A Note On Some Domination Parameters in Graph Products

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Abstract

In this paper, we study the domination number, the global domination number, the cographic domination number, the global cographic domination number and the independent domination number of all the graph products which are non-complete extended p-sums (NEPS) of two graphs.

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1 Introduction

We consider only finite, simple graphs $G = (V, E)$ with $|V| = n$ and $|E| = m$.

A set $S \subseteq V$ of vertices in a graph G is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A dominating set S is a minimal dominating set if no proper subset of S is a dominating set. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G [4]. A dominating set S is global dominating if S dominates both G and G^c. The global domination number $\gamma_g(G)$ of a graph G is the minimum cardinality of a global dominating set in G [10].

A graph which does not have P_4 - the path on four vertices, as an induced subgraph is called a cograph. A set $S \subseteq V$ is called a cographic dominating set if S dominates G and the subgraph induced by S is a cograph [9]. The minimum cardinality of a cographic dominating set is called the cographic domination number, $\gamma_{cd}(G)$. A set $S \subseteq V$ is called a global cographic dominating set if it dominates both G and G^c and the subgraph induced by S is a cograph. The minimum cardinality of a global cographic dominating set is called the global cographic domination number, $\gamma_{gcd}(G)$ [9]. A set $S \subseteq V$ is independent if no two vertices of S are adjacent in G. A set $S \subseteq V$ is called an independent dominating set if S is an independent set which dominates G. The minimum cardinality of an independent dominating set is called the independent domination number, $\gamma_i(G)$ [4].

A graphical invariant σ is supermultiplicative with respect to a graph product $\times$, if given any two graphs G and H $\sigma(G \times H) \geq \sigma(G)\sigma(H)$ and submultiplicative if $\sigma(G \times H) \leq \sigma(G)\sigma(H)$. A class $\mathcal{C}$ is called a universal multiplicative class for σ on $\times$ if for every graph H, $\sigma(G \times H) = \sigma(G)\sigma(H)$ whenever $G \in \mathcal{C}$ [8].

Let $\mathcal{B}$ be a non-empty subset of the collection of all binary n-tuples which does not include $(0, 0, ..., 0)$. The non-complete extended p-sum (NEPS) of graphs $G_1, G_2, ..., G_p$ with basis $\mathcal{B}$ denoted by NEPS$(G_1, G_2, ..., G_p; \mathcal{B})$, is the graph with vertex set $V(G_1) \times V(G_2) \times ... \times V(G_p)$, in which two vertices $(u_1, u_2, ..., u_p)$ and $(v_1, v_2, ..., v_p)$ are adjacent if and only if there exists $(\beta_1, \beta_2, ..., \beta_p) \in \mathcal{B}$ such that u_i is adjacent to v_i in G_i whenever $\beta_i = 1$ and $u_i = v_i$ whenever $\beta_i = 0$. The graphs $G_1, G_2, ..., G_p$ are called the factors of NEPS [2]. Thus, the NEPS of graphs generalizes the various types of graph products, as discussed in detail in the next section.

In this paper, we study the domination number, the global domination number, the cographic domination number, the global cographic domination number and the independent domination number of NEPS of two graphs.

All graph theoretic terminology and notations not mentioned here are from [1].

2 NEPS of two graphs

There are seven possible ways of choosing the basis $\mathcal{B}$ when $p = 2$.

$\mathcal{B}_1 = \{(0, 1)\}$

$\mathcal{B}_2 = \{(1, 0)\}$

$\mathcal{B}_3 = \{(1, 1)\}$

$\mathcal{B}_4 = \{(0, 1), (1, 0)\}$

$\mathcal{B}_5 = \{(0, 1), (1, 1)\}$

$\mathcal{B}_6 = \{(1, 0), (1, 1)\}$

$\mathcal{B}_7 = \{(0, 1), (1, 0), (1, 1)\}$

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $|V_i| = n_i$ and $|E_i| = m_i$ for $i = 1, 2$.

The NEPS$(G_1, G_2; \mathcal{B}_1)$ is n_1 copies of G_2 and the NEPS$(G_1, G_2; \mathcal{B}_2) =$ NEPS$(G_2, G_1; \mathcal{B}_1)$.

