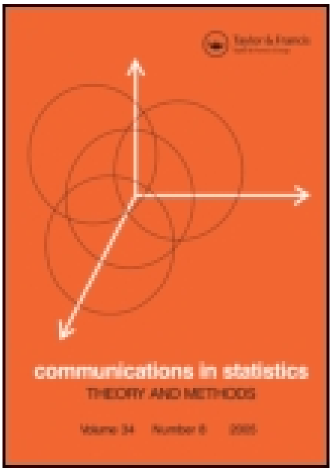

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Distribution Theory

Some Properties of Weighted Distributions in the Context of Repairable Systems

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In this article we introduce some structural relationships between weighted and original variables in the context of maintainability function and reversed repair rate. Furthermore, we prove some characterization theorems for specific models such as power, exponential, Pareto II, beta, and Pearson system of distributions using the relationships between the original and weighted random variables.

Keywords Maintainability function; Reversed hazard rate; Weighted distributions.

Mathematics Subject Classification 62E10; 62N05.

1. Introduction

The concept of weighted distributions was introduced by Rao (1965) in connection with modeling statistical data, in situations where the usual practice of employing standard distributions for the purpose was not found appropriate. A formal definition of a weighted distribution is obtained by considering a probability space $(\Omega, \Im, P)$ and a random variable (rv) $X : \Omega \to H$, where $H = (a, b)$ is an interval of the real line. In the continuous case, if $f(x)$ is the probability density function (pdf) of X and $w(\cdot)$ a non negative function satisfying $\mu_w = E(w(X)) < \infty$, then the rv Y with pdf

$$f_w(x) = \frac{w(x)f(x)}{\mu_w}, \quad a < x < b \tag{1.1}$$

is said to have weighted distribution, corresponding to the distribution of X. The definition in the discrete case is analogs. The concept of weighted distribution has been employed in various practical problems such as analysis of family size, study of

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albinism, human heredity, aerial survey and visibility bias, line transcend sampling, renewal theory, cell cycle analysis and pulse labeling, efficacy of early screening for disease, etiological studies, statistical ecology, and reliability modeling. For detailed investigation of research in this area, we refer to Patil and Rao (1977), Gupta and Kirmani (1990), Navarro et al. (2001), Nair and Sunoj (2003), and Pakes et al. (2003).

Motivated by the applicability of weighted distributions in various fields, Gupta and Keating (1986) found it worthwhile to investigate the structural relationships between the distributions of X and Y in the context of reliability. Later, Gupta and Kirmani (1990) gave a detailed account of the survey of literature and extended the study of weighted distributions in the context of reliability and life testing. They considered the reliability measures useful in modeling life length studies such as reliability function, failure rate, and mean residual life function (MRLF) of the rv Y for comparison. The major relationships established by them are

$$S_w(x) = \frac{E(w(X) \mid X > x)}{\mu_w} S(x) \tag{1.2}$$

$$h_w(x) = \frac{w(x)}{E(w(X) \mid X > x)} h(x) \tag{1.3}$$

$$r_w(x) = \frac{r(x)}{w(x) + A(x)} \int_x^b \frac{w(t) + A(t)}{r(t)} \exp\left(-\int_x^t \frac{du}{r(u)}\right) dt \tag{1.4}$$

where $S(x)$, $h(x)$, and $r(x)$ are survival function, failure rate, and MRLF of X, respectively, defined by $S(x) = P(X > x)$, $h(x) = \frac{f(x)}{S(x)}$, and $r(x) = E(X - x \mid X > x) = \frac{1}{S(x)} \int_x^\infty S(t)dt$, and $S_w(x)$, $h_w(x)$, and $r_w(x)$ are the corresponding functions for Y with $A(x) = E(w(X) - w(x) \mid X > x)$, the mean residual weight. When $w(x) = x$ and $a > 0$, the weighted distribution (1.1) then termed as the length biased distribution denoted by $f^*(x)$.

Even if an extensive work has been carried out on weighted distributions using various reliability measures such as failure rate, MRLF, and vitality function in univariate and multivariate set up, very little has been explored in the context of repair time models. Accordingly, in the present article, we obtain some structural relationships between the original and weighted random variables and prove characterization theorems arising out of it for models such as power, exponential, Pareto II, beta, and Pearson system of distributions.

