

CONTROL POLICIES IN REPAIR OF k -out-of- n SYSTEMS

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C E R T I F I C A T E

This is to certify that the thesis entitled '**CONTROL POLICIES IN REPAIR OF k -out-of- n SYSTEMS**' is a bonafide record of the research work carried out by **Mr. Sathian M K** under my supervision in the department of Mathematics, Cochin University of Science and Technology. The results embodied in the thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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DECLARATION

I, **Sathian M K** hereby declare that this thesis entitled '**CONTROL POLICIES IN REPAIR OF k -out-of- n SYSTEMS**' contains no material which had been accepted for any other Degree or Diploma in any University or Institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due reference are made.

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To
My
Parents and Teachers

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Chapter 1

Introduction

1.1. Queueing theory

All of us have experienced the annoyance of having to wait in line. Queueing is quite common in many fields, as there is more demand for service than availability of facility for service. Over the years, the subject found its applications in areas like telecommunications, Traffic flow, Computer systems, ATM facilities, Computing etc. and forced researchers study Queueing models extensively. Queueing theory was developed to provide models to predict the behaviour of systems that attempt to provide service for randomly arising demands in a natural way. The first problem of queueing theory arose in telephone calls and Erlang was the first who treated congestion problems in the beginning of 20th century.

The basic characteristics of a queueing system are the following:

Arrival pattern of customers.

It describes the way customers arrive and join a queuing system. Arrival pattern is often random with two adjacent arrivals generally spaced by random intervals called the inter-arrival time. The arrival pattern is described by means of a probability distribution of the inter-arrival time. Arrival may also occur in batches instead of one at a time.

If the queue is too long a customer may decide not to enter it upon arrival. This customer behaviour is called balking. A customer may enter the queue, but after some time lose patience and decide to leave. This is known as reneging. Another case is, when there is more than one queue, customers have the tendency to switch from one to another which is called jockeying.

Service Pattern.

This describes the manner in which the service is rendered. As in case of arrivals, the service also is provided in single or in batches. The probability distribution of the service time describes the service pattern.

Queue discipline.

Queue discipline refers to the rule in which customers are selected for service when a queue has formed. Some of the most commonly used disciplines are first come first served (FCFS), last come first served (LCFS), random service selection (RSS) i.e., selection for service in random order independent of the time of arrival; there are cases in which customers are given priorities upon entering the system, those with higher priority are selected first.

System capacity.

A queuing system can be finite or infinite. In certain queuing process there is a limitation on the length of the queue i.e., customers are not allowed to enter if the queue has reached a certain length. These are called finite queuing systems. If there is no restriction on the length of the queue then it is called an infinite capacity queuing system.

Number of service channels.

A queuing system can be single or a multiserver system. In a multiserver queuing system there are several parallel servers to serve a single line/several waiting lines.

Number of service stages.

A queuing system may have only a single stage of service. But as an example of a service with several stages of service, consider the physical examination procedure, where each patient proceeds through various stages of medical examination, like throat check up, eye test, blood test etc.

1.2. Basic Concepts

Here we give a brief description of the modelling tools/techniques applied in the thesis. For more details on these topics one can refer Karlin and Taylor [10] or Latouche and Ramaswami [16].

1.2.1. Stochastic process. A family of random variables $\{X(t), t \in T\}$, where T is an index set, is called a stochastic process. The index t is often referred to as time. When T is a countable set, $\{X(t), t \in T\}$ is said to be a discrete-time process, whereas if T is an interval of the real line, it is called a continuous-time process. For instance,

$\{X_n, n = 0, 1, \dots\}$ is a discrete time stochastic process indexed by the set of non negative integers, while $\{X(t), t \geq 0\}$ is a continuous time process indexed by non negative real numbers.

1.2.2. Markov Process.

A Markov process is a stochastic process $\{X(t), t \in T\}$ that satisfies the condition

$$Pr\{X(t_n) \leq x_n / X(t_{n-1}) = x_{n-1}, \dots, X(t_1) = x_1\} = Pr\{X(t_n) \leq x_n / X(t_{n-1}) = x_{n-1}\},$$

for any set of n time points $t_1 < t_2 < \dots < t_n$ in the index set or the time range of the process and x_1, x_2, \dots, x_n are elements of the state space. That is the stochastic process $\{X(t), t \in T\}$ that changes states according to a transition rule that only depends on the current state but not the past is called a Markov process.

1.2.3. Exponential distribution.

A continuous random variable X is said to follow exponential distribution with parameter μ if its probability density function is given by

$$f(x; \mu) = \begin{cases} \mu e^{-\mu x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and $\mu > 0$. One of the most important properties of the exponential distribution is the memoryless property: $Pr(X > x + y / X > x) = P(X > y)$ for $x, y \geq 0$. In making a mathematical model for a real life phenomenon we often assume that certain random variables associated with the problem under study are exponentially distributed.

1.2.4. Renewal Process.

A counting process $\{N(t), t \geq 0\}$ with independently and identically distributed inter-arrival times is called a renewal process. Consider a renewal process $\{N(t), t \geq 0\}$ having

inter arrival times X_1, X_2, \dots with distribution function F . Let $S_n = \sum_{i=1}^n X_i, n \geq 1; S_0 = 0$. Then we have $N(t) = \max\{n : S_n \leq t\}$ and the distribution of $N(t)$ is given by $Pr\{N(t) = n\} = F_n(t) - F_{n+1}(t)$ where F_n is the n -fold convolution of F with itself. The Poisson process is a renewal process where F is an exponential distribution.

1.2.5. Poisson Process.

A Poisson process $\{X(t), t \geq 0\}$ is a renewal process having rate λ if

- (i) $X(0) = 0$.
- (ii) The process has stationary and independent increments.
- (iii) $P\{X(h) = 1\} = \lambda h + o(h)$.
- (iv) $P\{X(h) \geq 2\} = o(h)$.

It follows from the definition that for all $s, t \geq 0$,

$$P\{(X(t+s) - X(s)) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, n = 0, 1, \dots$$

For a Poisson process having parameter λ the inter arrival time has an exponential distribution with mean $1/\lambda$.

1.2.6. Continuous-time Phase type (PH) distributions.

Consider a Markov process on the states $\{1, 2, \dots, m+1\}$ with infinitesimal generator matrix $Q = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$ where the $m \times m$ matrix T satisfies $T_{ii} < 0$ for $1 \leq i \leq m$ and $T_{ij} \geq 0$ for $i \neq j$; T^0 is an $m \times 1$ column matrix such that $T\mathbf{e} + T^0 = 0$, where \mathbf{e} is a column matrix of 1's of appropriate order. Let the initial probability vector of Q be (α, α_{m+1}) ,

where α is a $1 \times m$ dimensional row vector and α_{m+1} is a scalar such that $\alpha \mathbf{e} + \alpha_{m+1} = 1$. Also assume that the states $1, 2, \dots, m$ are all transient so that absorption in to the state $m + 1$ from any initial state is certain. For eventual absorption into the absorbing state, starting from every initial state, it is necessary and sufficient that T is non singular.

The probability distribution $F(\cdot)$ of time until absorption in the state $m + 1$ corresponding to the initial probability vector (α, α_{m+1}) is given by $F(x) = 1 - \alpha e^{(Tx)} \mathbf{e}$, $x \geq 0$.

Definition 1.2.1. A probability distribution $F(\cdot)$ is a distribution of phase type(*PH*-distribution) if and only if it is the distribution of time until absorption of a finite Markov process described above. The pair (α, T) is called a representation of $F(\cdot)$.

For *PH*-distribution $F(\cdot)$ with representation (α, T) ,

- (i) The distribution $F(\cdot)$ has a jump at $x = 0$ of magnitude α_{m+1} .
- (ii) The corresponding probability density function $f(\cdot)$ is given by $f(x) = \alpha \exp(Tx) T^0$.
- (iii) The Laplace-Stieltjes transform $f^{(s)}$ of $F(\cdot)$ is given by

$$F(s) = \alpha_{m+1} + \alpha(sI - T)^{-1} T^0, \text{ for } \text{Re}(s) \geq 0.$$

- (iv) The moments about origin are given by $\mu'_k = (-1)^k k! (\alpha T^{-k} \mathbf{e})$ for $k \geq 0$.

When $m = 1$ and $T = [-\lambda]$, the underlying *PH*-distribution is exponential.

1.2.7. *PH*-renewal process.

A renewal process whose inter-renewal times have a *PH* distribution is called a *PH*-renewal process. To construct a *PH*-renewal process we consider a continuous

time Markov chain with state space $\{1, 2, \dots, m + 1\}$ having infinitesimal generator $Q = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$. The $m \times m$ matrix T is taken to be nonsingular so that absorption to the state $m + 1$ occurs with probability 1 from any initial state. Let $(\alpha, 0)$ be the initial probability vector. When absorption occurs in the above chain we say a renewal has occurred. Then the process immediately starts anew in one of the states $\{1, 2, \dots, m\}$ according to the probability vector α . Continuation of this process gives a non terminating stochastic process called *PH*-renewal process.

1.3.

1.3.1. Level Independent Quasi-Birth – Death (LIQBD) process.

A level independent quasi birth and death process is a Markov process on the state space $S = \{(i, j) : i \geq 0, j = 1, 2, \dots, m\}$ and with infinitesimal generator matrix Q given by

$$Q = \begin{bmatrix} B_0 & A_0 & & & & \\ B_1 & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & \dots & \dots & \dots \\ & & & & \dots & \dots & \dots \\ & & & & & \dots & \dots \end{bmatrix} \quad (1.3.1)$$

The above matrix is obtained by partitioning the state space S as $S = \bigcup_i \Delta_i$ where $\Delta_i = \{(i, j) / j = 1, 2, \dots, m\}$. The states in Δ_i are said to be in level i . The states within

the levels are called phases. The matrix B_0 denotes the transition rates within level 0, matrix B_1 denotes the transition rates from level 1 to level 0. A_2, A_1 and A_0 denote transition rates from level i to $(i - 1)$, i and $(i + 1)$ respectively.

1.3.2. Matrix Analytic Method.

Even though the Queueing models such as $M/M/1, M/M/\infty$ and $G/G/1$ are well studied and are well tractable using the methods like Method of generating functions, Laplace Transforms etc., they fail to provide numerical tractability analysis of such queueing models especially when we assume the distribution of inter-arrival time or service time is to be not non-exponential.

Matrix analytic approach to stochastic models was introduced by M.F Neuts to provide an algorithmic analysis for queueing models. The following brief discussion gives an account of the method of solving an LIQBD using the matrix geometric method. For a detailed description, we refer to Neuts [17], Latouche and Ramaswami [16].

Let $x = (x_0, x_1, x_2, \dots)$, be the steady state vector, where x_i 's are partitioned as $x_i = (x(i, 0), x(i, 1), x(i, 2), \dots, x(i, m))$, m being the number of phases with in levels.

Let $x_i = x_0 R^i, i \geq 1$. Then from $xQ = 0$ we get

$$x_0 A_0 + x_1 A_1 + x_2 A_2 = 0$$

$$x_0 A_0 + x_0 R A_1 + x_0 R^2 A_2 = 0$$

$$x_0 (A_0 + R A_1 + R^2 A_2) = 0.$$

Choose R such that $R^2 A_2 + R A_1 + A_0 = 0$.

Also we have $x_0B_0 + x_1B_1 = 0$, which gives

$$x_0B_0 + x_0RB_1 = 0$$

$$\text{i.e., } x_0(B_0 + RB_1) = 0.$$

First we take x_0 as the steady state vector of $B_0 + RB_1$. Then x_i , for $i \geq 1$ can be found using the formulae; $x_i = x_0R^i$ for $i \geq 1$. Now the steady state probability distribution of the system is obtained by dividing each x_i , with the normalizing constant $[x_0 + x_1 + \dots]\mathbf{e} = x_0(I - R)^{-1}\mathbf{e}$.

The above discussion leads to the following theorem.

Theorem 1.3.1. *The process represented by matrix Q is positive recurrent if and only if the minimal non negative solution R of the matrix quadratic equation*

$$R^2A_2 + RA_1 + A_0 = 0 \tag{1.3.2}$$

has spectral radius less than 1 and the finite system of equations

$$x_0(B_0 + RB_1) = 0,$$

$$x_0(I - R)^{-1}\mathbf{e} = 1$$

has a unique solution x_0 . If the matrix $A = A_0 + A_1 + A_2$ is irreducible, then $sp(R) < 1$ if and only if $\pi A_0\mathbf{e} < \pi A_2\mathbf{e}$, where π is the stationary probability vector of $A = A_0 + A_1 + A_2$.

The stationary probability vector $x = (x_0, x_1, \dots)$ of Q is given by $x_i = x_0R^i$ for $i \geq 1$.

To find the solution R of equation (1.3.2), we use the iterative procedure.

1.3.3. Level Dependent Quasi Birth Death (LDQBD) Process.

A level dependent Quasi-Birth – Death process is a Markov process on a state space $S = \{(i, j), i \geq 0, J = 1, 2, \dots, n\}$ with infinitesimal generator matrix Q given by

$$P = \begin{bmatrix} A_{10} & A_{00} & & & & \\ A_{21} & A_{11} & A_{01} & & & \\ & A_{22} & A_{12} & A_{02} & & \\ & & A_{23} & A_{13} & A_{03} & \\ & & & \dots & \dots & \dots \\ & & & & \dots & \dots & \dots \\ & & & & & \dots & \dots \end{bmatrix}. \quad (1.3.3)$$

The state space S is partitioned in to different levels i where level i is given by $\Delta_i = \{(i, j)/i \geq 0, j = 1, 2, \dots, n\}$. Here the transitions take place only to the adjacent levels for $i \geq 1$. But the transition rate depends on the level i , unlike in the LIQBD, and therefore the spatial homogeneity of the associated process is lost.

A special class of LDQBD's is those which arise in retrial queueing models (when the retrial rate at any instant depends on the number of customers in the orbit).

1.3.4. Neuts-Rao Truncation method.

Since the repeating structure is lost in LDQBD, its analysis is much more involved. However Neuts and Rao [19] suggested a truncation procedure using which certain class of LDQBD's which include retrial models can be made to have a repeating structure from a certain level N^{th} , where N is sufficiently large. For giving a brief idea of their

method, we assume that $n_i = m$ for every $i \geq N$ so that each level $\geq N$ contains the same number of states. Note that this is the case in most of the retrial queueing models. To apply Nuets-Rao Truncation, we take $A_{1i} = A_{1N}$, $A_{2i} = A_{2N}$ and $A_{0i} = A_{0N}$ for all $i \geq N$. In the case of the retrial queues this is equivalent to assuming that retrial rate remains constant whenever the number of orbital customers exceeds a certain limit N .

Define $A_N = A_{0N} + A_{1N} + A_{2N}$ and $\pi_N = (\pi_N(0, 0), \pi_N(0, 1), \pi_N(0, 2), \dots, \pi_N(0, m))$ be the steady state vector of the matrix A_N . Then the relations $\pi_N A_N = 0$ together with $\pi_N \mathbf{e} = 1$ when solved give the various components of π_N . The truncated system is stable if and only if $\pi_N A_{2N} \mathbf{e} > \pi_N A_{0N} \mathbf{e}$ and the original system is stable if $\lim_{n \rightarrow \infty} \frac{\pi A_{0N} \mathbf{e}}{\pi A_{2N} \mathbf{e}} < 1$.

Having described the tools for analysis, we move on to provide a review of the work done in the theme of the present thesis.

1.4. Review of related works

An n component system is called a k -out-of- n system if at least k components are in operational state. Application of such systems can be seen in many real-world phenomena. For instance almost all our machines, of different complexity, are subjected to failure. One would expect a machine to work as a whole, even if some of its components have failed. The best example is that of an aircraft engine. A thorough reliability check is required to ensure the safety of passengers even in some unforeseen situations. Considering another example, once can't expect to run a good emergency service like a hospital meeting minimum requirements. We would expect a hospital to run even if some of its doctors/nurses/other staff is on leave. However, keeping these extra resources could be

costly and not even feasible in some cases: it may not be possible to keep an extra engine in an aircraft. A probabilistic study of a real world system, as k -out-of- n system, often helps to develop an optimal strategy for maintaining high system reliability.