In the NEPS$(G_1, G_2; \mathcal{B}_j)$ two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if,

- $j = 3$: u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2. This is same as the tensor product [1] of G_1 and G_2.
- $j = 4$: $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or u_1 is adjacent to u_2 in G_1 and $v_1 = v_2$. This is same as the cartesian product [1] of G_1 and G_2.
- $j = 5$: Either $u_1 = u_2$ or u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2.
- $j = 6$: This is same as NEPS$(G_2, G_1; \mathcal{B}_5)$.
- $j = 7$: Either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or u_1 is adjacent to u_2 in G_1 and $v_1 = v_2$ or u_1 is adjacent to u_2 in G_1 and v_1 is adjacent to v_2 in G_2. This is same as the strong product [1] of G_1 and G_2.

3 Domination in NEPS of two graphs

3.1 NEPS with basis $\mathcal{B}_1$ and $\mathcal{B}_2$

The value of γ(NEPS$(G_1, G_2; \mathcal{B}_1)$), γ_g(NEPS$(G_1, G_2; \mathcal{B}_1)$), γ_{cd}(NEPS$(G_1, G_2; \mathcal{B}_1)$), γ_{gcd}(NEPS$(G_1, G_2; \mathcal{B}_1)$), γ_i(NEPS$(G_1, G_2; \mathcal{B}_1)$) are $n_1.\gamma(G_2)$, $n_1.\gamma_g(G_2)$, $n_1.\gamma_{cd}(G_2)$, $n_1.\gamma_{gcd}(G_2)$ and $n_1.\gamma_i(G_2)$ respectively and the case of NEPS$(G_1, G_2; \mathcal{B}_2)$ follows similarly.

3.2 NEPS with basis $\mathcal{B}_3$

In [3] it was conjectured that $\gamma(G \times H) \geq \gamma(G)\gamma(H)$, where $\times$ denotes the tensor product of two graphs. But, the conjecture was disproved in [6] by giving a realization of a graph G such that $\gamma(G \times G) \leq \gamma(G)^2 - k$ for any non-negative integer k.

Theorem 1. *There exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_3)) - \sigma(G_1)\sigma(G_2) = k$ for any positive integer k, where σ denotes any of the domination parameters γ, γ_{cd} or γ_i.*

Proof. Let G_1 be the graph defined as follows. Let $u_{11}u_{12}u_{13}$, $u_{21}u_{22}u_{23}$, ..., $u_{k1}u_{k2}u_{k3}$ be k distinct P_3 s and let u_{j1} be adjacent to $u_{j+1,1}$ for $j = 1, 2, ..., k-1$. Then $\sigma(G_1) = k$. Let G_2 be K_2. Then, $\sigma(G_2) = 1$. Also, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_3)) = 2k$. Therefore, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_3)) - \sigma(G_1)\sigma(G_2) = k$. □

Theorem 2. *The γ_g and γ_{gcd} are neither submultiplicative nor supermultiplicative with respect to the NEPS with basis $\mathcal{B}_3$. Moreover, given any integer k there exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_3)) - \sigma(G_1)\sigma(G_2) = k$, where σ denotes γ_g or γ_{gcd}.*

Proof. **Case 1.** $k \leq 0$ is even.

Let $G_1 = K_n$ and $G_2 = K_2$. Then, $\sigma(G_1) = n$ and $\sigma(G_2) = 2$. But, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_3)) = 2$. Therefore, the required difference is $2-2n$ which can be zero or any negative even integer.

Case 2. $k < 0$ is odd or $k = 1$.

Let $G_3 = P_3$ and G_1 be as in Case 1. Then $\sigma(G_3) = 2$. Also, $\sigma(\text{NEPS}(G_1, G_3; \mathcal{B}_3)) = 3$. Therefore, the required difference is $3 - 2n$ which can be one or any negative odd integer.

Case 3. $k > 1$.

