Equienergetic self-complementary graphs

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Abstract

In this paper equienergetic self-complementary graphs on p vertices for every $p = 4k,\ k \geq 2$ and $p = 24t + 1,\ t \geq 3$ are constructed.

1 Introduction

Let G be a graph with $|V(G)| = p$ and let A be an adjacency matrix of G. The eigenvalues of A are called the eigenvalues of G and form the spectrum of G denoted by $spec(G)$ [4]. The energy [3] of G, $E(G)$ is the sum of the absolute values of its eigenvalues. The properties of $E(G)$ are discussed in detail in [7, 8, 9]. Two non-isomorphic graphs with identical spectrum are called cospectral and two non-cospectral graphs with the same energy are called equienergetic. In [2] and [5], a pair of equienergetic graphs on p vertices where $p \equiv 0(mod\ 4)$ and $p \equiv 0(mod\ 5)$ are constructed respectively. In [10] we have extended the same for $p = 6,\ 14,\ 18$ and for every $p \geq 20$. In [12] two classes of equienergetic regular graphs have been obtained and in [11], the energies of some non-regular graphs are studied .

In this paper, we provide a construction of equienergetic self-complementary graphs for every $p = 4k,\ k \geq 2$ and $p = 24t+1,\ t \geq 3$. The energies of some special classes of self-complementary graphs are also discussed.

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All graph theoretic terminologies are from [1, 4].

We use the following lemmas in this paper.

Lemma 1. *[4] Let G be a graph with an adjacency matrix A and spec$(G) = \{\lambda_1, \lambda_2, \dots, \lambda_p\}$. Then* $\det A = \prod_{i=1}^{p} \lambda_i$. *Also for any polynomial $P(x)$, $P(\lambda)$ is an eigenvalue of $P(A)$ and hence* $\det P(A) = \prod_{i=1}^{p} P(\lambda_i)$.

Lemma 2. *[4] Let M, N, P and Q be matrices with M invertible. Let $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$. Then $|S| = |M|\,|Q - PM^{-1}N|$ and if M and P commutes then $|S| = |MQ - PN|$ where the symbol $|.|$ denotes determinant.*

Lemma 3. *[12] Let G be an $r-$ regular connected graph, $r \geq 3$ with spec$(G) = \{r, \lambda_2, \dots, \lambda_p\}$. Then* spec$(L^2(G)) = \begin{pmatrix} 4r-6 & \lambda_2 + 3r - 6 & .. & \lambda_p + 3r - 6 & 2r-6 & -2 \\ 1 & 1 & .. & 1 & \frac{p(r-2)}{2} & \frac{pr(r-2)}{2} \end{pmatrix}$,
$E(L^2(G)) = 2pr(r-2)$ and $E(\overline{L^2(G)}) = (pr-4)(2r-3) - 2$.

Lemma 4. *[4] Let G be an $r-$ regular connected graph on p vertices with A as an adjacency matrix and $r = \lambda_1, \lambda_2, \dots, \lambda_m$ as the distinct eigenvalues. Then there exists a polynomial $P(x)$ such that $P(A) = J$ where J is the all one square matrix of order p and $P(x)$ is given by $P(x) = p \times \frac{(x-\lambda_2)(x-\lambda_3)\dots(x-\lambda_m)}{(r-\lambda_2)(r-\lambda_3)\dots(r-\lambda_m)}$, so that $P(r) = p$ and $P(\lambda_i) = 0$, for all $\lambda_i \neq r$.*

Let G be an $r-$ regular connected graph. Then the following constructions [6]result in self-complementary graphs H_i, $i = 1$ to 4.

Construction 1. *H_1 : Replace each of the end vertices of P_4, the path on 4 vertices by a copy of G and each of the internal vertices by a copy of $\overline{G}$. Join the vertices of these graphs by all possible edges whenever the corresponding vertices of P_4 are adjacent.*

Construction 2. *H_2 : Replace each of the end vertices of P_4, the path on 4 vertices by a copy of $\overline{G}$ and each of the internal vertices by a copy of G. Join the vertices of these graphs by all possible edges whenever the corresponding vertices of P_4 are adjacent.*