A k -out-of- n system further be classified as follows:

The system is called 'COLD' if the operational components do not fail while the system is in down state. It is called 'HOT' if operational components continue to deteriorate at the same rate while the system is down as when it is up. The system is called 'WARM' if the deterioration rate while the system is up differs from that when it is down. An extensive study of k -out-of- n systems can be seen in Krishnamoorthy *et al.* [15], Chakravarthy, Krishnamoorthy and Ushakumari [6].

In today's world, due to collaboration between different companies in different countries and also due to some government policies for reducing unnecessary additional use of global resources for a better tomorrow, sharing of resources between national/multi-national companies have become more common. For example, a mobile tower may be shared by different telecom companies. A transporting system may choose deliver goods along with passengers for additional income. A car service station may choose to serve customers other than those of its main dealer. However, a system entertaining customers other than its main customers may lead to dissatisfaction of its own customers, which may be very costly in some situations. For example, it is hard to imagine an aircraft overloaded with goods in addition to the passengers. For this reason, studies on k -out-of- n systems where external customers are also entertained, have gained attention in the literature. Dudin *et al.* [9], Krishnamoorthy *et al.* [12, 13] are among such studies. In

[9], the external customers are sent to an orbit and where there they can try to access the idle server. Once selected for service, an external customer is assumed to get a non-preemptive service. Numerically, they show that providing service to external customers in this fashion is economical to the system in comparison with the decrease in the reliability caused due to external service. In [12] it is assumed that the external customers, finding the service station busy on arrival, are directed to a pool of infinite capacity. They also assume that if the size of the buffer of internal customers is less than L , a pooled customer is selected for service with some probability p . In [13], a finite pool and an orbit of infinite capacity accommodate the external customers in such a manner that external customers join the orbit with some probability and from there try to enter the pool. The external customers are selected for service from the pool. The internal customers (failed components) are served based on an N -policy in the sense that the repair of the failed components start only on the accumulation of N -components. In addition they assume that the on-going service of an external customer is not pre-empted on accumulation of N -failed components. As in [9] and [12, 13] also indicates a decrease in the server idle probability, and an increase in the overall system revenue.

The first paper that introduced the concept of orbital customers in to reliability is by Krishnamoorthy and Ushakumari [14]. In that paper, the authors assumed that a failed component is sent to an orbit, if it finds the server busy. The authors studied the COLD, HOT, WARM variants of the problem. Ushakumari and Krishnamoorthy [21] generalized the above model by assuming arbitrarily distributed service time. Bocharov *et al.*

[5] discuss a retrial queueing system with a finite waiting space, where the customers in the waiting space have priority over customers in the orbit.

A T -policy refers to calling the server to the system after the elapse of a random time T . Queueing systems where the service is according to a T -policy have been extensively studied. We refer to Artalejo [2] for some references on such studies. Krishnamoorthy and Rekha [11], Ushakumari and Krishnamoorthy [22] are among the studies of k -out-of- n systems where the repair is under T -policy. In [11], it was assumed that the server is called to the system either when the random time T expires or when the number of failed components reaches $n - k$, whichever event occurs first. In [22], it was assumed that the server is called whenever the maximum of an exponentially distributed duration T and the sum of $N(1 \leq N \leq n - k)$ random variables is realized.

Queues with postponed work was introduced in Deepak *et al.* [8]; the Doctoral thesis of Ajayakumar [1] exclusively deals with queues with postponed work. We refer to the paper Chitra Devi *et al.* [7] for some references of queues with postponed demand. The idea of search for customers was introduced by Neuts and Ramalhoto [18]. The concept of orbital search was introduced by Artalejo *et al.* [4], where for utilizing the server idle time in a retrial queueing system, the server makes a search at a service completion epoch with some probability and picks a customer randomly from the orbit for the next service. Because of the importance of this notion, this work was followed by several other contributions. We refer to the paper Artalejo [3], Phung-Duc [20] for more references on such studies.

Postponement of work is a common phenomena. This may be to attend a more important job than the one being processed at present or for a break or due to lack of quorum (in case of bulk service, or when N -policy for service is applied) and so on. Queueing systems with postponed work is investigated in Deepak, Joshua and Krishnamoorthy [8].

1.5. An Outline of the Present Work

This thesis is divided into seven chapters including the present introductory chapter.

In second chapter we study reliability of a k -out-of- n system with a single repairman, who also renders service to external customers. We introduce an N -policy, in which repair of internal customers (failed components) is started only on accumulation of N failed components. The service to external customers is of pre-emptive nature, in the sense that their service can be interrupted in between on accumulation of N failed components. It is assumed that an external customer, who on arrival finds the server busy with an external customer, joins a queue of infinite capacity; where as an external customer who finds the server busy with an internal customer leaves the system forever. The failure times of the components follow an exponential distribution; the arrival of external customers is according to a Poisson process and service times of the internal and external customers follow non-identical phase type distributions. Using matrix-analytic methods we discuss system stability and steady state distribution. A special case of the model where the underlying distributions are all exponential has been considered. Explicit expression for the stability condition and a product form solution for the steady

state have been obtained for this case. Several system performance measures have been obtained explicitly. Analysis of a cost function indicates that N -policy does help to optimize the system revenue maintaining high system reliability.

In the third chapter we consider two k -out-of- n systems with single server who provides service to external customers also. Both models assume an N -policy that the repair of failed components (main customers) start only on the accumulation of N of them. When not repairing failed components, the server attends external customers (if there is any) who arrive according to a Poisson process. Once started, the repair of failed components is continued until all the components become operational. Service of external customers is non pre-emptive in nature. When there are at least N failed components in the system and or when the server is busy with failed components, external customers are not allowed to join the system. Otherwise, in the first model they are assumed to join an infinite capacity queue of external customers; whereas in the retrial model, they join an orbit of infinite capacity. Life time distribution of components, service time distribution of main and external customers and the inter retrial time distribution of orbital customers in the second model are all assumed to follow exponential distributions. Steady state analysis has been carried out for both models and several important system performance measures based on the steady state distribution derived. A numerical study comparing the current models with those in which external customers are not considered has been carried out. This suggests that rendering service to external customers helps to utilize the server idle time profitably, without affecting the system reliability.

In the fourth chapter we study a k -out-of- n system with a single server who offers service also to external customers according to T -policy. The server attends external customers only (if there is any) until the realization of the time T . If there is at least one failed component present at the moment of realization of time T , the external customer in service will get pre-empted and the server is switched on to the service of main customers; otherwise the server continues at his present status and the clock T restarts. The failure times of the components and realization times follow exponential distribution; the arrival of external customers is according to a Poisson process and service times of the internal and external customers follow non-identical exponential distributions. Explicit expression for stability condition has been obtained and steady state analysis has been carried out. A numerical study of several important performance measures and a comparison of the current model with the one in which no external customers are allowed has been carried out.

The fifth chapter describes a k -out-of- n system with single server extending service to external customers also. It has a finite buffer of capacity $n - k + 1$ where the failed components of the main system wait for service in the order of their arrival and a pool of external customers with infinite capacity. At the end of a service if there are external customers in the pool, the system operates as follows: if the queue in the buffer is empty an external customer from the pool is transferred to the buffer with probability 1 and immediately starts its service; if the queue size in the buffer (transition level) is less than L , a pre-assigned number ($1 \leq L \leq n - k + 1$), then again an external customer from the pool is transferred to the head of the queue in the buffer with probability p and

immediately starts service; if there are between L and $n - k + 1$ failed components in the buffer, the customer at the head of the queue in the buffer enters in to the service process. We assume that if an external customer on arrival finds a busy server with main customers, he joins the pool with probability γ , $0 \leq \gamma \leq 1$. When no external customers are present, the server attends main customers if there is any. Inter arrival times of failed components of the main system and external customers follow exponential distribution with different parameters. The service process of main customers and external customers has the same phase type distribution. Explicit expression for stability condition has been obtained and the steady state distribution and several important performance measures have been studied numerically. A numerical comparison of the current model with those in which no external customers are allowed has been carried out.

In Chapter 6 we study a retrial model discussed in chapter 2 with the assumption that at service completion epochs of external customers or at the moment of service completion of last main customer from the time of start of service of main customers, the server makes a search and selects an external customer (if any) randomly from the orbit for the next service with a given probability. Arrival process of failed components has inter-arrival times exponentially distributed and that of external customers is according to a Poisson process. Service time of both main and external customers are exponentially distributed with different parameters and are also independent. Stability of this model has been discussed and the analysis of the steady state distribution and several performance measures has been carried out numerically. Also the current model is compared

numerically with a k -out-of- n system with repair in which no external customers are allowed.

In the seventh chapter we study reliability of a k -out-of- n system with a single server which provides an essential and several inessential (by mistake) service with given probabilities. Contrary to assumptions on models in previous chapters, here no external customers are provided service. The essential service time and the components life time follow exponential distribution of different parameters and the duration of service in the inessential states has a phase type distribution. The effect of inessential service to the failed components on the system reliability has been studied. Several important performance measures have been studied numerically.

Chapter 2

Reliability of a k -out-of- n system with repair by a single server extending service to external customers with pre-emption

Abstract

In this chapter we study the reliability of a k -out-of- n system, with a single technician, who also renders service to external customers besides repairing the failed components in the system. For optimizing the revenue from external service without compromising the system reliability, we introduce the N -policy, in which the repair of the

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internal customers (failed components) starts only on accumulation of N failed components. The service to external customers is of preemptive nature, in the sense that their service can be interrupted on accumulation of N failed components. It is assumed that an external customer, who finds the server busy with an external customer on his/her arrival, joins a queue of infinite capacity; whereas an external customer who finds the server busy with an internal customer leaves the system forever. The failure times of the components follow an exponential distribution; the arrival of external customers is according to a Poisson process and the service times of the internal and external customers follow non-identical phase-type distributions. Using matrix-analytic methods, we discuss the system stability and steady state distribution. A special case of the model where the underlying distributions are all exponential has been considered for studying the effect of the service to external customers and the N -policy on the system reliability. Explicit expression for the stability condition and a product form solution for the steady state have been obtained for this case. Also several system performance measures have been obtained explicitly. Analysis of a cost function indicates that N -policy does help to optimize the system revenue maintaining high system reliability.

2.1. Introduction

In the present chapter, we study a k -out-of- n system, where the sever offers service to external customers for additional income. For optimizing the revenue by way of providing external service, maintaining a high system reliability, we introduce an N -policy

in which the service of the failed components starts on accumulation of N failed components. The service to the external customers is of preemptive nature in the sense that their service may be interrupted in between on accumulation of N failed components. The external customers join a queue of infinite capacity on finding a busy server. The current study differs from that in [13] in that, here the pool (waiting space) of external customers is of infinite capacity and here there is no orbit of retrying customers. Also in contrast to [13], the service of external customers is of preemptive in nature here. It may seem that the model under discussion has stronger assumptions than [13]; but the objective here is to check whether we can get more details of the system, like its stability condition, steady state probability distribution etc. by strengthening some assumptions. It turns out that, our objective is achieved, in the sense that an explicit steady state distribution of the underlying Markov chain has been obtained.

This chapter is arranged as follows: In section 2.2, we perform the Stochastic Modeling of the above problem and in section 2.3, we perform the steady state analysis of the underlying Markov chain after finding a necessary and sufficient condition for the stability of the system. Section 2.4, discusses a special case of the model discussed in Section 2.2, where the service time distributions are assumed to follow exponential distribution. In section 2.5 we conduct a numerical study of the model discussed in Section 2.4 and compares it with a model in which no external customers are allowed. Section 7.3 concludes the discussion.

2.2. Modeling and Analysis

In this chapter we study the reliability of a k -out-of- n system with repair by a single repair facility which also provides service to external customers. The system consists of two parts.

- (1) A main queue consisting of customers (failed components of the k -out-of- n system) and
- (2) A queue of external customers.

A k -out-of- n system is in the up state (working state) as long as at least k components are in operational state. Otherwise the system is in the down state.

The arrival process.

Arrival of main customers have inter-occurrence time exponentially distributed with parameter λ_i when the number of operational components of the k -out-of- n system is i . By taking $\lambda_i = \frac{\lambda}{i}$ we notice that the failure rate is a constant λ . Arrival of external customers have inter-occurrence time exponentially distributed with parameter $\bar{\lambda}$. Arrival of external customers is temporarily halted while serving the main customers (the failed components of the k -out -of- n system).

The service process.

Commencement of service to the failed components of the main system is governed by the N -policy, that is at the epoch the system starts with all components operational, the server starts attending one by one the customers from the queue of external customers

(if there is any waiting). At the epoch when the accumulated number of failed components of the main system reaches N , the external customer in service will get pre-empted and the server is switched on to the service of main customers. Service times of main customers and external customers follow phase-type distributions with representations (α, S) and (β, T) of orders m_1 and m_0 respectively.

Objective.

To maximize the reliability of a k -out-of- n system with repair by a single server, who provides service to external customers also, based on N -policy.

The Markov Chain.

Let $X_1(t)$ denotes at time t number of external customers in the system including the one getting service (if any) ,

$X_2(t)$ denotes the server status at time t defined as;

$$X_2(t) = \begin{cases} 0, & \text{if the server is idle or serving an external customer} \\ 1, & \text{if the server is busy with a failed component.} \end{cases}$$

$X_3(t)$ denotes number of main customers in the system at time t including the one getting service (if any). $X_4(t)$ denotes the phase of the service process.

Let $X(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$ then $\{X(t), t \geq 0\}$ is a continuous time Markov chain

on the state space whose levels are designated

$$l(0) = \{(0, 0, j_1)/0 \leq j_1 \leq N - 1\} \cup \{(0, 1, j_1, j_2)/1 \leq j_1 \leq n - k + 1, 1 \leq j_2 \leq m_1\},$$

$$l(i) = l(i, 0) \cup l(i, 1),$$

$$l(i, 0) = \{(i, 0, j_1, j_2)/0 \leq j_1 \leq N - 1, 1 \leq j_2 \leq m_0\}$$

$$l(i, 1) = \{(i, 1, j_1, j_2)/1 \leq j_1 \leq n - k + 1, 1 \leq j_2 \leq m_1\}.$$

In the sequel,

- (i) I_n denotes the identity matrix of order n ;
- (ii) I denotes an identity matrix of appropriate size;
- (iii) e_n denotes a $n \times 1$ column matrix of 1's
- (iv) e denotes a column matrix of 1's of appropriate order;
- (v) E_n denotes a square matrix of order n defined as

$$E_n(i, j) = \begin{cases} -1, & \text{if } i = j; 1 \leq i \leq n \\ 1, & \text{if } j = i + 1; 1 \leq i \leq n - 1 \\ 0, & \text{otherwise} \end{cases}$$

- (vi) $E'_n =$ Transpose of E_n
- (vii) $r_n(i)$ denotes a $1 \times n$ row matrix whose i^{th} entry is 1 and all other entries are zeros
- (viii) $C_n(i) =$ Transpose of $r_n(i)$
- (ix) \otimes denotes Kronecker product of matrices

$$(x) S^0 = -Se, T^0 = -Te.$$

The infinitesimal generator matrix of $\{X(t)\}$ is given by

$$Q = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_0 & & & & & & & \\ \tilde{A}_2 & A_1 & A_0 & & & & & & \\ & A_2 & A_1 & A_0 & & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & \\ & & & & & & \cdot & \cdot & \end{bmatrix}, \text{ where } \tilde{A}_1 = \begin{bmatrix} \tilde{A}_{00} & \tilde{A}_{01} \\ \tilde{A}_{10} & \tilde{A}_{11} \end{bmatrix}$$

$$\tilde{A}_{00} = \lambda E_N - \bar{\lambda} I_N, \tilde{A}_{01} = [C_N(N) \otimes r_{n-k+1}(N)] \otimes \lambda \alpha, \tilde{A}_{10} = [C_{n-k+1}(1) \otimes r_N(1)] \otimes S^0,$$

$$\tilde{A}_{11} = I_{n-k+1} \otimes S + (E'_{n-k+1} + I_{n-k+1}) \otimes (S^0 \alpha)$$

$$+ [E_{n-k+1} + C_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1)] \otimes \lambda I_{m_1};$$

$$A_1 = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix};$$

$$A_{00} = E_N \otimes \lambda I_{m_0} + I_N \otimes (T - \bar{\lambda} I_{m_0}), A_{01} = [C_N(N) \otimes r_{n-k+1}(N)] \otimes (\lambda e_{m_0} \alpha);$$

$$A_{10} = [C_{n-k+1}(1) \otimes r_N(1)] \otimes (S^0 \beta), \quad A_{11} = \tilde{A}_{11};$$

$$\tilde{A}_0 = \begin{bmatrix} I_N \otimes (\bar{\lambda} \beta) & 0 \\ 0 & 0 \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} I_N \otimes T^0 & 0 \\ 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} I_N \otimes (\bar{\lambda} I_{m_0}) & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} I_N \otimes (T^0 \beta) & 0 \\ 0 & 0 \end{bmatrix}$$

2.3. Steady State Analysis

2.3.1. Stability condition.

Let $A = A_0 + A_1 + A_2$ and π be the steady state vector of A . That is π satisfies the equations

$$\pi A = 0 \quad \text{and} \quad (2.3.1)$$

$$\pi e = 1. \quad (2.3.2)$$

Partitioning π as $\pi = (\pi_0, \pi_1)$, equation (2.3.1) gives

$$\pi_0 \left[E_N \otimes \lambda I_{m_0} + I_N \otimes (T + T^0 \beta) \right] + \pi_1 A_{10} = 0 \quad (2.3.3)$$

$$\pi_0 A_{01} + \pi_1 A_{11} = 0. \quad (2.3.4)$$

From equation (2.3.4), $\pi_1 = -\pi_0 A_{01} A_{11}^{-1}$.