Let G_3 be as in Case 2. Let G_4 be the graph defined as follows. Let $u_{11}u_{12}u_{13}$, $u_{21}u_{22}u_{23}$, ..., $u_{k1}u_{k2}u_{k3}$ be k distinct P_3 s and let u_{j1} be adjacent to $u_{j+1,1}$ for $j = 1, 2, ..., k-1$. Then $\sigma(G_4) = k$. Also, $\sigma(\text{NEPS}(G_4, G_3; \mathcal{B}_3)) = 3k$. Therefore, the required difference is k. □

3.3 NEPS with basis $\mathcal{B}_4$

Vizing's conjecture [11]. The domination number is supermultiplicative with respect to the cartesian product i.e; $\gamma(G\square H) \geq \gamma(G)\gamma(H)$.

Remark 3. *There are infinitely many pairs of graphs for which equality holds in the Vizing's conjecture [7].*

Remark 4. *Vizing's type inequality does not hold for cographic, global cographic and independent domination numbers. For example, let G be the graph obtained by attaching k pendant vertices to each vertex of a path on four vertices. Then, $\gamma_{cd}(G) = \gamma_{gcd}(G) = k + 3$ and $\gamma_{cd}(G\square G) = \gamma_{gcd}(G\square G) = 16k + 8$. For $k \geq 10$, $\gamma_{cd}(G\square G) \leq \gamma_{cd}(G)^2$.*

Theorem 5. *There exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_4)) - \sigma(G_1)\sigma(G_2) = k$ for any positive integer k, where σ denotes any of the domination parameters γ, γ_{cd} or γ_i.*

Proof. Let $G_1 = P_n$ and $G_2 = K_2$. Then, $\sigma(G_1) = \lfloor \frac{n+2}{3} \rfloor$ [4] and $\sigma(G_2) = 1$. Also, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_4)) = \lfloor \frac{n+2}{2} \rfloor$ [5]. Therefore, for any positive integer k, if we choose $n = 6k - 2$ the claim follows. □

Theorem 6. *The γ_g and γ_{gcd} are neither submultiplicative nor supermultiplicative with respect to the NEPS with basis $\mathcal{B}_4$. Moreover, given any integer k there exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_4)) - \sigma(G_1)\sigma(G_2) = k$, where σ denotes γ_g or γ_{gcd}.*

Proof. **Case 1.** $k \leq 0$ is even.

Let $G_1 = K_n$ and $G_2 = K_2$. Then, $\sigma(G_1) = n$ and $\sigma(G_2) = 2$. But, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_4)) = 2$. Therefore, the required difference is $2-2n$ which can be any positive even integer.

Case 2. $k < 0$ is odd.

Let $G_3 = P_3$ and G_1 be as in Case 1. Then $\sigma(G_3) = 2$. Also, $\sigma(\text{NEPS}(G_1, G_3; \mathcal{B}_4)) = 3$. Therefore, the required difference is $3 - 2n$ which can be any negative odd integer.

Case 3. $k \geq 1$.

Let $G_4 = P_n$ and $G_5 = P_4$. Then, $\sigma(G_4) = \lfloor \frac{n+2}{3} \rfloor$ and $\sigma(G_5) = 2$. For any positive integer k, if we choose $n = 3k+4$, then $\sigma(\text{NEPS}(G_4, G_5; \mathcal{B}_4)) = n$. (Note that the value is $n+1$ only when $n = 1, 2, 3, 5, 6, 9$ [5]). Therefore the required difference is k. $\square$

3.4 NEPS with basis $\mathcal{B}_5$ and $\mathcal{B}_6$

Theorem 7. *There exist graphs G_1 and G_2 such that $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_5)) - \sigma(G_1)\sigma(G_2) = k$ for any positive integer k, where σ denotes any of the domination parameters γ, γ_{cd} or γ_i.*

Proof. Let $G_1 = P_n$ and $G_2 = K_2$. Then $\sigma(G_1) = \lfloor \frac{n+2}{3} \rfloor$ and $\sigma(G_2) = 1$. Also, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_5)) = \lfloor \frac{n+2}{2} \rfloor$. For a positive integer k, if we choose $n = 6k - 2$ then the difference is k. Hence, the theorem. $\square$