2. Maintainability and Reversed Repair Rate

Maintainability of a system provides a measure of the repairability of the system, when it fails or it is the probability of repairing a failed component/system in a specified period of time (see Rao, 1992). Various probability distributions may be used to present an item's repair time data. Once the repair time distributions are identified, the corresponding maintainability function may be obtained. The maintainability functions are used to predict the probability that a repair, beginning at time $x = 0$, will be accomplished in a time x. Therefore the maintainability function, $F(x)$, for any distribution is given by $F(x) = P(X \leq x)$, where X is the repair time. When X represents the repair time, the usual distribution function is termed as the maintainability function as it gives the probability that required maintenance will be successfully completed in a given time period. Then the reversed repair rate is defined as

$$\lambda(x) = \frac{f(x)}{F(x)} \tag{2.1}$$

where $f(x)$ is the pdf of the repair time and $F(x)$ is the maintainability function, which is absolutely continuous. The interval of support of F being (a, b) with $a = \inf\{x : F(x) > 0\}$ and $b = \sup\{x : F(x) < 1\}$. When X represents the repair time of a component/system, the probability that it is repaired during the time $(x - \varepsilon, x)$ (ε is a small positive number) is approximately equal to $\varepsilon\lambda(x)$, whereas when X represents the lifetime, $\lambda(x)$ then termed as reversed hazard rate (RHR) (see for example, Block et al., 1998; Shaked and Santhikumar, 1994). In survival analysis, RHR has been found to be important for the estimation of survival function in the presence of left censored observations (see Anderson et al., 1993; Kalbfleish and Lawless, 1989). Let

$$m(x) = E(w(X) \mid X \leq x) = \frac{1}{F(x)} \int_a^x w(t)f(t)dt. \tag{2.2}$$

It is well known that $\lambda(x)$ and $m(x)$ uniquely determine $F(x)$ by

$$F(x) = \exp\left(-\int_x^b \lambda(t)dt\right) = \exp\left(-\int_x^b \frac{dm(t)}{w(t) - m(t)}\right). \tag{2.3}$$

See expression (1.2) in Navarro and Ruiz (1996) and Remark 2.2 in Navarro et al. (1998), respectively.

Recently, Nair et al. (2005) proved some characterization theorems using a relationship between RHR and conditional expectation (2.3). Further, Nair and Asha (2004) gave a review of the literature on RHR and developed certain identities connecting the distribution function, density function, and survival function in terms of RHR and the hazard rate.

We now study the structure of the weighted random variable Y in comparison to the original variable X.

$$F_w(x) = P(Y \leq x) = \int_a^x f_w(t)dt = \frac{m(x)}{\mu_w} F(x) \tag{2.4}$$

where $\mu_w = Ew(X) < \infty$. The corresponding reversed repair rate becomes

$$\lambda_w(x) = \frac{f_w(x)}{F_w(x)} = \frac{w(x)}{m(x)} \lambda(x). \tag{2.5}$$

3. Characterizations

In this section we prove some theorems useful to compare the original and weighted rv and also prove characterization theorems for specific models such as power, exponential, Pareto II, beta, and Pearson system of distributions using the relationships between the original and weighted rv.

Theorem 3.1. (a) $X \underset{lr}{\leq} Y(\geq)$ *if, and only if, w is non decreasing (non increasing).*
(b) $X \underset{rfr}{\leq} Y(\geq)$ *if, and only if, m is non decreasing (non increasing).*
(c) $X \underset{st}{\leq} Y(\geq)$ *if, and only if,* $\mu_w \leq m(x)(\geq)$.

Here $\underset{lr}{\leq}$, $\underset{rfr}{\leq}$, and $\underset{st}{\leq}$ denotes the likelihood ratio, the reversed failure rate, and the stochastic orders. It is well known (see p. 402 in Navarro et al., 1997) that

$$X \underset{lr}{\leq} Y \Rightarrow X \underset{rfr}{\leq} Y \Rightarrow X \underset{st}{\leq} Y.$$

Results (a) and (c) were given by Gupta and Kirmani (1990). Result (b) can be obtained immediately from (2.4) and the definition of the reversed hazard rate order.

Theorem 3.2. *If* $\alpha(x) = \frac{F_w(x)}{F(x)}$ *and* $\mu_w = Ew(X)$, *then*

$$F(x) = \exp\left(-\int_x^b \frac{\alpha'(t)}{\frac{w(t)}{\mu_w} - \alpha(t)} dt\right). \tag{3.1}$$

Theorem 3.3. *If* $\beta(x) = \frac{\lambda(x)}{\lambda_w(x)}$, *then*

$$F(x) = \exp\left(-\int_x^b \frac{\beta'(t)w(t) + \beta(t)w'(t)}{(1-\beta(t))w(t)} dt\right). \tag{3.2}$$

The proofs of Theorems 3.2 and 3.3 are similar to that of Theorems 1 and 2 in Navarro et al. (2001).