Substituting in equation (2.3.3), we get

$$\pi_0 \left[E_N \otimes \lambda I_{m_0} + I_N \otimes (T + T^0 \beta) \right] - \pi_0 A_{01} A_{11}^{-1} A_{10} = 0 \quad (2.3.5)$$

We notice that $A_{10} = (-A_{11} e)(r_N(1) \otimes \beta)$ and therefore $-A_{11}^{-1} A_{10} = e(r_N(1) \otimes \beta)$

$$\begin{aligned} -A_{01} A_{11}^{-1} A_{10} &= (C_N(N) \otimes \lambda e_{m_0})(r_N(1) \otimes \beta) \\ &= (C_N(N) \otimes r_N(1)) \otimes (\lambda e_{m_0} \beta). \end{aligned} \quad (2.3.6)$$

Thus equation (2.3.5) reduce to

$$\boldsymbol{\pi}_0 \left[E_N \otimes \lambda I_{m_0} + (C_N(N) \otimes r_N(1)) \otimes (\lambda e_{m_0} \beta) + I_N \otimes (T + T^0 \beta) \right] = 0. \quad (2.3.7)$$

Further partitioning $\boldsymbol{\pi}_0 = (\pi_{0,0}, \pi_{0,1}, \dots, \pi_{0,N-1})$, equation (2.3.7) give rise to the following set equations

$$\pi_{0,0} (T + T^0 \beta - \lambda I_{m_0}) + \pi_{0,N-1} \lambda e_{m_0} \beta = 0 \quad (2.3.8)$$

$$\pi_{0,i} \lambda I_{m_0} + \pi_{0,i+1} (T + T^0 \beta - \lambda I_{m_0}) = 0, 0 \leq i \leq N - 1. \quad (2.3.9)$$

Postmultiply both sides of equation (2.3.8) and (2.3.9) by the column vector e , we get

$$\pi_{0,0} (T + T^0 \beta - \lambda I_{m_0} + \lambda e_{m_0} \beta) = 0 \quad (2.3.10)$$

$$\pi_{0,i} e = \pi_{0,i+1} e, 0 \leq i \leq N - 1. \quad (2.3.11)$$

And equation (2.3.10) gives

$$\pi_{0,0} = a \eta \quad (2.3.12)$$

where η is the steady state vector of the generator matrix $T + T^0 \beta - \lambda I_{m_0} + \lambda e_{m_0} \beta$ and ‘ a ’ is a constant.

Now equation (2.3.9) gives

$$\pi_{0,i} = (-1)^i a \lambda^i \eta (T + T^0 \beta - \lambda I_{m_0})^{-i}, 0 \leq i \leq N - 1. \quad (2.3.13)$$

Equation (2.3.13) determines the vector π_0 up to the multiplicative constant.

It follows from equations (2.3.11) and (2.3.13) that

$$\begin{aligned}\pi A_0 \mathbf{e} &= \bar{\lambda} \pi_0 \mathbf{e} \\ &= \bar{\lambda} a N \\ \pi A_2 \mathbf{e} &= \sum_{i=0}^{N-1} \pi_{0,i} T^0 \\ &= a \sum_{i=0}^{N-1} (-1)^i \lambda^i \eta (T + T^0 \beta - \lambda I_{m_0})^{-i} T^0.\end{aligned}$$

Here $\pi A_0 \mathbf{e} < \pi A_2 \mathbf{e}$ becomes

$$N \bar{\lambda} < \sum_{i=0}^{N-1} (-1)^i \lambda^i \eta (T + T^0 \beta - \lambda I_{m_0})^{-i} T^0.$$

This leads to the following theorem for the stability of the system.

Theorem 2.3.1. *The Markov chain $\{X(t)\}$ is stable if and only if*

$$N \bar{\lambda} < \sum_{i=0}^{N-1} (-1)^i \lambda^i \eta (T + T^0 \beta - \lambda I_{m_0})^{-i} T^0.$$

2.3.2. Steady State Vector.

The steady state vector \mathbf{x} is partitioned as $\mathbf{x} = (x_0, x_1, x_2, \dots)$ satisfies the equations

$$x_0 \tilde{A}_1 + x_1 \tilde{A}_2 = 0$$

$$x_0 \tilde{A}_0 + x_1 A_1 + x_2 A_2 = 0$$

$$x_i A_0 + x_{i+1} A_1 + x_{i+2} A_2 = 0, i \geq 1.$$

Matrix theoretic approach (See Neuts [17]) gives

$$x_i = x_1 R^{i-1}, \quad i \geq 1 \quad (2.3.14)$$

where R is the minimal non negative solution of the matrix quadratic equation

$$R^2 A_2 + R A_1 + A_0 = 0. \quad (2.3.15)$$

It then follows that

$$x_1 = -x_0 \tilde{A}_0 (A_1 + R A_2)^{-1} \quad (2.3.16)$$

and that x_0 satisfies the system of equations

$$x_0 \left(\tilde{A}_1 - \tilde{A}_0 (A_1 + R A_2)^{-1} \tilde{A}_2 \right) = 0. \quad (2.3.17)$$

From the structure of the matrix A_0 , it follows that the R matrix has the form

$$R = \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \quad (2.3.18)$$

where R_1 is a square matrix of order Nm_0 and R_2 is a matrix of order $Nm_0 \times (n-k+1)m_1$.

$$R^2 = \begin{bmatrix} R_1^2 & R_1 R_2 \\ 0 & 0 \end{bmatrix}$$

Equation (2.3.15) then reduces to the following equations

$$R_1^2 (I_N \otimes T^0 \beta) + R_1 A_{00} + R_2 A_{10} + I_N \otimes \bar{\lambda} I_{m_0} = 0 \quad (2.3.19)$$

$$R_1 A_{01} + R_2 A_{11} = 0 \quad (2.3.20)$$

$$\text{Equation (2.3.20) gives } R_2 = -R_1 A_{01} A_{11}^{-1} \quad (2.3.21)$$

which when substituted in Equation (2.3.19) gives

$$R_1^2 (I_N \otimes T^0 \beta) + R_1 A_{00} - R_1 A_{01} A_{11}^{-1} A_{10} + \bar{\lambda} I_{Nm_0} = 0$$

$$\text{i.e., } R_1^2 (I_N \otimes T^0 \beta) + R_1 (A_{00} - A_{01} A_{11}^{-1} A_{10}) + \bar{\lambda} I_{Nm_0} = 0.$$

Using equation (2.3.6), the above equation can be rewritten as

$$R_1^2 (I_N \otimes T^0 \beta) + R_1 [A_{00} + (C_N(N) \otimes r_N(1)) \otimes (\lambda e_{m_0} \beta)] + \bar{\lambda} I_{Nm_0} = 0. \quad (2.3.22)$$

Solving equation (2.3.22), we get R_1 and hence the steady state vector of $\{X(t)\}$. For Solving equation (2.3.22) we use Logarithmic reduction algorithm (refer Latouche and Ramaswami [16]).

2.4. A Special Case

We now concentrate on a special case of the problem discussed in Section 2.2 where the service time distributions of main and external customers follow exponential distributions with parameters μ and $\bar{\mu}$ respectively. As expected, this resulted in arriving

at explicit expression for the stability condition, steady state distribution and several performance measures.

2.4.1. The Markov Chain Model.

With $X_1(t)$, $X_2(t)$ and $X_3(t)$ having same definition as in section 2.2, $\tilde{X}(t) = (X_1(t), X_2(t), X_3(t))$ is a continuous time Markov chain on the state space

$$\{(j_1, 0, j_2) | j_1 \geq 0; 0 \leq j_2 \leq N - 1\} \cup \{(j_1, 1, j_2) | j_1 \geq 0; 0 \leq j_2 \leq n - k + 1\}.$$

Arranging the states lexicographically and then partitioning the state space into levels i , where each level i corresponds to the collection of states with number of external customers in the system including the one getting service (if any) at time t as i . We get the infinitesimal generator of the above chain as

$$\tilde{Q} = \begin{bmatrix} F_{10} & F_0 & & & & \\ F_2 & F_1 & F_0 & & & \\ & F_2 & F_1 & F_0 & & \\ & & \dots & \dots & \dots & \\ & & & \dots & \dots & \dots \end{bmatrix} \quad (2.4.1)$$

The entries of the matrix are described below.

The transition from level i to level $i + 1$ is represented by the matrix

$$F_0 = \begin{bmatrix} \bar{\lambda} I_N & \mathbf{0}_{N \times n-k+1} \\ \mathbf{0}_{(n-k+1) \times N} & \mathbf{0}_{(n-k+1) \times (n-k+1)} \end{bmatrix}.$$

The transition from level i to level $i - 1$ is represented by the matrix

$$F_2 = \begin{bmatrix} \bar{\mu}I_N & 0_{N \times n-k+1} \\ 0_{(n-k+1) \times N} & 0_{(n-k+1) \times (n-k+1)} \end{bmatrix}.$$

The transition within level 0 to level 0 is represented by the matrix

$$F_{10} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

$$\text{where } B_1 = \lambda E_N - \bar{\lambda}I_N;$$

B_2 is a $N \times (n - k + 1)$ matrix whose $(N, N)^{\text{th}}$ entry is λ and all other entries are zeroes.

B_3 is a $(n - k + 1) \times N$ matrix whose $(1, 1)^{\text{th}}$ entry is μ and all other entries are zeroes.

$$B_4 = \lambda E_{n-k+1} + \mu E'_{n-k+1} + \lambda C_{n-k+1}(n - k + 1) \otimes r_{n-k+1}(n - k + 1).$$

The transitions within level i , $i \geq 1$, is represented by matrix

$$F_1 = \begin{bmatrix} D_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

$$\text{where } D_1 = \lambda E_N - (\bar{\lambda} + \bar{\mu})I_N.$$

2.4.2. Steady State Analysis.

First we derive the condition for stability of the system.

2.4.2.1. Stability condition.

Consider the generator matrix

$$F = F_0 + F_1 + F_2 = \begin{bmatrix} H_1 & H_2 \\ H_3 & B_4 \end{bmatrix},$$

where $H_1 = \lambda E_N$.

H_2 is a $N \times (n - k + 1)$ matrix whose (N, N) th entry is λ and all other entries are zeroes.

H_3 is a $(n - k + 1) \times N$ matrix whose $(1, 1)$ th entry is μ and all other entries are zeroes.

The stationary probability vector $\tilde{\Pi} = (\tilde{\pi}_{(0,0)}, \tilde{\pi}_{(0,1)}, \dots, \tilde{\pi}_{(0,N-1)}, \tilde{\pi}_{(1,1)}, \dots, \tilde{\pi}_{(1,N)} \dots, \tilde{\pi}_{(1,n-k+1)})$ of the generator matrix A satisfies the equations $\tilde{\Pi}F = 0$ and $\tilde{\Pi}e = 1$.

$\tilde{\Pi}F = 0$ gives the following equations

$$\tilde{\pi}_{(0,i)} = \tilde{\pi}_{(0,0)}, 1 \leq i \leq N - 1 \quad \text{and}$$

$$\tilde{\pi}_{(1,i)} = \begin{cases} \alpha_i \tilde{\pi}_{(0,0)}, & \text{where } \alpha_i = \sum_{j=1}^i (\lambda/\mu)^j, i = 1, 2, \dots, N \\ \beta_i \tilde{\pi}_{(0,0)}, & \text{where } \beta_i = \sum_{j=1-N+1}^i (\lambda/\mu)^j, i = N + 1, \dots, n - k + 1 \end{cases}$$

The normalizing condition $\tilde{\Pi}e = 1$ gives $\tilde{\pi}_{(0,0)} = \frac{1}{\varphi - \psi}$, where

$$\varphi = N + \frac{(\mu^{N-2} - \lambda^{N-2})\lambda}{(\mu - \lambda)\mu^N} \left\{ N + \frac{\lambda(\mu^{n-k+1-N} - \lambda^{n-k+1-N})}{\mu^{n-k+1-N}(\mu - \lambda)} \right\}$$

and

$$\psi = \frac{(\mu - \lambda)(\mu^{N-1} - (N - 1)\lambda^N) + \lambda\mu(\mu^{N-2} - \lambda^{N-2})}{\mu^{N-1}(\mu - \lambda)}$$

Thus we arrive at the following

Theorem 2.4.1. *The process $\{\tilde{X}(t), t \geq 0\}$ is positive recurrent if and only if $\bar{\lambda} < \bar{\mu}$.*

Proof. It is well known (see Neuts [17]) that the Markov chain with infinitesimal generator \tilde{Q} is stable if and only if $\tilde{\pi}F_0e < \tilde{\pi}F_2e$, that is if and only if the left drift rate exceeds that to the right.

We have $\tilde{\pi}F_0e = N\bar{\lambda}\tilde{\pi}_{(0,0)}$ and $\tilde{\pi}F_2e = N\bar{\mu}\tilde{\pi}_{(0,0)}$. Thus $\{\tilde{X}(t), t \geq 0\}$ is positive recurrent if and only if $\bar{\lambda} < \bar{\mu}$. □

2.4.2.2. Steady State Distribution.

Here using the steady state vector $\tilde{\Pi}$ of the generator matrix F , we proceed construct the steady state vector $\tilde{X} = (\tilde{X}(0), \tilde{X}(1), \tilde{X}(2), \dots)$ of the Markov chain $\{\tilde{X}(t), t \geq 0\}$ by defining, $\tilde{X}(i) = \eta \left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^i \tilde{\Pi}$, for $i \geq 0$, where η is a positive constant to be found out.