Construction 3. H_3 : *Replace each of the end vertices of the non-regular self-complementary graph F on 5 vertices by a copy of $\overline{G}$, each of the vertices of degree 3 by a copy of G and the vertex of degree 2 by K_1. Join the vertices of these graphs by all possible edges whenever the corresponding vertices of F are adjacent.*

Construction 4. H_4 : *Consider the regular self-complementary graph $C_5 = v_1v_2v_3v_4v_5v_1$, the cycle on 5 vertices. Replace the vertices v_1 and v_5 by a copy of $\overline{G}$, v_2 and v_4 by a copy of G and v_3 by K_1. Join the vertices of these graphs by all possible edges whenever the corresponding vertices of C_5 are adjacent.*

Note:-For all non self-complementary graphs G, Constructions 1 and 2 yield non- isomorphic graphs and for any graph G, $H_1(G) = H_2(\overline{G})$.

2 Equienergetic self-complementary graphs

In this section, we construct a pair of equienergetic self complementary graphs, first for $p = 4k,\ k \geq 2$ and then for $p = 24t + 1,\ t \geq 3$.

Theorem 1. *Let G be an $r-$ regular connected graph on p vertices with $spec(G) = \{r, \lambda_2, \ldots, \lambda_p\}$ and H_1 be the self-complementary graph obtained by Construction 1.Then*

$$E(H_1) = 2\left[E(G) + E(\overline{G}) - (p-1)\right] + \sqrt{(2p-1)^2 + 4\left\{(p-r)^2 + r\right\}} + \sqrt{1 + 4\left(p^2 + r + r^2\right)}\ .$$

Proof. Let G be an $r-$ regular connected graph on p vertices with an adjacency matrix A, $spec(G) = \{r, \lambda_2, \ldots, \lambda_p\}$ and H_1 be the self-complementary graph obtained by Construction 1. Then the adjacency matrix of H_1 is $\begin{bmatrix} A & J & 0 & 0 \\ J & \overline{A} & J & 0 \\ 0 & J & \overline{A} & J \\ 0 & 0 & J & A \end{bmatrix}$, so that the characteristic equation of H_1 is

$$\begin{vmatrix} \lambda I - A & -J & 0 & 0 \\ -J & \lambda I - \overline{A} & -J & 0 \\ 0 & -J & \lambda I - \overline{A} & -J \\ 0 & 0 & -J & \lambda I - A \end{vmatrix} = 0.$$

that is $\begin{vmatrix} -J & \lambda I - \overline{A} & 0 & -J \\ \lambda I - \overline{A} & -J & -J & 0 \\ -J & 0 & \lambda I - A & 0 \\ 0 & -J & 0 & \lambda I - A \end{vmatrix} = 0$, by a sequence of elementary transformations.

But, the last expression by virtue of Lemma 2 is

$$\left| J^2(\lambda I - A)^2 - \left[(\lambda I - A)\left(\lambda I - \overline{A}\right) - J^2\right]^2 \right| = 0$$

and so $\prod_{i=1}^{p} \left\{ \langle P(\lambda_i)\rangle^2 (\lambda - \lambda_i)^2 - \left[(\lambda - \lambda_i)(\lambda - P(\lambda_i) + 1 + \lambda_i) - \langle P(\lambda_i)\rangle^2\right]^2 \right\} = 0$ by Lemmas 1 and 4.

Now, corresponding to the eigenvalue r of G, the eigenvalues of H_1 are given by

$$\left\{ p^2 (\lambda - r)^2 - \left[(\lambda - r)(\lambda - p + 1 + r) - p^2\right]^2 \right\} = 0 \text{ by Lemmas 1 and 4.}$$

That is $\left[\lambda^2 + \lambda - (r^2 + r + p^2)\right]\left[\lambda^2 - (2p-1)\lambda - \left\{(p-r)^2 + r\right\}\right] = 0$

So $\lambda = \dfrac{-1 \pm \sqrt{1 + 4(p^2 + r + r^2)}}{2}; \dfrac{2p - 1 \pm \sqrt{(2p-1)^2 + 4\left\{(p-r)^2 + r\right\}}}{2}$

The remaining eigenvalues of H_1 satisfy $\prod_{i=2}^{p} \left[(\lambda - \lambda_i)(\lambda + 1 + \lambda_i)\right]^2 = 0.$

