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ON THE MEDIAN AND THE ANTIMEDIAN OF A COGRAPH

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Abstract: In this paper, the median and the antimedian of cographs are discussed. It is shown that if G_1 and G_2 are any two cographs, then there is a cograph that is both Eulerian and Hamiltonian having G_1 as its median and G_2 as its antimedian. Moreover, the connected planar and outer planar cographs are characterized and the median and antimedian graphs of connected, planar cographs are listed.

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1. Introduction

Cographs-complement reducible graphs are graphs that can be reduced to edgeless graphs by taking complements within components. These graphs were systematically studied by Corneil, Lerchs and Stewart-Burlingham [3] and are recursively defined as follows:

1. K_1 is a cograph;

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2. If G is a cograph, so is its complement $\overline{G}$; and

3. If G and H are cographs, so is their disjoint union, $G \cup H$.

The *join (sum)* of two graphs G and H, denoted by $G+H$ is defined as the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{u, v\}$ where $u \in V(G)$ and $v \in V(H)$. Then $G + H = \overline{(\overline{G} \cup \overline{H})}$. Hence the condition (3) in the definition of cographs can be replaced by:

(3$'$) If G and H are cographs, so is their join.

Among the many characterizations for a graph G to be a cograph, the following [8] are used in this paper.

G is P_4-free (that is, G has no induced subgraph isomorphic to P_4, the path on four vertices).

For every induced subgraph H of G, H or $\overline{H}$ is disconnected.

Every connected induced subgraph of G has diameter at most 2.

Complete bipartite graphs and multipartite graphs are examples of cographs. Cographs are extensively studied in [2, 4, 5, 7, 9].

Royle [11] has proved that the rank of a cograph is equal to the number of distinct non zero rows of its adjacency matrix. Larrion et al [7] have studied in detail the clique operator on cographs and proved that a cograph is clique convergent if and only if it is clique Helly.

The median of a graph was first studied in [6], where it has been proved that the median graphs of trees are K_1 or K_2 (see also [10]). Slater [12] proved that every connected finite graph is the median graph of some connected graph, and obtained the median graph of 2-trees. Yushmanov [14] showed that the median of a Ptolemaic graph is complete.

Bielak and Syslo [1] introduced the notion of antimedian graphs and proved that every graph is the antimedian graph of some graph.

2. Median of a Cograph

Definition 1. (see [10]) Let $G = (V, E)$ be a connected graph with the metric $d(u, v)$, the length of a $u - v$ geodesic. If $D(v) = \sum\{d(u, v), u \in V\}$, then the set of vertices v such that $D(v)$ is minimum are called median vertices of G. The subgraph $M(G)$ of G induced by median vertices is called the median graph of G. A graph G is a median graph if there exists a connected graph H such that $M(H) = G$. A graph G is self median if $M(G) = G$.

Theorem 1. *The median graph of a connected cograph is the subgraph induced by the vertices of maximum degree in G.*

Proof. Since G is a connected cograph, $d(v,u) \leq 2$ for any pair of vertices u, v of G. Let the degree of v in G be d. Then, these d vertices are at a distance 1 from v. So, there are $p-1-d$ vertices u in G such that $d(v,u) = 2$ and $D(v) = d + 2(p-1-d) = 2(p-1) - d$. Hence the vertices in G such that $D(v)$ is minimum are those for which the degree is maximum. □

Note. The median graph of a cograph is also a cograph. A cograph is self median if and only if it is regular.