First we will prove that \tilde{X} satisfies the equation $\tilde{X}\tilde{Q} = 0$. For this, notice that we can decompose the infinitesimal generator matrix \tilde{Q} as $\tilde{Q} = \tilde{Q}_1 + \tilde{Q}_2$, where

$$\tilde{Q}_1 = \begin{bmatrix} F & & & & \\ & F & & & \\ & & F & & \\ & & & \dots & \\ & & & & \dots \end{bmatrix}$$

and

$$\tilde{Q}_2 = \begin{bmatrix} -F_0 & F_0 & & & \\ F_2 & \overline{F}_1 & F_0 & & \\ & F_2 & \overline{F}_1 & F_0 & \\ & & \dots & \dots & \dots \\ & & & \dots & \dots & \dots \end{bmatrix},$$

where each entry is a square matrix of order $N + n - k + 1$ listed as:

$$\overline{F}_1 = \begin{bmatrix} -(\overline{\lambda} + \overline{\mu})I_N & 0_{N \times n-k+1} \\ 0_{(n-k+1) \times N} & 0_{(n-k+1) \times (n-k+1)} \end{bmatrix}.$$

Since $\tilde{\Pi}F = 0$ and $\tilde{X}(i) = \eta \left(\frac{\tilde{\lambda}}{\tilde{\mu}} \right)^i \tilde{\Pi}$, we have

$$\tilde{X}\tilde{Q}_1 = 0. \quad (2.4.2)$$

Now,

$$\tilde{X}\tilde{Q}_2 = \left[\tilde{X}(0)(-F_0) + \tilde{X}(1)F_2, \tilde{X}(0)F_0 + \tilde{X}(1)\overline{F}_1 + \tilde{X}(2)F_2, \tilde{X}(1)F_0 + \tilde{X}(2)\overline{F}_1 + \tilde{X}(3)F_2, \dots \right].$$

Notice that $(-F_0) + \frac{\tilde{\lambda}}{\tilde{\mu}}F_2 = 0$ and

$$F_0 + \left(\frac{\tilde{\lambda}}{\tilde{\mu}} \right) \overline{F}_1 + \left(\frac{\tilde{\lambda}}{\tilde{\mu}} \right)^2 F_2 = F_0 + \left(\frac{\tilde{\lambda}}{\tilde{\mu}} \right) \left(\overline{F}_1 + \frac{\tilde{\lambda}}{\tilde{\mu}} F_2 \right)$$

$$\begin{aligned}
&= F_0 - \frac{\bar{\lambda}}{\bar{\mu}} F_2 \\
&= 0,
\end{aligned}$$

which leads us to $\bar{X}(0)(-F_0) + \bar{X}(1)F_2 = 0$ and

$$\begin{aligned}
\bar{X}(i)F_0 + \bar{X}(i+1)\bar{F}_1 + \bar{X}(i+1)F_2 &= \left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^i \bar{X}(0) \left[F_0 + \frac{\bar{\lambda}}{\bar{\mu}} F_1 + \left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^2 F_2 \right] \\
&= 0, \quad i = 0, 1, 2, 3, \dots
\end{aligned}$$

$$\text{Hence} \quad \bar{X}Q_2 = 0. \quad (2.4.3)$$

From (2.4.2) and (2.4.3), we have $\bar{X}\bar{Q}_1 + \bar{X}\bar{Q}_2 = 0$, which implies that $\bar{X}\bar{Q} = 0$.

Finally, $\bar{X}e = 1$ gives the unknown constant $\eta = \left(1 - \frac{\bar{\lambda}}{\bar{\mu}}\right)$.

Hence, $\bar{X} = (\bar{X}(0), \bar{X}(1), \bar{X}(2), \dots)$, where $\bar{X}(i) = \left(1 - \frac{\bar{\lambda}}{\bar{\mu}}\right) \left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^i \bar{\Pi}$ is the steady state vector for the matrix \bar{Q} and we have the following theorem;

Theorem 2.4.2. *Let $\bar{\Pi} = (\bar{\pi}_{(0,0)}, \bar{\pi}_{(0,1)}, \dots, \bar{\pi}_{(0,N-1)}, \bar{\pi}_{(1,1)}, \dots, \bar{\pi}_{(1,N)}, \dots, \bar{\pi}_{(1,n-k+1)})$ be the steady state vector for the matrix F , where*

$$\bar{\pi}_{(0,i)} = \bar{\pi}_{(0,0)}, \quad 1 \leq i \leq N-1 \quad \text{and}$$

$$\bar{\pi}_{(1,i)} = \begin{cases} \alpha_i \bar{\pi}_{(0,0)}, & \text{with } \alpha_i = \sum_{j=1}^i (\lambda/\mu)^j, \quad i = 1, 2, \dots, N \\ \beta_i \bar{\pi}_{(0,0)}, & \text{for } \beta_i = \sum_{j=1-N+1}^i (\lambda/\mu)^j, \quad i = N+1, \dots, n-k+1 \end{cases}.$$

Further $\bar{\pi}_{(0,0)} = \frac{1}{\varphi - \psi}$, where

$$\varphi = N + \frac{(\mu^{N-2} - \lambda^{N-2})\lambda}{(\mu - \lambda)\mu^N} \left\{ N + \frac{\lambda(\mu^{n-k+1-N} - \lambda^{n-k+1-N})}{\mu^{n-k+1-N}(\mu - \lambda)} \right\} \quad \text{and}$$

$$\psi = \frac{(\mu - \lambda)(\mu^{N-1} - (N-1)\lambda^N) + \lambda\mu(\mu^{N-2} - \lambda^{N-2})}{\mu^{N-1}(\mu - \lambda)}.$$

Then $\widetilde{X} = (\widetilde{X}(0), \widetilde{X}(1), \widetilde{X}(2), \dots)$, where $\widetilde{X}(i) = \left(1 - \frac{\bar{\lambda}}{\bar{\mu}}\right) \left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^i \widetilde{\pi}$ is the steady state probability vector for the Markov chain $\{\widetilde{X}(t), t \geq 0\}$.

2.4.3. Performance Measures.

Here we derive certain important performance measures of the system under study.

2.4.3.1. Busy period of the server with the failed components of the main system.

The busy period of the server with failed components starts the instant when N failed components accumulate and it ends when no failed components are left in the system. Let $T_N(i)$, for $i \geq 0$, denote the server busy period with failed components, which starts with i external customers in the system. Note that, the number of external customers does not affect the busy period of the server with the failed components. Hence, $T_N(i) = T_N$, for $i \geq 0$. For analyzing the time T_N , we consider the Markov chain $\{Y(t)\}$ with state space $\{0, 1, 2, \dots, N, N+1, \dots, n-k+1\}$ and infinitesimal generator given by:

$$B_N = \begin{bmatrix} 0 & 0 \\ -\widehat{B}_N e & \widehat{B}_N \end{bmatrix}, \quad \text{where}$$

$$\widehat{B}_N = \lambda E_{n-k+1} + \mu E'_{n-k+1}.$$

Note that $Y(t)$ denotes the number of failed components of the main system and $Y(t) = 0$ is considered as an absorbing state; so that the busy period T_N is the time until absorption in the Markov chain $\{Y(t)\}$, assuming that it starts at the state N . Hence, the busy period T_N has a phase type distribution with representation (ω, \widehat{B}_N) , where the probability vector $\omega = (0, \dots, 0, 1, 0, \dots, 0)$, with 1 appearing in the N^{th} position. The expected value of T_N is therefore given by $ET_N = -\omega(B_N^{-1})\mathbf{e}$ where \mathbf{e} is a column vector with $n - k + 1$ elements all equal to 1. Now for finding ET_N , let us partition the column vector $(\widehat{B}_N^{-1})\mathbf{e}$ as $(t_1, t_2, \dots, t_{n-k+1})^T$. Then the identity $\widehat{B}_N(\widehat{B}_N^{-1})\mathbf{e} = \mathbf{e}$ leads us to the following equations:

$$-(\lambda + \mu)t_1 + \lambda t_2 = 1$$

$$\mu t_{i-1} - (\lambda + \mu)t_i + \lambda t_{i+1} = 1, \text{ for } 2 \leq i \leq n - k$$

$$\mu t_{n-k} - \mu t_{n-k+1} = 1.$$

The above equations give

$$t_i - t_{i+1} = \frac{1}{\mu} \sum_{j=0}^{n-k-i} j = 0(\lambda/\mu)^j, 1 \leq i \leq n - k$$

$$t_{n-k} - t_{n-k+1} = \frac{1}{\mu} \quad \text{and} \quad -\mu t_1 = \sum_{j=0}^{n-k} (\lambda/\mu)^j.$$

Hence

$$ET_N = -t_N = \frac{1}{\mu} \left(N \sum_{j=0}^{n-k-N+1} (\lambda/\mu)^j + \sum_{j=n-k-N+2}^{n-k} (n - k + 1 - j)(\lambda/\mu)^j \right). \quad (2.4.4)$$

The expected value of the busy period of the server with failed components, which starts with an arbitrary number of external customers is given by

$$\begin{aligned}
 E_B &= ET_N \sum_{j_1=0}^{\infty} \tilde{x}(j_1, 0, N-1) \\
 &= \frac{1}{(\varphi - \psi)\mu} \frac{1}{\mu} \left(N \sum_{j=0}^{n-k-N+1} (\lambda/\mu)^j + \sum_{j=n-k-N+2}^{n-k} (n-k+1-j)(\lambda/\mu)^j \right). \quad (2.4.5)
 \end{aligned}$$

We sum up the above results in

Theorem 2.4.3. *The busy period of the server with the repair of the components of the k -out-of- n system has phase type distribution with representation (ω, \widehat{B}_N) . The expected length of the busy period is given by (2.4.5).*

2.4.3.2. Expected number of pre-emptions of an external customer who is taken for service.

Consider the Markov process $X_p(t) = (N_p(t), J(t))$, where $N_p(t)$ is the number of pre-emptions occurred upto time t (measured from the time he is taken for service) of a particular external customer who is taken for service and $J(t)$ is the number of failed components of the main system. Then $X_p(t)$ has the state space

$$\{(j_1, j_2) \mid j_1 = 0, 1, 2, \dots, 0 \leq j_2 \leq N-1\} \cup \{\Delta\}$$

where Δ is an absorbing state which denotes the service completion of the external customer. The infinitesimal generator of this process is

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ \tilde{T}^0 & \tilde{T} & \widehat{A}_0 & 0 & \cdots & \cdots & \cdots \\ \tilde{T}^0 & 0 & \tilde{T} & \widehat{A}_0 & \cdots & \cdots & \cdots \\ \tilde{T}^0 & 0 & 0 & \tilde{T} & \widehat{A}_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \text{ where } \tilde{T}^0 = \bar{\mu}e_N$$

$$\tilde{T} = \lambda E_N - \bar{\mu}I_N$$

and \widehat{A}_0 is an $N \times N$ matrix whose $(N, 1)^{\text{th}}$ entry is λ .

If p_{k_i} is the probability for k pre-emptions of an external customer who starts service with i failed components, then $p_{0_i} = (-\tilde{T}^{-1}\tilde{T}^0)_i = 1 - \left(\frac{\lambda}{\lambda + \bar{\mu}}\right)^{N-i}$, $0 \leq i \leq N - 1$ and for $k \geq 1$,

$$\begin{aligned} p_{k_i} &= \left((-\tilde{T}^{-1}\widehat{A}_0)^k (-\tilde{T}^{-1}\tilde{T}^0) \right) \\ &= \left(\frac{\lambda}{\lambda + \bar{\mu}} \right)^{N-i} \left(\frac{\lambda}{\lambda + \bar{\mu}} \right)^{N(k-1)} \left(1 - \left(\frac{\lambda}{\lambda + \bar{\mu}} \right)^N \right) \\ &= \left(\frac{\lambda}{\lambda + \bar{\mu}} \right)^{Nk-i} \left(1 - \left(\frac{\lambda}{\lambda + \bar{\mu}} \right)^N \right). \end{aligned}$$

Expected number of pre-emptions of an external customer, starting service with i failed components

$$= \sum_{k=0}^{\infty} k p_{k_i} = \left(1 - \left(\frac{\lambda}{\lambda + \bar{\mu}} \right)^N \right)^{-1} \left(\frac{\lambda}{\lambda + \bar{\mu}} \right)^{N-i}.$$

2.4.3.3. Expected waiting time of an external customer.

For computing the expected waiting time of an external customer who joins as the r^{th} customer in the queue of external customers, we consider the Markov process $X_w(t) = (J_1(t), S(t), J_2(t))$, where $J_1(t)$ is the rank of the external customer, $S(t) = 0$ if the server is busy with external customers and $S(t) = 1$ if the server is busy with a main customer. $J_2(t)$ is the number of main customers in the system. The rank $J_1(t)$ of an external customer is assumed to be ' l ' if it finds $l - 1$ external customers ahead of it. The rank of an external customer may decrease by 1 if an external customer ahead of it leaves the system after completing the service. Now consider the Markov process $X_w(t)$ for a tagged external customer who finds $l - 1$ external customers ahead of it while joining the system. The state space for this process is given by $\{*\} \cup \{(1, 2, \dots, l) \times (\{0\} \times \{0, 1, \dots, N - 1\}) \cup \{1\} \times \{1, 2, \dots, n - k + 1\}\}$, where $*$ is an absorbing state, which denotes the service completion of the tagged customer. The infinitesimal generator Q_w of this process is $Q_w = \begin{bmatrix} 0 & 0 \\ W_l^0 & W_l \end{bmatrix}$, where

$$W_l = \begin{bmatrix} w_{11} & & & & & \\ w_{22} & w_{12} & & & & \\ & w_{23} & w_{13} & & & \\ & & & & & \\ & & & & & \\ & & & & w_{2l} & w_{1l} \end{bmatrix}$$

with $w_{1i} = F_1 + F_0; 1 \leq i \leq l$

$w_{2i} = F_2; 1 \leq i \leq l$

$w_l^0 = C_l(1) \otimes (F_2 e)$

The waiting time of the tagged customer is the time until absorption in the Markov process $X_w(t)$. Let $E_W^{(i)}(l)$ denote the expected waiting time of a tagged customer who joins the system with rank l , who finds ‘ i ’ failed components. Defining the row vector $\tilde{\theta}_i$ as $\tilde{\theta}_i = r_l(l) \otimes r_{N+n-k+1}(i+1)$, $0 \leq i \leq N-1$. Then $E_W^{(i)}(l) = -\tilde{\theta}_i W_l^{-1} e$, $0 \leq i \leq N-1$. Let $E_W(l)$ be the $N \times 1$ column matrix whose $(i, 1)^{\text{th}}$ entry is $E_W^{(i-1)}(l)$. Taking the probability that an external customer see i external customers, j failed components and server busy with external customers on its arrival as $(1 - \frac{\bar{\lambda}}{\bar{\mu}}) (\frac{\bar{\lambda}}{\bar{\mu}})^i \frac{1}{\varphi - \psi}$, the expected waiting time of an arbitrary external customers is given by

$$\sum_{i=0}^{\infty} \left(1 - \frac{\bar{\lambda}}{\bar{\mu}}\right) \left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^i \frac{1}{\varphi - \psi} \sum_{j=0}^{N-1} E_W^{(j)}(i+1).$$

2.4.4. Other Performance measures.

(1) Fraction of time the system is down is given by,

$$P_{down} = \sum_{j_1=0}^{\infty} x(j_1, 1, n-k+1) = \frac{\lambda^{n-k+2-N} (\mu^N - \lambda^N)}{\mu^{n-k+1} (\mu - \lambda) (\varphi - \psi)}.$$

(2) System reliability defined as the probability that at least k components are operational

$$P_{rel} = 1 - P_{down} = 1 - \frac{\lambda^{n-k+2-N} (\mu^N - \lambda^N)}{\mu^{n-k+1} (\mu - \lambda) (\varphi - \psi)}.$$

(3) Average number of external units waiting in the queue is given by,

$$\begin{aligned}
 N_q &= \sum_{j_1=0}^{\infty} j_1 \sum_{j_3=1}^{n-k+1} X_{(j_1,1,j_3)} + \sum_{j_1=2}^{\infty} (j_1 - 1) \sum_{j_3=1}^{N-1} X_{(j_1,0,j_3)} \\
 &= \bar{\lambda} \left[\frac{1}{\bar{\mu} - \bar{\lambda}} - \frac{N}{\bar{\mu}(\varphi - \psi)} \right]
 \end{aligned}$$