Hence , $spec(H_1) = \begin{pmatrix} \frac{-1 \pm \sqrt{1+4(p^2+r+r^2)}}{2} & \frac{2p-1 \pm \sqrt{(2p-1)^2 + 4\left\{(p-r)^2 + r\right\}}}{2} & \underset{i=2 \text{ to } p}{\lambda_i} & \underset{i=2 \text{ to } p}{-1-\lambda_i} \\ 1 & 1 & 2 & 2 \end{pmatrix}.$

Now, the expression for $E(H_1)$ follows. □

Theorem 2. *Let G be an $r-$ regular connected graph on p vertices with spec$(G) = \{r, \lambda_2, \ldots, \lambda_p\}$ and H_2 be the self-complementary graph obtained by Construction 2. Then* $E(H_2) = 2\left[E(G) + E(\overline{G}) - (p-1)\right] + \sqrt{(2p-1)^2 + 4\left\{(p-r)^2 + r\right\}} + \sqrt{1 + 4\left(p^2 + r + r^2\right)}$.

Proof. Let A be an adjacency matrix of G. Then the adjacency matrix of H_2 is $\begin{bmatrix} \overline{A} & J & 0 & 0 \\ J & A & J & 0 \\ 0 & J & A & J \\ 0 & 0 & J & \overline{A} \end{bmatrix}$.
By a similar computation as in Theorem 1 in which A is replaced by $\overline{A}$, we get the characteristic equation of H_2 as
$\prod_{i=1}^{p} \left\{ \langle P(\lambda_i) \rangle^2 (\lambda - P(\lambda_i) + \lambda_i + 1)^2 - \left[(\lambda - \lambda_i)(\lambda - P(\lambda_i) + 1 + \lambda_i) - \langle P(\lambda_i) \rangle^2 \right]^2 \right\} = 0$, by Lemmas 1, 2 and 4.

Hence $spec(H_2) = \begin{pmatrix} \frac{2p-1 \pm \sqrt{1+4(p^2+r+r^2)}}{2} & \frac{-1 \pm \sqrt{(2p-1)^2 + 4\left\{(p-r)^2 + r\right\}}}{2} & \underset{i=2 \text{ to } p}{\lambda_i} & \underset{i=2 \text{ to } p}{-1 - \lambda_i} \\ 1 & 1 & 2 & 2 \end{pmatrix}$.
Now, the expression for $E(H_2)$ follows. □

Corollory 1.

1. *If $G = K_p$, then $E(H_1) = E(H_2) = 2(p-1) + \sqrt{1 + 4p^2} + \sqrt{8p^2 - 4p + 1}$.*
2. *If $G = K_{n,n}$, then $p = 2n$ and $E(H_1) = E(H_2) = 2(2p-3) + \sqrt{5p^2 - 2p + 1} + \sqrt{5p^2 + 2p + 1}$.*

Theorem 3. *For every $p = 4k$, $k \geq 2$, there exists a pair of equienergetic self-complementary graphs.*

Proof. Let H_1 and H_2 be the self-complementary graphs obtained from K_k as in Constructions 1 and 2. Then by Theorems 1 and 2, they are equienergetic on $p = 4k$ vertices. □

Theorem 4. *Let H_3 be the self-complementary graph obtained from K_p by Construction 3. Then $E(H_3) = 2(p-1) + \sqrt{4p^2 + 1} + \sqrt{8p^2 + 4p + 1}$.*

Proof. Let A be an adjacency matrix of K_p. Then by Construction 3, the adjacency matrix of

H_3 is $\begin{bmatrix} \overline{A} & J & 0_{p\times 1} & 0 & 0 \\ J & A & J_{p\times 1} & J & 0 \\ 0_{1\times p} & J_{1\times p} & 0 & J_{1\times p} & 0 \\ 0 & J & J_{p\times 1} & A & J \\ 0 & 0 & 0 & J & \overline{A} \end{bmatrix}$.