Theorem 8. *There exist graphs G_1 and G_2 such that $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_5)) - \sigma(G_1)\sigma(G_2) = k$ for any negative integer k, where σ denotes γ_g or γ_{gcd}.*

Proof. Let $G_1 = P_n$ and $G_2 = K_2$. Then $\sigma(G_1) = \lfloor \frac{n+2}{3} \rfloor$ and $\sigma(G_2) = 2$. Also, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_5)) = \lfloor \frac{n+2}{2} \rfloor$. Therefore, if we choose $n = 6k - 2$, the required difference is $-k$. $\square$

3.5 NEPS with basis $\mathcal{B}_7$

Theorem 9. *The γ, γ_i and γ_g are submultiplicative with respect to the NEPS with basis $\mathcal{B}_7$.*

Proof. Let $D_1 = \{u_1, u_2, ..., u_s\}$ be a dominating set of G_1 and $D_2 = \{v_1, v_2, ..., v_t\}$ be a dominating set of G_2. Consider the set $D = \{(u_1, v_1), (u_1, v_2), ..., (u_1, v_t), ..., (u_s, v_1), (u_s, v_2), ..., (u_s, v_t)\}$. Let (u, v) be any vertex in $\text{NEPS}(G_1, G_2; \mathcal{B}_7)$. Since D_1 is a γ-set in G_1, there exists at least one $u_i \in D_1$ such that $u = u_i$ or u is adjacent to u_i. Similarly, there exists at least one $v_j \in D_2$ such that $v = v_j$ or v is adjacent to v_j. Therefore, (u_i, v_j) dominates (u, v) in $\text{NEPS}(G_1, G_2; \mathcal{B}_7)$. Hence, $\gamma(\text{NEPS}(G_1, G_2; \mathcal{B}_7)) \leq \gamma(G_1)\gamma(G_2)$. $\square$

Similar arguments hold for the independent domination and global domination numbers also.

Note. The difference between $\gamma(G_1)\gamma(G_2)$ and $\gamma(\text{NEPS}(G_1, G_2; \mathcal{B}_7))$ can be arbitrarily large. Similar is the case for γ_i and γ_g. For, let G_1 be the graph, n copies of C_4 s with exactly one common vertex. Then, $\gamma(G_1) = \gamma_i(G_1) = n + 1$. Also, $\gamma(\text{NEPS}(G_1, G_1; \mathcal{B}_7)) \leq n^2 + 3$ and $\gamma_i(\text{NEPS}(G_1, G_1; \mathcal{B}_7)) \leq n^2 + 5$. Also, $\gamma_g(K_n) = n$, $\gamma_g(P_3) = 2$ and $\gamma_g(\text{NEPS}(G_2, G_3; \mathcal{B}_7)) = n + 2$, if $n > 1$.

Theorem 10. *The γ_{cd} and γ_{gcd} are neither submultiplicative nor supermultiplicative with respect to the NEPS with basis $\mathcal{B}_7$. Moreover, for any integer k there exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_7)) - \sigma(G_1)\sigma(G_2) = k$, where σ denotes γ_{cd} or γ_{gcd}.*

Proof. **Case 1.** $k \leq 0$.

Let G_1 be the graph P_3 with k pendant vertices each attached to all the three vertices of the P_3. Let G_2 be the graph P_4 with k pendant vertices each attached to all the four vertices of the P_4. So, $\sigma(G_1) = 3$ and $\sigma(G_2) = k + 3$. Also, $\sigma\text{NEPS}(G_1, G_2; \mathcal{B}_7)) = 2k + 10$. Therefore, the required difference is $1 - k$.

Case 2. $k \geq 0$.

Let G_1 be as in Case 1 and G_3 be the graph P_6 with k pendant vertices each attached to all the six vertices of the P_6. So, $\sigma(G_3) = k + 5$. Also, $\sigma\text{NEPS}(G_1, G_3; \mathcal{B}_7)) = 4k + 14$. Therefore, the required difference is $k - 1$. □

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