(4) Average number of failed components of the main system,

$$\begin{aligned}
 N_{fail} &= \sum_{j_3=0}^{N-1} J_3 \left(\sum_{j_1=0}^{\infty} X_{(j_1,0,j_3)} \right) + \sum_{j_3=0}^{n-k+1} j_3 \left(\sum_{j_1=0}^{\infty} X_{(j_1,1,j_3)} \right) \\
 &= \frac{1}{(\varphi - \psi)} \left\{ \frac{N(N-1)}{2} + \sum_{i=1}^{N-1} i \left(\sum_{j=1}^i (\lambda/\mu)^j \right) + \frac{\lambda(\mu^N - \lambda^N)}{\mu^N(\mu - \lambda)} \left(\sum_{i=N}^{n-k+1} i(\lambda/\mu)^{i-N} \right) \right\}
 \end{aligned}$$

(5) Average number of failed components waiting when the server is busy with external customers

$$\begin{aligned}
 &= \sum_{j_3=0}^{N-1} j_3 \left(\sum_{j_1=1}^{\infty} x_{(j_1,0,j_3)} \right) \\
 &= \frac{N(N-1)\bar{\lambda}}{2\bar{\mu}(\varphi - \psi)}
 \end{aligned}$$

(6) Expected number of external customers joining the system,

$$\begin{aligned}
 \theta_3 &= \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=0}^{N-1} X_{(j_1,0,j_3)} \right) \\
 &= N \frac{\bar{\lambda}}{(\varphi - \psi)}.
 \end{aligned}$$

(7) Expected number of external customers, on arrival, getting service directly

$$\begin{aligned}
 &= \bar{\mu} \sum_{j_3=0}^{N-1} x_{(0,0,j_3)} \\
 &= N \frac{(\bar{\mu} - \bar{\lambda})}{(\varphi - \psi)}.
 \end{aligned}$$

(8) Fraction of time the server is busy with external customers,

$$P_{ex.busy} = \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{N-1} x_{(j_1,0,j_3)} \right) = \frac{N \cdot \bar{\lambda}}{\bar{\mu}(\varphi - \psi)}.$$

(9) Probability that the server is found idle,

$$P_{idle} = \sum_{j_3=0}^{N-1} x_{(0,0,j_3)} = N \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}(\varphi - \psi)}.$$

(10) Probability that the server is found busy,

$$P_{busy} = 1 - P_{idle} = 1 - N \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}(\varphi - \psi)}.$$

(11) Expected loss rate of external customers,

$$\theta_4 = \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} x_{(j_1,1,j_3)} \right) = \bar{\lambda} \left(1 - \frac{N}{(\varphi - \psi)} \right).$$

(12) Expected service completion rate of external customers,

$$\begin{aligned}
 \theta_5 &= \bar{\mu} \sum_{j_1=0}^{\infty} \sum_{j_3=0}^{N-1} x_{(j_1,0,j_3)} \\
 &= \frac{N\bar{\mu}}{(\varphi - \psi)}.
 \end{aligned}$$

(13) Expected number of external customers in the system when the server is busy with external customers

$$\theta_6 = \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=0}^{N-1} x_{(j_1,0,j_3)} \right) = \frac{N\bar{\lambda}}{(\bar{\mu} - \bar{\lambda})(\varphi - \psi)}.$$

2.4.5. Another Special case.

Next we consider second special case of the problem discussed in section 4.1, where we take $N = 1$; that is the case where no special policy has been applied for providing service to external customers. Notice that in this case, at most importance is given to the failed components and an external customer can get service only when there are no failed components in the system. Further, an ongoing external customer's service may be pre-empted if a component of the system fails during the service of the former. Since in this case, knowing the number of external as well as the failed components is enough for determining the server status, the Markov chain becomes $\widehat{X}(t) = (X_1(t), X_3(t))$, with state space $\widetilde{S} = \{(j_1, j_2) | j_1 \geq 0, 0 \leq j_2 \leq n - k + 1\}$ and infinitesimal generator

$$\widehat{Q} = \begin{bmatrix} \widetilde{A}_{10} & \widetilde{A}_0 & & & & \\ \widetilde{A}_2 & \widetilde{A}_1 & \widetilde{A}_0 & & & \\ & \widetilde{A}_2 & \widetilde{A}_1 & \widetilde{A}_0 & & \\ & & \dots & \dots & \dots & \\ & & & \dots & \dots & \dots \end{bmatrix}, \text{ where}$$

$$\begin{aligned}\widetilde{A}_{10} &= \lambda E_{n-k+2} + \lambda C_{n-k+2}(n-k+2) \otimes r_{n-k+2}(n-k+2) \\ &+ \mu E'_{n-k+2} + (\mu - \bar{\lambda}) C_{n-k+2}(1) \otimes r_{n-k+2}(1);\end{aligned}$$

\widetilde{A}_0 is a $(n-k+2) \times (n-k+2)$ matrix whose $(1, 1)$ entry is $\bar{\lambda}$ and all other entries are zeroes;

\widetilde{A}_2 is a $(n-k+2) \times (n-k+2)$ matrix whose $(1, 1)$ entry is $\bar{\mu}$ and all other entries are zeroes;

$$\widetilde{A}_1 = \widetilde{A}_{10} - \bar{\mu} C_{n-k+2}(1) \otimes r_{n-k+2}(1).$$

Let $\widetilde{A} = \widetilde{A}_0 + \widetilde{A}_1 + \widetilde{A}_2$; then

$$\widetilde{A} = \lambda E_{n-k+2} + \lambda C_{n-k+2}(n-k+2) \otimes r_{n-k+2}(n-k+2) + \mu E_{n-k+2} + \mu C_{n-k+2}(1) \otimes r_{n-k+2}(1)$$

The stationary probability vector $\widehat{\Pi} = (\widehat{\pi}_{(0,0)}, \widehat{\pi}_{(0,1)}, \dots, \widehat{\pi}_{(0,N-1)}, \widehat{\pi}_{(1,1)}, \dots, \widehat{\pi}_{(1,N)}, \dots, \widehat{\pi}_{(1,n-k+1)})$ of the generator matrix \widetilde{A} is given by $\widehat{\pi}_{(1,i)} = \left(\frac{\lambda}{\mu}\right)^i \widehat{\pi}_{(0,0)}$, $i = 1, 2, \dots, n-k+1$, where

$$\widehat{\pi}_{(0,0)} = \frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

Here again, from the condition $\widehat{\pi} \widetilde{A}_0 e < \widehat{\pi} \widetilde{A}_2 e$, it can be easily verified that the necessary and sufficient condition for the stability of the Markov chain $\widehat{X}(t)$ is $\bar{\lambda} < \bar{\mu}$.

Applying the same technique as in section 4.2.2, we can easily prove that the vector $\widehat{X} = (\widehat{X}(0), \widehat{X}(1), \widehat{X}(2), \dots)$, with $\widehat{X}(i) = \left(1 - \frac{\bar{\lambda}}{\bar{\mu}}\right) \left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^i \widehat{\Pi}$, is the steady state probability vector for the matrix \widehat{Q} .

Performance Measures for the case $N = 1$

(1) Fraction of time the system is down,

$$P_{down} = \sum_{j_1=0}^{\infty} x(j_1, 1, n-k+1) = \frac{\lambda^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(2) System reliability,

$$P_{rel} = 1 - P_{down} = 1 - \sum_{j_1=0}^{\infty} x(j_1, 1, n-k+1) = \frac{\mu(\mu^{n-k+1} - \lambda^{n-k+1})}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(3) Average number of customers waiting in the queue,

$$\begin{aligned} N_q &= \sum_{j_1=2}^{\infty} X_{(j_1,0,1)} + \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=1}^{n-k+1} x(j_1, 1, j_3) \right) \\ &= \frac{\bar{\mu}}{(\bar{\mu} - \bar{\lambda})} \frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})} \left[\left(\frac{\bar{\lambda}}{\bar{\mu}} \right)^2 + \frac{\lambda(\mu^{n-k+1} - \lambda^{n-k+1})}{\mu^{n-k+1}(\mu - \lambda)} \right] \end{aligned}$$

(4) Average number of failed components,

$$N_{fail} = \sum_{j_3=1}^{n-k+1} J_3 \left(\sum_{j_1=0}^{\infty} X_{(j_1,1,j_3)} \right) = \frac{\lambda \mu^{n-k+2}}{(\mu - \lambda)(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(5) Expected number of external customers joining the system in unit time,

$$\theta_3 = \bar{\lambda} \sum_{j_1=0}^{\infty} x_{(j_1,0,0)} = \frac{\bar{\lambda} \mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(6) Expected number of external customers, on arrival, getting service directly

$$\begin{aligned}
 &= \bar{\mu}x_{(0,0,0)} \\
 &= \frac{(\bar{\mu} - \bar{\lambda})\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.
 \end{aligned}$$

(7) Fraction of time the server is busy with external customers,

$$\begin{aligned}
 P_{ex.busy} &= \sum_{j_1=0}^{\infty} x_{(j_1,0,0)} \\
 &= \frac{\bar{\lambda}\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.
 \end{aligned}$$

(8) Probability that the server is idle,

$$P_{idle} = x_{(0,0,j_3)} = \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}} \frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(9) Probability that the server is found busy,

$$P_{busy} = 1 - P_{idle} = 1 - \frac{(\bar{\mu} - \bar{\lambda})}{\bar{\mu}} \frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(10) Expected loss rate of external customers,

$$\theta_4 = \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} x_{(j_1,1,j_3)} \right) = \bar{\lambda} \frac{(\mu^{n-k+1} - \lambda^{n-k+1})}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(11) Expected service completion rate of external customers,

$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} x_{(j_1,0,0)} = \bar{\mu} \frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

(12) Expected number of external customers in the system when the server is busy with external customers

$$\theta_6 = \sum_{j_1=0}^{\infty} j_1 x_{(j_1,0,0)} = \bar{\lambda} \frac{\mu^{n-k+1}(\mu - \lambda)}{(\mu^{n-k+2} - \lambda^{n-k+2})}.$$

2.5. Numerical illustrations

Here, we perform a numerical study on the effect of the N -policy on the system performance. Unless otherwise stated, the parameter values for the numerical study are the following: $\bar{\lambda} = 3.2$, $\mu = 5.5$, $\bar{\mu} = 8$.

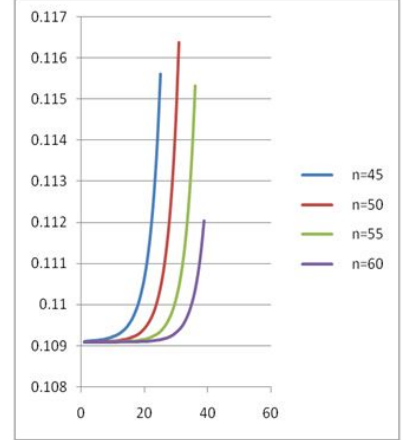
2.5.1. Effect of the N -policy on the probability that server is busy with external customers.

While studying a k -out-of- n system, where the server provides service to external customers also, the main purpose of N -policy is to provide improved attention to external customers for optimizing the system revenue. According to the N -policy considered here, the moment the number of failed components of the main system reaches N , the external customer's service ('if there is any') is pre-empted to attend the failed components. Hence, an increase in the value of N will extend the time during which external customers can get service and so it is expected that the probability that the server is

busy with external customers increases with an increase in the value of N . The column wise increase in Table 2.1 supports this intuition. The high service rate for the external customers, as compared to their arrival rate can be considered as the reason for the slow increase in the above probability. The row wise decrease in Table 2.1 points to the decrease in the probability that the server is busy with external customers with an increase in the total number of components in the system. We have the following reasoning for this behavior: With an increase in the total number of components n in the system, there can be more number of failed components in the system for a fixed N , which leads to an increase in the probability that the server is attending failed components, resulting in a decrease in the probability $P_{ex.busy}$. A closer scrutiny of Table 2.1 shows that, by increasing the policy level N with an increase in the number of components n , the same value for the fraction $P_{ex.busy}$ can be achieved as that when n has a lesser value. For example, when $n = 45$ and $N = 7$, $P_{ex.busy} = 0.10915$ and $P_{ex.busy} = 0.10909$, when $n = 60$ with the same N . Now with $n = 60$ and when N is increased to 25, we see that $P_{ex.busy} = 0.10915$. This suggests that, when n increases, the N -policy level can be adjusted in favor of the external customers, which was our objective while introducing the N policy. However, when N increases, it is probable that the server spends more time for failed components, once he starts attending them, which leads to a loss of the external customers who finds the server busy with internal customers. In Table 2.1, one can see that the probability $P_{ex.busy}$ has a lesser value when $n = 60$, $N = 30$ than in the case when $n = 45$, $N = 15$, which points to the loss of external customers. Another challenge here is that, while increasing the N -policy level, the system reliability is not affected significantly.

Table 2.1. Dependence of the probability $P_{ex.busy}$ on the N -policy level

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$
1	0.10910	0.10909	0.10909	0.10909
2	0.10910	0.10910	0.10909	0.10909
3	0.10912	0.10910	0.10909	0.10909
6	0.10914	0.10910	0.10909	0.10909
7	0.10915	0.10910	0.10909	0.10909
10	0.10922	0.10912	0.10910	0.10909
11	0.10925	0.10912	0.10910	0.10909
12	0.10929	0.10913	0.10910	0.10910
15	0.10952	0.10918	0.10911	0.10910
18	0.11002	0.10928	0.10913	0.10910
21	0.11118	0.10952	0.10918	0.10911
22	0.11185	0.10965	0.10921	0.10912
23	0.11275	0.10982	0.10925	0.10913
24	0.11397	0.11006	0.10929	0.10914
25	0.11562	0.11037	0.10935	0.10915
26		0.11078	0.10944	0.10917
27		0.11134	0.10955	0.10919
28		0.11209	0.10970	0.10922
29		0.11310	0.10989	0.10926
30		0.11448	0.11016	0.10932
31		0.11638	0.11051	0.10939



2.5.2. Effect of the N -policy on the system reliability.

In the previous section, we discussed how N -policy helps in longer duration of attention to external customers and the challenge there is the possibility of a decrease in the system reliability. Here we discuss how the N -policy level affects the system reliability P_{rel} . We study two cases with $\frac{\lambda}{\mu} < 1$ and $\frac{\lambda}{\mu} > 1$ respectively, results of which are given in Table 2.2(a) and (b) respectively. While studying the impact of the N -policy on the

system reliability, a decrease in P_{rel} is expected with an increase in value of N . Hence, the purpose of the Tables 2.2(a) and (b) is to show the magnitude of this impact. Table 2.2(a) shows that when $\frac{\lambda}{\mu} < 1$, $n = 45$ and when N increased from 3 to 25, there is a decrease in reliability of magnitude equal to 0.02. As the total number of components n increases, the magnitude of decrease in reliability reduces. This is because, when n increases, k being fixed, $n - k + 1$ increases; as a result, once the server starts attending the failed components on accumulation of N of them, he spends more time for the failed components, which maintains a high system reliability even when N increases. In Table 2.1 we have seen that as n increases, the probability $P_{ex.busy}$ decreases and that increasing the N -policy level can remedy this to some extent; Table 2.2(a) shows that the reliability of the system is not much affected by increasing the N -policy level. However, the magnitude of drop in the system reliability increases with the increase in N -policy level. Table 2.2(b) studies the system reliability when the failure rate of the components λ is larger than their repair rate μ . As expected, there is a drop in the system reliability compared to the case $\lambda < \mu$. Other behaviour of the system reliability are similar to that in Table 2.2(a).