Now, after a sequence of elementary transformations applied to the rows and columns and by Lemma 2, the characteristic equation is

$$\frac{1}{\lambda^{2p-1}}\left|\left[\{\lambda(\lambda I - A) - J\}(\lambda I - \overline{A}) - \lambda J^2\right]^2 - \left[(\lambda+1)(\lambda I - \overline{A})J\right]^2\right| = 0.$$

Since $G = K_p$ is connected and regular, by Lemmas 1 and 4 the characteristic equation of H_3 is

$$\lambda^{2p-1}(\lambda+1)^{2p-2}(\lambda^2+\lambda-p^2)\left[\lambda^2-(2p-1)\lambda-p(p+2)\right] = 0.$$

Hence $spec(H_3) = \begin{pmatrix} \frac{-1\pm\sqrt{4p^2+1}}{2} & \frac{2p-1\pm\sqrt{8p^2+4p+1}}{2} & -1 & 0 \\ 1 & 1 & 2p-2 & 2p-2 \end{pmatrix}$. Now, the expression for $E(H_3)$ follows. □

Theorem 5. *Let H_4 be the self-complementary graph obtained from K_p by Construction 4. Then $E(H_4) = 2\,(2p-1) + \sqrt{4p+1} + \sqrt{8p^2-4p+1}$.*

Proof. Let A be an adjacency matrix of K_p. Then by Construction 4, the adjacency matrix of H_4 is

$$\begin{bmatrix} \overline{A} & J & 0_{p\times 1} & 0 & J \\ J & A & J_{p\times 1} & 0 & 0 \\ 0_{1\times p} & J_{1\times p} & 0_{1\times 1} & J_{1\times p} & 0 \\ 0 & 0 & J_{p\times 1} & A & J \\ J & 0 & 0 & J & \overline{A} \end{bmatrix}$$

Now, after a sequence of elementary transformations applied to the rows and columns and by Lemma 2, the characteristic equation is

$$\frac{1}{\lambda^{2p-1}}\left|\left[\{\lambda(\lambda I - A) - J\}^2 + (\lambda-1)J^2\right]\left[(\lambda-1)J^2 + \left(\lambda I - \overline{A}\right)^2\right] - \lambda J^2\left[\lambda(\lambda I - A) - J + \lambda I - \overline{A}\right]^2\right| = 0$$

Since $G = K_p$ is connected and regular, by Lemma 4 the characteristic equation of H_4 is

$$\lambda^{(2p-2)} (\lambda+1)^{(2p-2)} (\lambda-2p) \left(\lambda^2+\lambda-p\right) \left(\lambda^2+\lambda-2p^2+p\right) = 0.$$

Hence $spec(H_4) = \begin{pmatrix} 2p & \frac{-1\pm\sqrt{4p+1}}{2} & \frac{2p-1\pm\sqrt{8p^2-4p+1}}{2} & -1 & 0 \\ 1 & 1 & 1 & 2p-2 & 2p-2 \end{pmatrix}$. Now, the expression for $E(H_4)$ follows.

□

Corollary 2. *Let G be a connected $r-$ regular graph on p vertices with $spec(G) = \{r, \lambda_2, \lambda_3, \ldots, \lambda_p\}$ and H be the self-complementary graph obtained as in Construction 4. Then* $E(H) = 2\left[E(G) + E(\overline{G}) - (p-1)\right] + \sqrt{1+4\left(p^2+r+r^2\right)} + T$ *where T is the sum of absolute values of roots of the cubic* $x^3 - (2p-1)\,x^2 - \left[p^2 - 2p\,(r-1) + r\,(r+1)\right] x + 2p\,(2p-r-1) = 0.$

Lemma 5. *There exists a pair of non-cospectral cubic graphs on $2t$ vertices, for every $t \geq 3$.*

Proof. Let G_1 and G_2 be the non-cospectral cubic graphs on six vertices labelled as $\{v_j\}$ and $\{u_j\}$, $j = 1$ to 6 respectively.

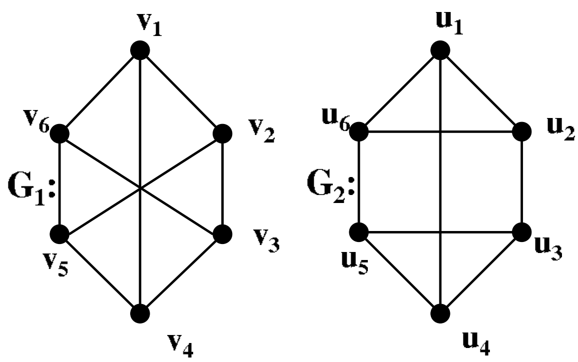

Figure 1: The graphs G_1 and G_2.