Theorem 2. *If M_1 and M_2 are the median graphs of connected cographs G_1 and G_2 respectively, then $M_1 \cup M_2$ is also the median graph of a connected cograph.*

Proof. Let us assume that $\Delta(G_1) \geq \Delta(G_2)$, say $\Delta(G_1) = \Delta(G_2) + \alpha$. Let $p_1 = |V(G_1)|$ and $p_2 = |V(G_2)|$ and let θ_1 and θ_2 be any non-negative integers such that

$$\alpha + p_1 + \theta_1 = p_2 + \theta_2 . \tag{A}$$

Let $t_1 = p_1 + 1 + \theta_1$ and $t_2 = p_2 + 1 + \theta_2$. Define $F = G_1 + \overline{K_{t_1}}$ and $H = G_2 + \overline{K_{t_2}}$. Then $\Delta(F) = \Delta(G_1) + t_1 = \Delta(G_2) + \alpha + t_1$ and $\Delta(H) = \Delta(G_2) + t_2$. By (A), $\Delta(F) = \Delta(H)$. Now, let $t_3 \geq |V(F)| + |V(H)|$ and $G = (F \cup H) + \overline{K}_{t_3}$. Then, it follows that $M_1 \cup M_2$ is the median of G. □

Lemma 1. *If there is an Eulerian cograph G of order p such that $M(G) = H$ then there exists an Eulerian cograph G^* of even order such that $M(G^*) = H$.*

Proof. It suffices to prove the result when p is odd. Form, $G^* = (G \cup K_1) + \overline{K}_{p+1}$. Then $|G^*| = 2(p+1)$, the degree of every vertex in G^* is even and $M(G^*) = H$. □

Theorem 3. *If M_1 and M_2 are the median graphs of Eulerian, connected cographs G_1 and G_2 respectively, then $M_1 \cup M_2$ is also the median graph of an Eulerian connected cograph.*

Proof. By Lemma 1, we can assume that $p_1 = |G_1|$ and $p_2 = |G_2|$ are even. The proof is along similar lines as of Theorem 2. Then, the α in the proof is also even. Also, θ_1, θ_2 can be chosen to be odd satisfying (A) and $\Delta(G_1) + t_1 = \Delta(G_2) + t_2$. Hence t_1 and t_2 are even. As $t_3 \geq |V(F \cup H)|$, it can also be chosen as even. Thus the degree of all the vertices is even and hence the graph constructed is Eulerian. □

3. The Main Theorem

Theorem 4. *Every cograph G is the median graph of some connected cograph.*

Proof. The proof is by induction on the number of vertices p of G. The result is true for all cographs with atmost three vertices. Assume that the result is true for all cographs with less than p vertices and G be a cograph with p vertices, $p \geq 4$.

Case I. G is connected. Since G is a cograph, $\overline{G}$ is disconnected. Let $C_1, C_2, \ldots, C_t$ be the components of $\overline{G}$. Then $G = G_1 + G_2$, where G_1 is $\overline{C}_1$ and $G_2 = \overline{C}_2 + \overline{C}_3 + \cdots + \overline{C}_t$. Now, both G_1 and G_2 have less than p vertices, and so by induction hypothesis, they are the median graphs of some connected cographs H_1, H_2. Choose numbers x_1, x_2 such that $\Delta(H_1) + x_1 = \Delta(H_2) + x_2$.

Now, consider the graph $H = A_1 + A_2$, where $A_1 = [(H_1 + \overline{K}_{x_1+\theta}) \cup \overline{K}_{q_1}]$ and $A_2 = [(H_2 + \overline{K}_{x_2+\theta}) \cup \overline{K}_{q_2}]$, where $\theta \geq \max(p_1, p_2)$, where $p_i = |V(H_i)|$ for $i = 1, 2$ and q_1, q_2 are chosen so that each A_i has the same number of vertices. Then, H is the required connected cograph.

Case II. G is disconnected. Let $C_1, C_2, \ldots, C_t$ be the components of G. Then each is a cograph with less than n vertices. So, by induction hypothesis, each C_i is the median graph of a connected cograph. Then by Theorem 2, $C_1 \cup C_2 \cup \cdots \cup C_t$ is the median graph of a cograph. □

Theorem 5. *Every cograph G is the median graph of some connected, Eulerian cograph.*

Proof. The proof is same as that of Theorem 4, where in x_1, θ, q_1 can be chosen to be even. Hence the degree of each vertex in H is even. □

4. Antimedian of a Cograph

Definition 2. (see [10]) The antimedian graph $AM(G)$ of a graph G is the subgraph induced by the vertices such that $D(v)$ is maximum.