Table 2.2. (a): Dependence of the system reliability on the N -policy level in the $\lambda < \mu$ case $\lambda = 4$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.999930799	0.999985933	0.999997139	0.999999404	0.999999881
3	0.999901652	0.999979973	0.999995947	0.999999166	0.999999821
5	0.999855518	0.999970615	0.999994040	0.999998808	0.999999762
9	0.999660194	0.999930918	0.999985933	0.999997139	0.999999404
13	0.999121249	0.999821544	0.999963701	0.999992609	0.999998510
17	0.997560024	0.999506116	0.999899626	0.999979556	0.999995828
21	0.992828071	0.998562694	0.999708474	0.999940693	0.999987960
25	0.977587163	0.995647013	0.999122441	0.999821782	0.999963760
26		0.994222760	0.998838782	0.999764323	0.999952078
29		0.986251056	0.997281969	0.999450147	0.999888241
31		0.974976659	0.995165646	0.999026358	0.999802291
34			0.984254420	0.996900022	0.999531090
35			0.978649259	0.995844364	0.999373376
38				0.989870846	0.998496175
39				0.986294508	0.979825020
40				0.981382251	0.972903130
41					0.996356070
45					0.987866700
46					0.983495116

Table 2.2. (b): Dependence of the system reliability on the N -policy level in the $\lambda > \mu$ case $\lambda = 6$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.907874525	0.911180377	0.913196325	0.914452970	0.915247083
3	0.907009840	0.910661936	0.912876606	0.914252222	0.915119767
5	0.906079888	0.910108566	0.912536800	0.914039671	0.914985061
9	0.904014528	0.908894181	0.911796451	0.913578153	0.914693415
11	0.902873158	0.908231616	0.911395609	0.913329482	0.914536774
13	0.901655436	0.907531500	0.910974264	0.913069129	0.914373279
17	0.898979187	0.906016290	0.910070777	0.912513614	0.914025128
21	0.895960152	0.904344857	0.909087002	0.911913455	0.913650930
25	0.892570674	0.902514517	0.908024848	0.911270797	0.913252294
26		0.902032018	0.907747209	0.911103785	0.913149118
29		0.900522947	0.906886399	0.910588324	0.912831187
31		0.89946568	0.906289339	0.910232842	0.912612915
34			0.905359924	0.909682870	0.912276387
35			0.905041218	0.909495234	0.912169460
38				0.908919990	0.911812007
39				0.908724248	0.911693335
40				0.908526540	0.911573648
41				0.908326924	0.911453009
45					0.910961330
46					0.910836279

2.5.3. Cost analysis.

In sections 1.5.1 and 1.5.2, we have seen that by increasing N , we can provide uninterrupted service over a long duration to more external customers and without compromising the system reliability significantly. However, the magnitude of decrease in the system reliability increases with N . Hence, it is worth finding whether there exists an optimal value for the N -policy level. For this, we construct the following cost function. Let C_1 be the cost per unit time incurred if the system is down; C_2 , the holding cost per unit time per external customer in the queue; C_3 is the cost incurred towards set up (instantaneous) of the server to serve main customers; C_4 be the cost due to loss of an external customer, C_5 , be the holding cost per unit time of one failed component and C_6 be the cost per unit idle time.

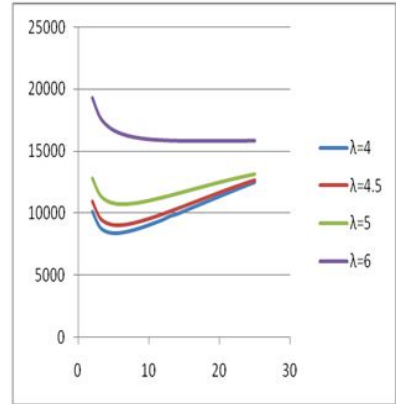
$$\text{Expected Cost per unit time} = C_1 \cdot P_{down} + C_2 \cdot N_q + C_4 \cdot \theta_4 + C_5 \cdot N_{fail} + \left(\frac{C_3}{E_B} \right) + C_6 \cdot P_{idle}.$$

Table 2.3 studies the variation of cost function as N varies. We study the cost function for different failure rates of the components. In all the 4 cases studied, for the various costs assumed, we get a concave nature for the cost curve, which gives an optimal value for N . Table 2.3 shows that when $\lambda < \mu$, the optimal values for N are 5,6 and 6 when λ equal to 4, 4.5 and 5 respectively; whereas when $\lambda = 6 > 5.5 = \mu$, we get a much higher optimal value 18 for N . This is as expected, since when λ is greater than μ , there will be a heavier traffic of failed components so that the server has to spend more time attending the failed components. Hence, the policy level N needs to be increased to a much higher value than in the $\lambda < \mu$ situation, for the system to earn maximum profit. Also note that

the optimal value of the cost function is much higher in the $\lambda > \mu$ case, when compared to the opposite situation.

Table 2.3. Variation in the cost function $n = 50, k = 20, C_1 = 2000, C_2 = 1000, C_3 = 1600, C_4 = 1000, C_5 = 500, C_6 = 100$

N	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$	$\lambda = 6$
2	10139.47	10923.82	12783.28	19330.75
3	8910.199	9663.817	11496.69	17827.49
4	8489.626	9199.57	10981.16	17095.61
5	8370.844	9038.382	10764.71	16671.58
6	8396.2	9024.268	10694.5	16401.28
7	8500.631	9092.232	10706.09	16218.57
8	8652.372	9210.307	10767.47	16090.46
12	9474.447	9919.942	11245.17	15847.97
14	9939.594	10337.5	11542.35	15805.31
16	10416.93	10769	11849.9	15786.79
17	10657.6	10986.45	12003.95	15783.35
18	10898.62	11203.53	12156.71	15782.6
19	11139.36	11419.23	12307.2	15783.87
20	11379.19	11632.57	12454.51	15786.61
23	12085.88	12248.03	12868.57	15799.16
24	12313.97	12441.01	12994.22	15803.64
25	12536.01	12625.49	13111.97	15807.79



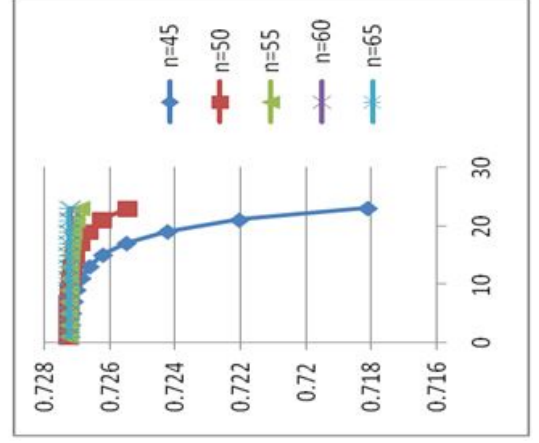
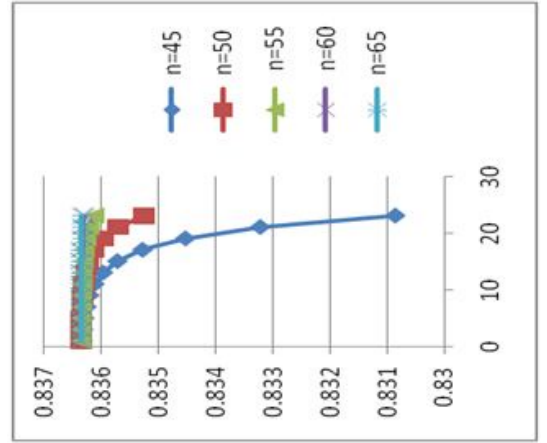
2.5.4. Comparison with a k -out- n system where no external customers are serviced.

Here we compare the model discussed above with another model where no external customers are allowed but N -policy is maintained. Notice that because of the assumption of the preemption of service of an external customer on accumulation of N failed components, the two systems will have the same reliability. The nature of the steady state distribution obtained in Theorem 2.4.2 further substantiates this claim.

Hence, it can be concluded that the external customers when allowed as in this study, utilizes the server idle time without affecting the performance of the k -out-of- n system. In Table 2.4, we present the results of the numerical study conducted for comparing the increase in the server busy probability, when external customers are allowed. In that Table, case 1 refers to the model discussed above and case 2 stands for k -out-of- n system where no external customers are allowed. Table 2.4 shows that when external customers are allowed, there is an increase, of magnitude 0.11, in the server busy probability.

Table 2.4. Variation in the server busy probability

Case 1		Case 2									
$\lambda = 4$		$\lambda = 4$									
N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$	N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.83633	0.83635	0.83636	0.83636	0.83636	1	0.72722	0.72726	0.72727	0.72727	0.72727
3	0.83632	0.83635	0.83636	0.83636	0.83636	3	0.7272	0.72726	0.72727	0.72727	0.72727
5	0.8363	0.83635	0.83636	0.83636	0.83636	5	0.72717	0.72725	0.72727	0.72727	0.72727
7	0.83626	0.83634	0.83635	0.83636	0.83636	7	0.72711	0.72724	0.72727	0.72727	0.72727
9	0.83621	0.83633	0.83635	0.83635	0.83635	9	0.72703	0.72722	0.72726	0.72727	0.72727
11	0.83612	0.83631	0.83634	0.83635	0.83635	11	0.72688	0.72719	0.72726	0.72727	0.72727
13	0.83597	0.83627	0.83634	0.83635	0.83635	13	0.72663	0.72714	0.72725	0.72727	0.72727
15	0.83572	0.83622	0.83632	0.83635	0.83635	15	0.72622	0.72706	0.72723	0.72726	0.72727
17	0.83528	0.83613	0.83631	0.83634	0.83635	17	0.7255	0.72691	0.7272	0.72726	0.72727
19	0.83453	0.83598	0.83627	0.83633	0.83634	19	0.72425	0.72666	0.72715	0.72725	0.72727
21	0.83322	0.83572	0.83622	0.83632	0.83634	21	0.72206	0.72623	0.72706	0.72723	0.72726
23	0.83086	0.83526	0.83613	0.8363	0.83634	23	0.71814	0.72546	0.72691	0.7272	0.72726



Chapter 3

Reliability of a k -out-of- n system with a single server extending non-preemptive service to external customers

3.1. Introduction

In the previous chapter we analysed a k -out-of- n system with repair of failed components under N -policy. The repair facility is also extended to external customers. However, we assumed pre-emption of service to external customers as soon as N failed components of the k -out-of- n system accumulated in a new cycle. In this chapter the pre-emption part is done away with. As a consequence the reliability of the k -out-of- n

⁰This Chapter is to be published as two papers titled: 1. Reliability of a k -out-of- n system with a single server extending non-preemptive service to external customers-Part I and 2. Reliability of a k -out-of- n system with a single server extending non-preemptive service to external customers-Part II in Electronic Journal "Reliability: Theory and Applications" (Gnedenko forum, September 2016)

system decreases if we retain the same N value that provided high system reliability in the previous chapter.

In this chapter, we consider two variants of the model in section 2.4 of chapter 2. In both models, we assume N -policy for starting repair of failed components. However, the priority given to main customers is reduced by assuming that an ongoing service of an external customer is not preempted when the number of failed components reaches N . This can be a serious compromise on the reliability of the k -out-of- n system. As in section 2.4 of chapter 2, it has been assumed that an external customer, arriving when the server is busy with service of main customers and/or when there are at least N failed components in the system, is not allowed to join the system. In the first model the external customer joins a queue of infinite capacity; where as in the second model it joins an orbit of infinite capacity and retries for service from there.

3.2. The queueing model

Here we consider a k -out-of- n system with a single server, offering service to external customers also. Commencement of service to failed components of the main system is governed by N -policy. That is at the epoch the system starts with all components operational, the server starts attending one by one the external customers (if there is any). When the number of failed components in the system is $\geq N$, the server in service of external customer (if there is any) is switched on to the service of the main customers after completing the ongoing service of the external customer. Arrival of main customers and external customers have inter occurrence times exponentially distributed with parameters λ and $\bar{\lambda}$ respectively. External customers are not allowed to join the system when the server is busy with main customers or when there is $\geq N$ failed components. An external customer, who on arrival finds an idle server is directly taken for

service. Service times of main and external customers follow exponential distribution with parameters μ and $\bar{\mu}$ respectively.

3.2.1. The Markov Chain. Let $X_1(t)$ = number of external customers in the system including the one getting service (if any) at time t ,

$X_2(t)$ = number of main customers in the system including the one getting service (if any) at time t ,

$$S(t) = \begin{cases} 0, & \text{if the server is idle or is busy with external customers} \\ 1, & \text{if the server is idle or is busy with main customers.} \end{cases}$$

Let $X(t) = (X_1(t), S(t), X_2(t))$ then $X = \{X(t), t \geq 0\}$ is a continuous time Markov chain on the state space

$$S = \{(0, 0, j_2)/0 \leq j_2 \leq N - 1\} \cup \{(j_1, 0, j_2)/j_1 \geq 1, 0 \leq j_2 \leq n - k + 1\}$$

$$\cup \{(j_1, 1, j_2)/j_1 \geq 0, 1 \leq j_2 \leq n - k + 1\}.$$

Arranging the states lexicographically and partitioning the state space into levels i , where each level i corresponds to the collection of the states with number of external customers in the system at any time t equal to i , we get an infinitesimal generator of the above chain as

$$Q = \begin{bmatrix} A_{10} & A_{00} & & & & \\ A_{20} & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & & \dots & \\ & & & & & \dots \end{bmatrix}$$

In order to describe the entries in the above matrix we introduce some notations below.

(i) I_m denotes an identity matrix of order m and I denotes an identity matrix of appropriate order.

(ii) e_m denotes a $m \times 1$ column matrix of 1s and e denotes a column matrix of 1s of appropriate order.

(iii) E_m denotes a square matrix of order m defined as

$$E_m(i, j) = \begin{cases} -1 & \text{if } j = i, 1 \leq i \leq m \\ 1 & \text{if } j = i + 1, 1 \leq i \leq m - 1 \\ 0 & \text{otherwise} \end{cases}$$

(iv) $E'_m = \text{Transpose}(E_m)$

(v) $r_m(i)$ denotes a $1 \times m$ row matrix whose i th entry is 1 and all other entries are zeros

(vi) $c_m(i) = \text{Transpose}(r_m(i))$

(vii) \otimes denotes Kronecker product of matrices.

The transition within level 0 is represented by the matrix

$$A_{10} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \text{ where}$$

$$B_1 = \lambda E_N - \bar{\lambda} I_N$$

B_2 is a $N \times (n - k + 1)$ matrix whose (N, N) th entry is λ and all other entries are zeroes.

B_3 is a $N \times (n - k + 1)$ matrix whose $(1, 1)$ th entry is μ and all other entries are zeroes.

$$B_4 = \lambda E_{n-k+1} + \lambda c_{n-k+1}(N-k+1) \otimes r_{n-k+1}(n-k+1) + \mu E'_{n-k+1}.$$

The transition from level 0 to level 1 is represented by the matrix

$$A_{00} = \begin{bmatrix} \bar{\lambda} I_N & O_{N \times (2n-2k+3-N)} \\ O_{(n-k+1) \times N} & O_{(n-k+1) \times (2n-2k+3-N)}. \end{bmatrix}$$

Transition from level 1 to 0 is represented by the matrix

$$A_{20} = \begin{bmatrix} \bar{\mu} I_N & O \\ O & H \\ O_{(n-k+1) \times N} & O \end{bmatrix} \text{ where } H = \begin{bmatrix} O_{(n-k+2-N) \times (N-1)} & \bar{\mu} I_{(n-k+2-N)}. \end{bmatrix}$$

Transition within level 1 is represented by the matrix

$$A_1 = \begin{bmatrix} H_{11} & H_{12} & 0 \\ 0 & H_{22} & 0 \\ H_{31} & 0 & B_4 \end{bmatrix} \text{ where}$$

$$H_{11} = B_1 - \bar{\mu} I_N, H_{12} = \lambda c_N(N) \otimes r_{n-k+2-N}(1),$$

$$H_{22} = \lambda E_{n-k+2-N} + \lambda c_{n-k+2-N}(n-k+2-N) \otimes r_{n-k+2-N}(n-k+2-N) - \bar{\mu} I_{n-k+2-N}.$$

H_{31} is an $(n-k+1) \times N$ matrix whose $(1, 1)^{\text{th}}$ entry is μ .

$$A_0 = \begin{bmatrix} \bar{\lambda} I_N & O_{N \times (2n-2k+3-N)} \\ O_{(2n-2k+3-N) \times N} & O_{(2n-2k+3-N) \times (2n-2k+3-N)}. \end{bmatrix}.$$

$$A_2 = \begin{bmatrix} \bar{\mu}I_N & O & O \\ O & O_{(n-k+2-N) \times (n-k+2-N)} & \tilde{H} \\ O_{(n-k+1) \times N} & O & O \end{bmatrix}.$$

where $\tilde{H} = \begin{bmatrix} O_{(n-k+2-N) \times (N-1)} & \bar{\mu}I_{(n-k+2-N)} \end{bmatrix}$.