Now replacing v_1 and u_1 in G_1 and G_2 by a triangle each we get two cubic graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ on eight vertices containing one and two triangles respectively as shown in Figure 2. Since the number of triangles in a graph is the negative of half the coefficient of λ^{p-3} in its characteristic polynomial [4], $\mathcal{H}_1$ and $\mathcal{H}_2$ are non-cospectral.

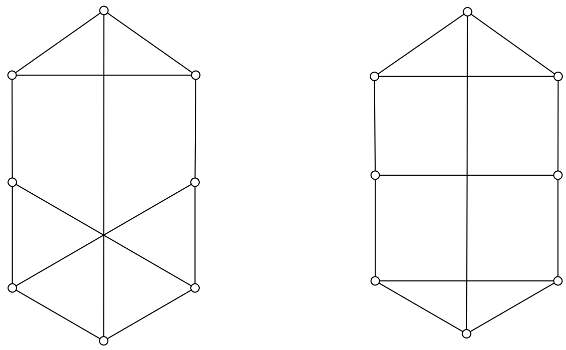

Figure 2: The graphs $\mathcal{H}_1$ and $\mathcal{H}_2$

Replacing any vertex in the newly formed triangle in $\mathcal{H}_1$ and $\mathcal{H}_2$ by a triangle we get two cubic graphs on ten vertices which are non co-spectral. Repeating this process $(t-3)$ times, we get two cubic graphs on $2t$ vertices containing one and two triangles respectively. Hence they are non cospectral. □

Theorem 6. *For every $p = 24t+1$, $t \geq 3$, there exists a pair of equienergetic self-complementary graphs.*

Proof. Let G_1 and G_2 be the two non co-spectral cubic graphs on $2t$ vertices given by Lemma 5. Let F_1 and F_2 respectively denote their second iterated line graphs. Then F_1 and F_2 have $6t$ vertices each and $6-$regular with $E(F_1) = E(F_2) = 12t$ and $E(\overline{F_1}) = E(\overline{F_2}) = 3(6t-4)-2$ by Lemma 3. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the self-complementary graphs obtained from F_1 and F_2 by Construction 4. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are on $p = 24t+1$ vertices and by Corollary 2, $E(\mathcal{F}_1) = E(\mathcal{F}_2) = 2(24t-13) + \sqrt{169+144t^2} + T$ where T is the sum of the absolute values of the roots of the cubic $x^3 - (12t-1)x^2 - 6(6t^2 - 10t + 7)x + 12t(12t-7) = 0$. □

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References

[1] R. Balakrishnan, *A Text Book of Graph Theory*, Springer (2000), zbl 938.05001.

[2] R. Balakrishnan, *The energy of a graph*, Linear Algebra Appl., **387** (2004), 287–295, zbl 1041.05046.

[3] C.A. Coulson, Proc. Cambridge Phil. Soc., **36** (1940), 201–203.

[4] D.M.Cvetkovic, M. Doob, H. Sachs, *Spectra of Graphs-Theory and Applications*, Academic Press, (1980), zbl 458.05042.

[5] D. Stevanović, *Energy and NEPS of graphs*, Linear Multilinear Algebra, **53** (2005), 67–74, zbl 1061.05060.

[6] A. Farrugia, *Self-complementary graphs and generalisations*:A comprehensive reference manual, M.Sc Thesis, University of Malta(1999).

[7] I. Gutman, *The energy of a graph*, Ber. Math. Statist. Sekt. Forschungszenturm Graz, **103** (1978), 1–22, zbl. 402.05040.

[8] I. Gutman, *The energy of a graph: old and new results*, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), Algebraic Combinatorics and Applications, Springer,(2000), 196-211, zbl.974.05054.

[9] I. Gutman, *Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology*, J. Serb. Chem. Soc., **70** (2005), 441-456.

[10] G. Indulal, A. Vijayakumar, *On a pair of equienergetic graphs*, MATCH Commun. Math. Comput. Chem., **55**(2006), 83 - 90, zbl 1106.05061.

[11] G.Indulal, A. Vijayakumar, *Energies of some non-regular graphs*, J.Math.Chem.(to appear).

[12] H.S.Ramane, I.Gutman, H.B. Walikar, S.B. Halkarni,*Another class of equienergetic Graphs*, Kragujevac.J.Math., **26**(2004), 15-18, zbl 1079.05057.