By the proof of Theorem 1, it follows that for a cograph G, $AM(G)$ is the subgraph induced by the vertices of minimum degree.

Theorem 6. *Any cograph H is the antimedian graph of a connected Eulerian cograph G of even order.*

Proof. Since, H is a cograph, $\overline{H}$ is also a cograph. Then, by Theorem 5, $\overline{H}$ is the median graph of some connected Eulerian cograph G_1 of even order. Hence, the vertices of $\overline{H}$ are the maximum degree vertices of G_1. Consider $\overline{G}_1$, then the vertices of H will be of minimum degree. The graph $\overline{G_1}$ will be disconnected. So, form the graph $G = \overline{G}_1 + K_1$ which will be a connected Eulerian cograph and $AM(G) = H$. □

5. The Median and the Antimedian of a Cograph

In this section we answer the following

Problem. Given two cographs G_1 and G_2, does there exist a connected cograph G such that $M(G) = G_1$ and $AM(G) = G_2$?

First, note that the regular connected cographs G have the property that $M(G) = AM(G) = G$. The next result shows that the number of such cographs for a fixed regularity is finite.

Lemma 2. *The number of vertices in a connected r-regular cograph is at most $2r$.*

Proof. Suppose this number exceeds $2r$. Since, G is connected, $\overline{G}$ is disconnected. So, $G = G_1 + G_2$, where G_1 and G_2 are also cographs and $V(G) = V(G_1) \cup V(G_2)$. Then, at least one of G_1 or G_2 has more than r vertices, which contradicts the fact that G is r-regular. □

Note. All connected r-regular cographs G on p vertices can be described as follows. By Lemma 2, $p \leq 2r$ and $G = G_1 + G_2$, where G_1 and G_2 are regular cographs, which are not necessarily connected, with regularity r_1, r_2 respectively, such that $r_1 + p_2 = r_2 + p_1 = r$ and $r_1p_1 \equiv r_2p_2 \equiv 0 (\mathrm{mod}\ 2)$.

Example.

r	G
1	K_2
2	$K_3,\ C_4$
3	$K_{3,3},\ K_4$
4	$K_{4,4},\ K_3 + 2K_2,\ \overline{K_2} + C_4$
5	$K_{5,5},\ 2K_2 + 2K_2,\ K_6$

Theorem 7. *Let G_1 and G_2 be two cographs. Then there is a Hamiltonian, Eulerian cograph G such that $M(G) = G_1$ and $AM(G) = G_2$.*

Proof. Let F_1 and F_2 be such that $M(F_1) = G_1, AM(F_2) = G_2, |F_1| =$

$p_1, |F_2| = p_2$ and let $H_1 = H_2 = K_{p_1p_2+1}$. Construct the graph $G = ([F_1 + 2H_1] \cup F_2) + 2H_2$, which is a cograph. By Lemma 1 and Theorem 6, F_1 and F_2 can be chosen to be Eulerian graphs of even order. Then G is the required graph which is Hamiltonian and Eulerian. □

6. Outer Planar and Planar Cographs

In this section, we characterize outer planar connected cographs and planar connected cographs. Using these results, we list all medians graphs and antimedian graphs of connected planar cographs. We say that a vertex is a universal vertex if its degree is $p - 1$. The following two theorems are well known [13].

Theorem 8. *A graph G is outer planar if and only if it has no subgraph that is a subdivision of K_4 or $K_{2,3}$.*

Theorem 9. *A graph G is planar if and only if it has no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Theorem 10. *If $G \neq C_4$ is a connected, outer planar cograph with p vertices then G has a universal vertex u. Moreover, $G - u$ is a forest, where each component is a path of length at most two.*

Proof. Since G is a cograph, $G = G_1 + G_2 + \cdots + G_t$ and $t \geq 2$. If $t \geq 4$, then G will contain a K_4. Therefore, $t = 2$ or 3.