3.3. Steady state analysis

3.3.1. Stability condition. Consider the generator matrix $A = A_0 + A_1 + A_2$

$$A = \begin{bmatrix} \lambda E_N & H_{12} & 0 \\ 0 & H_{22} & F_{23} \\ F_{31} & 0 & B_4 \end{bmatrix}$$

$$F_{23} = \begin{bmatrix} O_{(n-k+2-N) \times (N-1)} & \bar{\mu}I_{n-k+2-N} \end{bmatrix},$$

$$F_{31} = \mu c_{n-k+1}(1) \otimes r_N(1).$$

Let $\zeta = (\zeta_0, \zeta_1, \zeta_2)$ be the steady state vector of the generator matrix A , where

$$\zeta_0 = (\zeta_{(0,0)}, \zeta_{(0,1)}, \dots, \zeta_{(0,N-1)}), \zeta_1 = (\zeta_{(0,N)}, \zeta_{(0,N+1)}, \dots, \zeta_{(0,n-k+1)}),$$

$$\zeta_2 = (\zeta_{(1,1)}, \zeta_{(1,2)}, \dots, \zeta_{(1,n-k+1)}).$$

The Markov chain $\{X(t), t \geq 0\}$ is stable if and only if $\zeta A_0 e < \zeta A_2 e$

(see Neuts [17]).

It follows that $\zeta A_0 e = \bar{\lambda} \zeta_0 e$ and $\zeta A_2 e = \bar{\mu}(\zeta_0 e + \zeta_1 e)$. Therefore the stability condition becomes

$$\frac{\bar{\lambda}}{\bar{\mu}} \frac{\zeta_0 e}{(\zeta_0 e + \zeta_1 e)} < 1. \quad (3.3.1)$$

It follows from the relation $\zeta A = 0$ that

$$\zeta_0 \lambda E_N + \zeta_2 F_{31} = 0, \quad (3.3.2)$$

$$\zeta_0 H_{12} + \zeta_1 H_{22} = 0, \quad (3.3.3)$$

$$\zeta_1 F_{23} + \zeta_2 B_4 = 0. \quad (3.3.4)$$

From (3.3.4), it follows that

$$\zeta_2 = -\zeta_1 F_{23} B_4^{-1}. \quad (3.3.5)$$

Substituting this in (3.3.2) we get

$$\zeta_0 \lambda E_N - \zeta_1 F_{23} B_4^{-1} F_{31} = 0. \quad (3.3.6)$$

$$\lambda \zeta_0 e = (-\zeta_1 F_{23} B_4^{-1} F_{31})(-E_N^{-1} e). \quad (3.3.7)$$

Notice that the first column of the matrix F_{31} is $-B_4 e$ and all other columns of it are zero columns. This implies that the first column of the matrix $B_4^{-1} F_{31}$ is $-e$ and its all other columns are zero columns. Hence the first column of the matrix $-F_{23} B_4^{-1} F_{31}$ is $\bar{\mu} e$ and all other columns are zero columns. The first entry of the row matrix $-\zeta_1 F_{23} B_4^{-1} F_{31}$ is thus $\bar{\mu} \zeta_1 e$ and its all other entries are zeros. It can be seen that the first entry of the

column matrix $-E_N^{-1}e$ is N . These two facts together tell us that $(-\zeta_1 F_{23} B_4^{-1} F_{31})(-E_N^{-1}e)$ is $N\bar{\mu}\zeta_1 e$. Thus, equation (3.3.7) becomes

$$\lambda\zeta_0 e = N\bar{\mu}\zeta_1 e.$$

Adding $N\bar{\mu}\zeta_0 e$ on both sides of the above equation, we get

$$(\lambda + N\bar{\mu})\zeta_0 e = N\bar{\mu}(\zeta_0 e + \zeta_1 e),$$

which implies

$$\frac{\zeta_0 e}{(\zeta_0 e + \zeta_1 e)} = \frac{N\bar{\mu}}{(\lambda + N\bar{\mu})}.$$

Hence the stability condition (3.3.1) becomes

$$\frac{\bar{\lambda}}{\bar{\mu}} \frac{N\bar{\mu}}{(\lambda + N\bar{\mu})} < 1.$$

3.3.2. Computation of steady state vector. Let $\pi = (\pi(0), \pi(1), \pi(2), \dots)$ the steady state vector of the Markov chain X where $\pi(0) = (\pi_{(0,0)}, \pi_{(0,1)})$, with $\pi_{(0,0)} = (\pi_{(0,0,0)}, \pi_{(0,0,1)}, \dots, \pi_{(0,0,N-1)})$ and $\pi_{(0,1)} = (\pi_{(0,1,1)}, \dots, \pi_{(0,1,n-k+1)})$. For

$$i \geq 1, \pi(i) = (\pi_{(i,0)}, \tilde{\pi}_{(i,0)}, \pi_{(i,1)}),$$

$$\pi_{(i,0)} = (\pi_{(i,0,0)}, \pi_{(i,0,1)}, \dots, \pi_{(i,0,N-1)}),$$

$$\tilde{\pi}_{(i,0)} = (\pi_{(i,0,N)}, \pi_{(i,0,N+1)}, \dots, \pi_{(i,0,n-k+1)}),$$

$$\pi_{(i,1)} = (\pi_{(i,1,1)}, \pi_{(i,1,2)}, \dots, \pi_{(i,1,n-k+1)}).$$

Now from $\pi Q = 0$, we can write

$$\pi_{(0,0)}B_1 + \pi_{(0,1)}B_3 + \pi_{(1,0)}\bar{\mu}I_N = 0, \quad (3.3.8)$$

$$\pi_{(0,0)}B_2 + \pi_{(0,1)}B_4 + \tilde{\pi}_{(1,0)}H = 0, \quad (3.3.9)$$

For $i \geq 1$,

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i,1)}H_{31} + \pi_{(i+1,0)}\bar{\mu}I_N = 0, \quad (3.3.10)$$

$$\pi_{(i,0)}H_{12} + \tilde{\pi}_{(i,0)}H_{22} = 0, \quad (3.3.11)$$

$$\pi_{(i,1)}B_4 + \tilde{\pi}_{(i+1,0)}\tilde{H} = 0. \quad (3.3.12)$$

From (3.3.11), we get, for $i \geq 1$

$$\tilde{\pi}_{(i,0)} = -\pi_{(i,0)}H_{12}(H_{22}^{-1}). \quad (3.3.13)$$

From (3.3.12), we get

$$\pi_{(i,1)} = -\tilde{\pi}_{(i+1,0)}\tilde{H}(B_4^{-1}). \quad (3.3.14)$$

Substituting (3.3.13) in (3.3.14), we get

$$\pi_{(i,1)} = \pi_{(i+1,0)}H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1}). \quad (3.3.15)$$

Substituting (3.3.15) in (3.3.10), we get

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i+1,0)}H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31} + \pi_{(i+1,0)}\bar{\mu}I_N = 0. \quad (3.3.16)$$

We notice that the first column of the matrix H_{31} is $-B_4e$ and all other columns of H_{31} are zero columns. Hence the first column of the matrix $(B_4^{-1})H_{31}$ is $-e$ and its all other columns are zero columns. This tells us that the first column of the matrix $\tilde{H}(B_4^{-1})H_{31}$ is $-\bar{\mu}e$ and all other columns are zeros. But $-\bar{\mu}e$ is $H_{22}e$ and hence the first column of the matrix $(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31}$ is e and all other columns are zeros. This fact leads us to conclude that the first column of the matrix $H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31}$ is $H_{12}e = \lambda c_N(N)$ and all other columns are zeros. In other words

$$H_{12}(H_{22}^{-1})\tilde{H}(B_4^{-1})H_{31} = \lambda c_N(N) \otimes r_N(1).$$

Now equation (3.3.16) becomes

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i+1,0)}\lambda c_N(N) \otimes r_N(1) + \pi_{(i+1,0)}\bar{\mu}I_N = 0.$$

That is

$$\pi_{(i-1,0)}\bar{\lambda}I_N + \pi_{(i,0)}H_{11} + \pi_{(i+1,0)}(\lambda c_N(N) \otimes r_N(1) + \bar{\mu}I_N) = 0. \quad (3.3.17)$$

Now from equation (3.3.9), we can write

$$\pi_{(0,1)} = -\pi_{(0,0)}B_2(B_4^{-1}) - \tilde{\pi}_{(1,0)}H(B_4^{-1}). \quad (3.3.18)$$

However, from equation (3.3.13), we have

$$\tilde{\pi}_{(1,0)} = -\pi_{1,0}H_{12}(H_{22}^{-1}). \quad (3.3.19)$$

Hence equation (3.3.18) becomes

$$\pi_{(0,1)} = -\pi_{(0,0)}B_2(B_4^{-1}) + \pi_{(1,0)}H_{12}(H_{22}^{-1})H(B_4^{-1}). \quad (3.3.20)$$

Substituting (3.3.20) in (3.3.8), we get

$$\pi_{(0,0)}B_1 + (-\pi_{(0,0)}B_2(B_4^{-1}) + \pi_{(1,0)}H_{12}(H_{22}^{-1})H(B_4^{-1}))B_3 + \pi_{(1,0)}\bar{\mu}I_N = 0. \quad (3.3.21)$$

Since the first column of the matrix B_3 is $-B_4e$, a similar reasoning as for equation (3.3.16) leads us to write:

$$-B_2(B_4^{-1})B_3 = \lambda c_N(N) \otimes r_N(1).$$

$$H_{12}(H_{22}^{-1})H(B_4^{-1})B_3 = \lambda c_N(N) \otimes r_N(1).$$

Hence equation (3.3.21) becomes

$$\pi_{(0,0)}(B_1 + \lambda c_N(N) \otimes r_N(1)) + \pi_{(1,0)}(\lambda c_N(N) \otimes r_N(1) + \bar{\mu}I_N) = 0. \quad (3.3.22)$$

Equations (3.3.17) and (3.3.22) shows that the vector $\hat{\pi} = (\pi_{(0,0)}, \pi_{(1,0)}, \pi_{(2,0)}, \dots)$ satisfies the relation $\tilde{\pi}\tilde{Q} = 0$, where \tilde{Q} is a generator matrix defined as

$$\tilde{Q} = \begin{bmatrix} \tilde{A}_{10} & \tilde{A}_0 & & & & \\ \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & & & \\ & \tilde{A}_2 & \tilde{A}_1 & \tilde{A}_0 & & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \end{bmatrix}$$

In the above, $\tilde{A}_{10} = B_1 + \lambda c_N(N) \otimes r_N(1)$, $\tilde{A}_0 = \bar{\lambda}I_N$, $\tilde{A}_1 = H_{11}$ and $\tilde{A}_2 = \lambda c_N(N) \otimes r_N(1) + \bar{\mu}I_N$. Hence the vector $\hat{\pi}$ is a constant multiple of the steady state vector $\tau = (\tau(0), \tau(1), \dots)$ of the generator matrix \tilde{Q} . The vector τ can be obtained by applying the matrix analytic methods (see Neuts [17]) as

$$\tau(i) = \tau(0)R^i, \quad i \geq 0, \quad (3.3.23)$$

where the matrix R is the minimal non-negative solution of the matrix quadratic equation:

$$A_0 + RA_1 + R^2A_2 = 0. \quad (3.3.24)$$

Equation (3.3.23) implies

$$\pi(0, 0) = \mathcal{K}\tau(0),$$

$$\pi(i, 0) = \pi(0, 0)R^i, \quad i \geq 0.$$

Now the vector $\hat{\pi}$ is obtained up to a constant \mathcal{K} as $\hat{\pi} = \mathcal{K}\tau$, the other component vectors $\tilde{\pi}_{(i,0)}, i \geq 1, \pi_{(i,1)}, i \geq 0$ of π can be obtained from the equations (3.3.13), (3.3.14) and (3.3.20), up to the constant \mathcal{K} , which is finally obtained from the normalizing condition $\pi e = 1$.

3.4. Performance measures

3.4.1. Busy period of the server with the failed components of the main system. Let T_i denote the server busy period with failed components which starts with i failed components and with j external customers in the system. Consider the absorbing Markov chain $Y = \{Y(t), t \geq 0\}$, where $Y(t)$ is the number of failed components

of the main system, with the state space $\{0, 1, 2, \dots, N, N + 1, \dots, n - k + 1\}$ and having infinitesimal matrix given by

$$\tilde{H}_{BF} = \begin{bmatrix} 0 & 0 \\ -H_{BF}e & H_{BF} \end{bmatrix},$$

where $H_{BF} = \lambda E_{n-k+1} + \lambda c_{n-k+1}(n - k + 1) \otimes r_{n-k+1} + \mu E'_{n-k+1}$.

Note that $Y(t) = 0$ is an absorbing state. T_i is the time until absorption in the Markov chain $\{Y(t)\}$ assuming that it starts at the state i . The expected value ET_i of T_i is therefore the i th entry of the column matrix $-H_{BF}^{-1}e$ as given by (Krishnamoorthy et al. [13]):

$$ET_i = \frac{1}{\mu} \left(i \sum_{j=0}^{n-k+1-i} \left(\frac{\lambda}{\mu}\right)^j + \sum_{j=n-k+2-i}^{n-k} (n - k + 1 - j) \left(\frac{\lambda}{\mu}\right)^j \right)$$

We notice that once the service of failed components starts, the external customers has no effect on it. Define

$$P_f(N) = \pi_{(0,0,N-1)} + \sum_{j=1}^{\infty} \pi_{(j,0,N)} \quad \text{and}$$

$$P_f(i) = \sum_{j=1}^{\infty} \pi_{(j,0,i)} \quad \text{for } N < i \leq n - k + 1$$

$P_f(i)$ will then denote the system steady state probability just before starting service to failed components with i number of failed components. The expected length of the busy period of the server with failed components is then given by

$$E_{\hat{H}} = \frac{\sum_{i=N}^{n-k+1} P_f(i) ET_i}{\sum_{i=N}^{n-k+1} P_f(i)}.$$

3.4.2. Other performance measures.

(1) Fraction of time the system is down,

$$P_{down} = \sum_{j_1=0}^{\infty} \pi_{(j_1,0,n-k+1)} + \sum_{j_1=0}^{\infty} \pi_{(j_1,1,n-k+1)}.$$

(2) System reliability, $P_{rel} = 1 - P_{down}$.