Theorem 3.4. *Let* $A^*(x) = E(x - X \mid X \leq x)$. *If X is increasing reversed repair rate and* $\frac{A^*(x)}{x}$ *is non decreasing, then Y is also increasing reversed repair rate.*

Proof. X is increasing reversed repair rate implies

$$\lambda(x_1) \leq \lambda(x_2) \quad \text{for all } x_1 \leq x_2,$$

i.e., $\left(\frac{x_1 - A^*(x_1)}{x_1}\right)\lambda^*(x_1) \leq \left(\frac{x_2 - A^*(x_2)}{x_2}\right)\lambda^*(x_2)$ for all $x_1 \leq x_2$

$$\Rightarrow \lambda^*(x_1) \leq \frac{\left(1 - \frac{A^*(x_2)}{x_2}\right)}{\left(1 - \frac{A^*(x_1)}{x_1}\right)}\lambda^*(x_2)$$

$$\Rightarrow \lambda^*(x_1) \leq \lambda^*(x_2) \quad \text{for all } x_1 \leq x_2. \qquad \square$$

Theorem 3.5. *The ratio of the relationship*

$$\frac{F_w(x)}{F(x)} = 1 - x(1 + Cx)\lambda(x) \tag{3.3}$$

if and only if X has Pareto II distribution with

$$F(x) = 1 - (1 + ax)^{-c}; \quad x > 0, \ a, c > 0 \tag{3.4}$$

for $C > 0$, *exponential distribution with*

$$F(x) = 1 - e^{-ax}; \quad x > 0, \ a > 0 \tag{3.5}$$

for $C = 0$, *or beta distribution with*

$$F(x) = 1 - (1 - ax)^c; \quad 0 < x < \frac{1}{a}, \ c > 0 \tag{3.6}$$

for $C < 0$.

The proof can be obtained from Theorem 3 in Ruiz and Navarro (1994).

Further, Table 1 provides some characterization theorems for power and beta distributions based on their functional forms of $\alpha(x)$ and $\beta(x)$.

We now prove a characterization theorem that provide the relationships between reversed repair rate and right truncated moments of the original and weighted rv for the Pearson system of distributions.

Let X be an absolutely continuous rv with pdf $f(x)$, the distribution of X is said to be a member of the Pearson System of distributions if $f(x)$ satisfies the differential equation

$$\frac{f'(x)}{f(x)} = \frac{-(x + a)}{b_0 + b_1 x + b_2 x^2} \tag{3.7}$$

where a, b_0, b_1, and b_2 are real valued. The shape of the distribution depends on the values of the parameters a, b_0, b_1, and b_2.

In a study on Pearson system of distributions, Sankaran and Nair (1993) derived the conditions under which Pearson and Ord families are form-invariant with respect to the length-biased sampling. According to them, the members of the Pearson system satisfying the differential equation (3.7) with $b_2 \neq 1$, have the same type of distributions for Y if and only if $b_0 = 0$. Later, Asadi (1998) extended this result to size-biased sampling of order α (i.e., $w(x) = x^\alpha$).

Theorem 3.6. *Let* X *be a non negative, non degenerate rv with density function* $f(x)$ *and suppose that* $w(x) = x^\alpha$. *Then* X *is a member of Pearson system of distributions of the form* (3.7) *with* $b_0 = 0$ *and* $\lim_{x \to a}(b_1 x + b_2 x^2) f(x) = 0$ *if and only if*

$$\frac{\lambda_w(x)}{\lambda(x)} = K \frac{(\mu_\alpha - m_\alpha(x))}{(\mu^* - m^*(x))} \tag{3.8}$$

where $K = \frac{1-(\alpha+2)b_2}{1-2b_2}$, $1 - 2b_2 \neq 0$, $\mu_\alpha = E(X^\alpha)$, *and* $m_\alpha(x) = E(X^\alpha \mid X \leq x)$.

Table 1
Different functional forms of $\alpha(x)$ and $\beta(x)$

$w(x) = x$	$F(x)$	$\alpha(x)$	$\beta(x)$
power	$\left(\frac{x}{p}\right)^d$	Cx	C
beta	$(px - q)^d$	$A + Bx$	$\frac{Ax+B}{x}$

The proof of Theorem 3.6 is similar to the analogous reversed result given in Asadi (1998).

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