Case 1. $t = 2$. Let $|V(G_1)| = p_1$ and $|V(G_2)| = p_2$. If $p_1 = 1$ then G has a universal vertex. If $p_1 \geq 2$ then $p_2 < 3$. Otherwise G will contain a supergraph of $K_{2,3}$. If $p_2 = 1$ then G has a universal vertex. If $p_2 = 2$, by a similar argument it follows that $p_1 = 2$. In that case $G = C_4, K_4 - \{e\}$ or K_4.

Case 2. $t = 3$. Let $|V(G_i)| = p_i$, for $i = 1, 2, 3$. If $p_1 = 1$ then G has a universal vertex. If $p_1 \geq 3$, then G will contain a supergraph of $K_{2,3}$. If $p_1 = 2$ then at least one of p_2 or p_3 must be one. Otherwise G will contain a supergraph of $K_{2,3}$.

Thus, in any case G has a universal vertex.

Let u be the universal vertex of G. If $G - \{u\}$ contains a cycle, then G will contain a wheel. So, $G - \{u\}$ is a forest. If $G - \{u\}$ contains a vertex of degree 3, then G will contain a supergraph of $K_{2,3}$. Therefore, each component of $G - \{u\}$ is a path and since G is a cograph, its length cannot exceed two. □

Theorem 11. *If G is a connected planar cograph, then G has a universal vertex u and $G - u$ is an outer planar cograph or G has two vertices u, v such*

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that both are joined to all other vertices. In the later case, the components of $G - \{u, v\}$ are singletons, edges, paths on 3 vertices, 3-cycles or 4- cycles. If $G - \{u, v\}$ has a 3-cycle or a 4-cycle then G is precisely $K_3 + \{u, v\}$ or $C_4 + \{u, v\}$ respectively.

Proof. Let $G = G_1 + G_2 + \cdots + G_t$ and $|V(G_i)| = p_i$ for $i = 1, 2, \ldots, t$. Also, let $p_1 \leq p_2 \leq \cdots \leq p_t$. Since G is a cograph, $t \geq 2$. If $t \geq 5$, then G will contain a K_5. Therefore $t = 2, 3$ or 4.

Case 1. $t = 2$. If $p_1 \geq 3$, then G will contain a supergraph of $K_{3,3}$. If $p_1 = 2$ then the two vertices, say u, v of G_1 are adjacent to all other vertices in G. Note that $G - \{u, v\}$ is G_2. If a vertex of $w \in V(G_2)$ has degree greater than or equal to three, then G will contain a supergraph of $K_{3,3}$. Therefore, each component of G_2 is either a path or a cycle. But, G_2 cannot have P_4 as an induced subgraph and hence the components of G_2 are singletons, edges, P_3, K_3 or C_4. If G_2 has K_3 or C_4 as one component, then there cannot be another vertex adjacent to both u and v. Otherwise G will contain a subdivision of K_5. Then G is $K_3 + \{u, v\}$ or $C_4 + \{u, v\}$ respectively.

If $p_1 = 1$ then G has a universal vertex.

Case 2. $t = 3$ or 4. If $p_1 \geq 2$, then G will contain a supergraph of $K_{3,3}$. If $p_1 = 1$ then G has a universal vertex.

If u is a universal vertex of G, then $G - \{u\}$ is outer planar. Otherwise, $G - \{u\}$ has a subdivision of K_4 or $K_{2,3}$ and then G will contain a subdivision of K_5 or $K_{3,3}$. □

Corollary 1. *The median graph of a planar, connected cograph is one of the following graphs $K_1, K_2, K_3, K_4, \overline{K}_2, K_4 - e, C_4, K_{1,2}, C_4 + \overline{K}_2$.*

Corollary 2. *The antimedian graph of a planar, connected cograph is one of the following graphs $K_1, K_2, K_3, K_4, \overline{K}_n, nK_2, C_4, C_4 + \overline{K}_2$.*

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