(x+y) = \frac{\partial}{\partial y}\Gamma(x+y) \quad \text{and} \quad \Gamma''_y(x+y) = \frac{\partial^2}{\partial y^2}\Gamma(x+y).$$

In Table 1, we summarize the simulation results on parameter estimation.

Table 1
Simulated CLSE for gamma PAR(1) model.

Parameters		$n = 100$		$n = 500$		$n = 1000$	
θ	α	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
0.5	0.4	0.5008 (0.1657)	0.3501 (0.1072)	0.5132 (0.0741)	0.3930 (0.0665)	0.50656 (0.0471)	0.3968 (0.0499)
	0.6	0.4794 (0.2463)	0.5252 (0.1269)	0.4771 (0.1087)	0.5733 (0.0782)	0.4940 (0.0816)	0.5923 (0.0601)
	0.8	0.5631 (0.6336)	0.6919 (0.1120)	0.4775 (0.1861)	0.7598 (0.0652)	0.5077 (0.1481)	0.7792 (0.0509)
	0.9	0.51036 (0.5252)	0.7709 (0.1025)	0.4965 (0.2817)	0.8641 (0.0499)	0.5009 (0.2475)	0.8674 (0.0390)
	0.95	0.5362 (0.7811)	0.7994 (0.1210)	0.4771 (0.3376)	0.9163 (0.0345)	0.4616 (0.295)	0.9267 (0.0262)
1.0	0.4	0.9717 (0.1770)	0.3656 (0.0959)	0.9967 (0.0778)	0.3952 (0.0462)	1.0066 (0.0574)	0.3948 (0.0363)
	0.6	1.0144 (0.2499)	0.5509 (0.1131)	0.9955 (0.1189)	0.6088 (0.0601)	0.9983 (0.0719)	0.5925 (0.0440)
	0.8	1.0077 (0.4445)	0.7232 (0.0748)	0.9800 (0.2144)	0.7712 (0.0556)	0.9557 (0.1521)	0.7948 (0.0439)
	0.9	1.082 (0.8095)	0.8143 (0.0709)	1.0078 (0.2667)	0.8701 (0.0398)	0.9958 (0.2106)	0.8890 (0.0299)
	0.95	0.9477 (0.9036)	0.8675 (0.0626)	1.0429 (0.5098)	0.9243 (0.0287)	0.9977 (0.3134)	0.9387 (0.0227)
2.0	0.4	1.9971 (0.2314)	0.3824 (0.0949)	2.0051 (0.1069)	0.3924 (0.0438)	2.0236 (0.0754)	0.3959 (0.0326)
	0.6	2.0405 (0.3473)	0.5523 (0.0938)	1.9976 (0.1364)	0.5862 (0.0523)	2.0036 (0.1062)	0.5972 (0.0356)
	0.8	2.1461 (0.5349)	0.7308 (0.0792)	1.9587 (0.2381)	0.7825 (0.0458)	2.0097 (0.1534)	0.7905 (0.0355)
	0.9	2.2278 (0.8939)	0.8380 (0.0658)	2.0317 (0.3067)	0.8802 (0.0349)	2.0095 (0.2535)	0.8931 (0.0236)
	0.95	1.9856 (1.0655)	0.8703 (0.0611)	1.9504 (0.4545)	0.9292 (0.0271)	1.9771 (0.3564)	0.9405 (0.0183)

Some remarks on the table are required at this stage. We generated a sample of size n for specified value of the parameters α, θ using the accept–reject method described in Section 4 for $n = 100, 500, 1000$ and obtained CLSE by solving Eqs. (5.4)–(5.5). For different parametric combinations of $\theta = 0.5,\ 1.0,\ 2.0$ and $\alpha = 0.4,\ 0.6,\ 0.8,\ 0.9,\ 0.95$ we repeated the estimation 100 times and the mean values are presented in the Table along with the standard error in the parenthesis. Note that the estimates are better for smaller values of α and they tend to the corresponding parameter values as the sample size increases.

Remark 5.1. Maximum likelihood estimates (MLEs) of the model parameters are preferred whenever we have a manageable likelihood function. Billingsley (1961) established that the MLE of the parameter vector of a stationary Markov sequence is Consistent and Asymptotically Normal (CAN) under a set of regularity conditions on the one-step transition density function. Obtaining the MLEs for the gamma PAR(1) model is difficult due to the complex structure of the transition density given by (3.8) and we will try to solve this problem in our future research.

6. Concluding remarks

In this paper, we studied the properties of product autoregressive models in view of their applications in financial time series to model stochastic volatilities and stochastic conditional durations. Apart from exploring their probabilistic properties, we also illustrate the existence of explicit solution for gamma PAR(1) model. Method of conditional least squares is proposed to obtain consistent and asymptotically normal estimates of the parameters. Detailed studies on maximum likelihood estimation and the modeling of stochastic volatility using the product models will be discussed in the forthcoming papers.

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