(3) Average number of external customers waiting in the queue,

$$N_q = \sum_{j_i=0}^{\infty} j_i \left(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,1,j_3)} \right) + \sum_{j_1=1}^{\infty} (j_1 - 1) \left(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)} \right).$$

(4) Average number of failed components of the main system,

$$N_{fail} = \sum_{j_3=0}^{n-k+1} j_3 \left(\sum_{j_1=0}^{\infty} \pi_{(j_1,0,j_3)} \right) + \sum_{j_3=1}^{n-k+1} j_3 \left(\sum_{j_1=0}^{\infty} \pi_{(j_1,1,j_3)} \right).$$

(5) Average number of failed components waiting when server is busy with external customers

$$= \sum_{j_3=0}^{n-k+1} j_3 \left(\sum_{j_1=1}^{\infty} \pi_{(j_1,0,j_3)} \right).$$

(6) Expected number of external customers joining the system,

$$\theta_3 = \bar{\lambda} \left\{ \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{N-1} \pi_{(j_1,0,j_3)} \right) + \sum_{j_1=0}^{N-1} \pi_{(0,0,j_3)} \right\}.$$

(7) Expected number of external customers on its arrival gets service directly

$$= \sum_{j_3=0}^{N-1} \pi_{(0,0,j_3)}.$$

(8) Fraction of time the server is busy with external customers,

$$P_{ext,busy} = \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)} \right).$$

(9) Probability that server is found idle,

$$P_{idle} = \sum_{j_3=0}^{N-1} \pi_{(0,0,j_3)} = N\pi_{(0,0,0)}.$$

(10) Probability that the server is found busy,

$$P_{busy} = 1 - \sum_{j_3=0}^{N-1} \pi_{(0,0,j_3)} = 1 - N\pi_{(0,0,0)}.$$

(11) Expected loss rate of external customers,

$$\theta_4 = \bar{\lambda} \left\{ \sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} \pi_{(j_1,1,j_3)} \right) + \sum_{j_1=1}^{\infty} \left(\sum_{j_3=N}^{n-k+1} \pi_{(j_1,0,j_3)} \right) \right\}.$$

(12) Expected service completion rate of external customers,

$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)} \right).$$

(13) Expected number of external customers when server is busy with external customers,

$$\theta_6 = \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=0}^{n-k+1} \pi_{(j_1,0,j_3)} \right).$$

3.5. Numerical Study of the Performance of the System

3.5.1. The Effect of N Policy on the Server Busy Probability. The main purpose of introducing N -policy while studying a k -out-of- n system with a single server offering service to external customers, in a non pre-emptive nature, was optimization of the system revenue, by utilizing the server idle time, without compromising the reliability of the system much. From Tables 3.1 and 3.2, it follows that there is an increase in the server busy probability, when external customers are allowed. 3.3 tells that there is an increase in the fraction of time that the server is busy with external customers with an increase in N . Hence, it can be concluded that the N -policy has helped in improving the attention towards external customers slightly. Now, we want to check whether the introduction of the N -policy has badly affected the system reliability.

3.5.2. The effect of N policy on system reliability. We study two cases $\lambda < \mu$ and $\lambda > \mu$. We expected a decrease in P_{rel} with an increase in N . This is because as N increases, the server spends more time for external customers, which we thought might cause a decrease in the system reliability. This was verified from Table 3.4, where we assumed $\lambda < \mu$. However, Table 3.4 shows very high system reliability over 95%. The magnitude of decrease in reliability was found lesser when the total number of components n was high. In short Table 3.4 shows that reliability of the system is not much affected by increasing N -policy level. In Table 3.5 where it was assumed that the component failure rate λ is greater than their service rate μ , it was again found that P_{rel} decreases with increase in N and that the magnitude of decrease is not high. More importantly, the reliability of the system was found less than 91.5%. To check whether this was actually due to the introduction of external customers, we compared the

system reliability of the current model with that of a k -out-of- n system where no external customers are entertained. Table 6 shows that allowing external customers in the system has only a narrow effect on the system reliability and the decrease in reliability is actually due to the assumption $\lambda > \mu$.

3.5.3. Analysis of a Cost function. Table 3.1 shows that as N increases, even though the server busy probability increases first, it decreases as N crosses some value. Note that the overall server busy probability is the sum of the server busy probability with external customers and the server busy probability with main customers. Table 3.3 shows that the fraction of time server remaining busy with external customers is ever increasing with N . Now as N increases, there is a decrease in the server busy probability with main customers. Hence, the above said behavior of the overall server busy probability can be concluded to be due to the conflicting nature of the two entities constituting it. This behavior of the server busy probability lead us to construct a cost function in the hope of finding an optimal value for the N -policy level defined as follows:

$$\text{Expected cost per unit time} = C_1 \cdot P_{\text{down}} + C_2 \cdot N_q + C_4 \cdot \theta_4 + C_5 \cdot N_{\text{fail}} + \frac{C_3}{E_{\hat{H}}} + C_6 \cdot P_{\text{idle}}$$

In the above, C_1 denote the cost per unit time incurred if the system is down, C_2 denote the holding cost per unit time per external customer in the queue, C_3 denote the cost incurred for starting failed components service, C_4 denote the cost due to loss of 1 external customer, C_5 denote the holding cost per unit time of one failed component, C_6 denote the cost per unit time if the server is idle. We study the cost function for various failure rates of the components, which is presented in Table 3.7. In all the 4 cases studied, we obtained an optimal value for N .

Table 3.1. Variation in the server busy probability when external customers are allowed $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.823494	0.823522	0.823528	0.823529	0.823529
3	0.829935	0.829973	0.829981	0.829983	0.831354
5	0.832187	0.832243	0.832254	0.832256	0.832891
7	0.833255	0.833338	0.833355	0.833358	0.833717
9	0.833839	0.833968	0.833994	0.834	0.83423
11	0.834162	0.834367	0.834408	0.834417	0.834577
13	0.834295	0.834627	0.834695	0.834708	0.834827
15	0.834239	0.834789	0.8349	0.834923	0.835093
17	0.833936	0.834861	0.835047	0.835085	0.835224
19	0.833252	0.834829	0.835146	0.835211	0.835329
21	0.831922	0.834652	0.835196	0.835306	0.835413
23	0.829445	0.834239	0.835184	0.835375	0.83548
25	0.824871	0.833426	0.83508	0.835412	0.83553

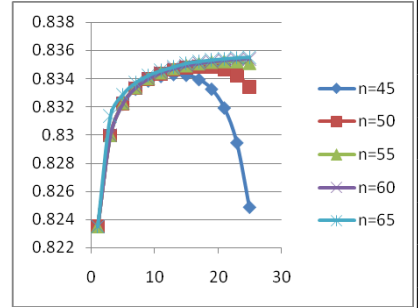


Table 3.2. Variation in the server busy probability when external customers are not allowed $k = 20, \lambda = 4, \mu = 5.5$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.72722	0.72726	0.72727	0.72727	0.72727
3	0.7272	0.72726	0.72727	0.72727	0.72727
5	0.72717	0.72725	0.72727	0.72727	0.72727
7	0.72711	0.72724	0.72727	0.72727	0.72727
9	0.72703	0.72722	0.72726	0.72727	0.72727
11	0.72688	0.72719	0.72726	0.72727	0.72727
13	0.72663	0.72714	0.72725	0.72727	0.72727
15	0.72622	0.72706	0.72723	0.72726	0.72727
17	0.7255	0.72691	0.7272	0.72726	0.72727
19	0.72425	0.72666	0.72715	0.72725	0.72727
21	0.72206	0.72623	0.72706	0.72723	0.72726
23	0.71814	0.72546	0.72691	0.7272	0.72726

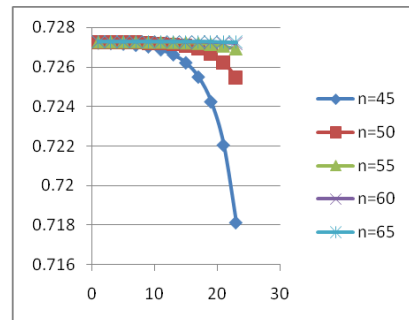


Table 3.3. Effect of the N -policy level on the fraction of time server is busy with external customers with $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$

N	n=40	n=45	n=50	n=55	n=60
1	0.096351	0.096276	0.096261	0.096257	0.096257
2	0.100557	0.100464	0.100445	0.100441	0.10044
3	0.102853	0.10274	0.102717	0.102712	0.102711
4	0.104255	0.104117	0.104089	0.104083	0.104082
5	0.105198	0.105028	0.104993	0.104986	0.104985
6	0.105882	0.105672	0.105629	0.105621	0.105619
7	0.106413	0.106153	0.1061	0.106089	0.106087
8	0.106853	0.106528	0.106462	0.106449	0.106446
9	0.107241	0.106832	0.106749	0.106733	0.106729
10	0.107605	0.107088	0.106984	0.106963	0.106958
11	0.107968	0.107313	0.10718	0.107153	0.107148
12	0.108354	0.107517	0.107348	0.107314	0.107307
13	0.108786	0.107711	0.107495	0.107451	0.107442
14	0.109291	0.107904	0.107626	0.10757	0.107559
15	0.109905	0.108106	0.107747	0.107675	0.10766
17	0.111651	0.108581	0.107976	0.107854	0.107829
19	0.114606	0.109249	0.108092	0.108008	0.107966
21		0.110301	0.108216	0.108153	0.10808
23		0.112079	0.10851	0.108308	0.108182
25		0.115216	0.108928	0.1085	0.108281
27			0.110699	0.108771	0.108387
29			0.112652	0.109196	0.108516
31			0.116153	0.10991	0.108697
33				0.111158	0.108978
35				0.113399	0.109446

Table 3.4. Variation in the system reliability with increase in N ($\lambda < \mu$ case) $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$

N	n=40	n=45	n=50	n=60	n=65
1	0.99963	0.99993	0.99998	1	1
2	0.99957	0.99991	0.99998	1	1
3	0.99948	0.99989	0.99998	1	1
4	0.99937	0.99987	0.99997	1	1
5	0.99924	0.99985	0.99997	1	1
6	0.99907	0.99981	0.99996	1	1
7	0.99885	0.99977	0.99995	1	1
8	0.99856	0.99971	0.99994	1	1
9	0.9982	0.99964	0.99993	1	1
10	0.99778	0.99954	0.99991	1	1
11	0.99712	0.99942	0.99988	1	1
12	0.99633	0.99926	0.99985	0.99999	1
13	0.9953	0.99905	0.99981	0.99999	1
14	0.99395	0.99878	0.99975	0.99999	1
15	0.99217	0.99843	0.99968	0.99999	1
17	0.98668	0.99736	0.99947	0.99998	1
19	0.97689	0.9955	0.99909	0.99996	0.99999
21	0.95915	0.99223	0.99844	0.99994	0.99999
23		0.98638	0.9973	0.99989	0.99998
25		0.97578	0.99528	0.99981	0.99996
27			0.99165	0.99966	0.99993
29			0.98509	0.9994	0.99988
31			0.97315	0.99894	0.99979
33				0.99862	0.99962
35					0.99932
37					0.99878
39					0.99781
41					0.99604

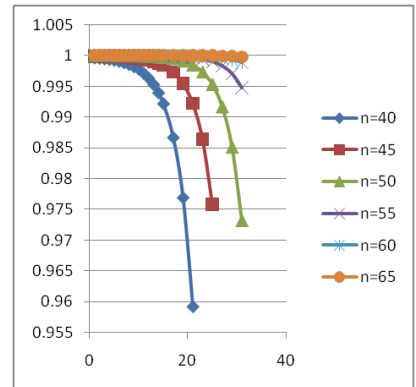


Table 3.5. Variation in the system reliability with increase in N ($\lambda > \mu$ case) $\lambda = 6, \mu = 5.5, \bar{\lambda} = 3.2, \bar{\mu} = 8$

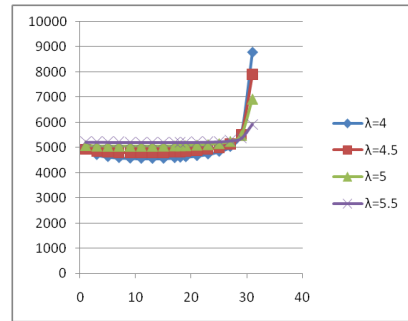
N	n=40	n=50	n=55	n=60
1	0.90191	0.91106	0.91312	0.91441
2	0.90118	0.91081	0.91297	0.91431
3	0.90041	0.91055	0.91281	0.91421
4	0.89961	0.91028	0.91264	0.91411
5	0.89876	0.91	0.91247	0.914
6	0.89758	0.90971	0.91229	0.91389
7	0.89696	0.90941	0.91211	0.91377
8	0.896	0.9091	0.91192	0.91366
9	0.895	0.90878	0.91173	0.91354
10	0.89396	0.90845	0.91153	0.91341
11	0.89287	0.90812	0.91133	0.91329
12	0.89174	0.90777	0.91112	0.91316
13	0.89055	0.90741	0.9109	0.91303
14	0.88932	0.90705	0.91068	0.91289
15	0.88804	0.90667	0.91046	0.91275
16	0.8867	0.90628	0.91	0.91261
17	0.88531	0.90589	0.90951	0.91247
18	0.88386	0.90548	0.90901	0.91232
19	0.88235	0.90507	0.90848	0.91217
21	0.88079	0.90464	0.90794	0.91186
23	0.87916	0.90421	0.90738	0.91155
25		0.90331	0.90679	0.91122
27		0.90237	0.9062	0.91088
29		0.90139	0.90558	0.91053
31		0.90036	0.90494	0.91018
33		0.8993	0.90462	0.90981
35				0.90944
37				0.90905
39				0.90866
41				0.90827

Table 3.6. Variation in the system reliability with increase in N (case when no external customers are allowed) $k = 20, \lambda = 6, \mu = 5.5$

N	n=40	n=45	n=50	n=55	n=60	n=65
1	0.902225375	0.907874465	0.911180377	0.913196206	0.914452851	0.915246844
3	0.900740206	0.90700978	0.910661995	0.912876785	0.914252281	0.915119886
5	0.899092674	0.906079888	0.910108447	0.912536681	0.914039791	0.914984941
7	0.897301137	0.905082345	0.909519434	0.91217649	0.913814664	0.914842606
9	0.895354867	0.904014587	0.908894181	0.911796391	0.913578033	0.914693356
11	0.893241525	0.902873158	0.908231676	0.911395431	0.913329422	0.914536655
13	0.890948415	0.901655376	0.907531381	0.910974264	0.913069129	0.914373219
15	0.888461053	0.900358438	0.906793237	0.910532713	0.912796974	0.914202273
17	0.885763168	0.898979008	0.906016231	0.910070777	0.912513793	0.914025187
19	0.882836878	0.897513986	0.905200183	0.909588754	0.912219048	0.913841009
21		0.895959914	0.904344797	0.909087062	0.911913395	0.91365093
23		0.894313395	0.903449655	0.908565581	0.911597252	0.913454473
25		0.892570376	0.902514458	0.908024669	0.911270797	0.913252354
27			0.901538968	0.907464802	0.910934329	0.913044453
29			0.900522768	0.90688622	0.910588205	0.912831426
31			0.899465442	0.90628922	0.910232782	0.912613034
33				0.905673981	0.90986824	0.912389636
35				0.905041099	0.909495115	0.912161767
37					0.909113765	0.911929727
39					0.908724129	0.911693275
41					0.908326745	0.911452949
43						0.911208868
45						0.910961211

Table 3.7. Analysis of a cost function for finding optimal N value $n = 50$, $k = 20$, $\mu = 5.5$, $\bar{\lambda} = 3.2$, $\bar{\mu} = 8$, $C_1 = 2000$, $C_2 = 20$, $C_3 = 800$, $C_4 = 1000$, $C_5 = 10$, $C_6 = 200$

N	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$	$\lambda = 6$
1	4925.877	4937.695	5079.029	5226.181
3	4710.059	4856.852	5057.425	5221.212
5	4630.354	4825.835	5050.332	5218.775
7	4591.702	4812.151	5048.243	5216.965
9	4571.3	4806.745	5048.411	5215.313
11	4561.086	4806.248	5049.849	5213.713
13	4558.217	4809.556	5052.345	5212.268
15	4563.915	4817.604	5056.578	5211.373
17	4588.216	4835.444	5064.896	5211.922
18	4605.19	4846.938	5070.21	5212.65
19	4624.185	4859.68	5076.196	5213.701
21	4670.646	4890.628	5091.4	5217.34
23	4735.585	4934.206	5114.597	5224.719
25	4837.829	5004.721	5155.522	5240.069
27	5032.125	5144.138	5241.815	5274.736
29	5546.901	5525.659	5482.957	5371.341
31	8780.95	7911.995	6932.789	5918.758



3.6. The retrial model

Here we consider a variant of the model discussed in section 3.2 by assuming that an arriving external customer either gets immediate service if it finds the server is idle at that time or joins an orbit of infinite capacity, if the server is busy with external customers with $\leq N - 1$ failed components of the k -out-of- n system. As in the model discussed in section 3.2, the external customers are not allowed to join the orbit when the server is busy with failed components of the system. An orbital customer retries for service with inter-retrial time following an exponential distribution with parameter θ . All other assumptions and parameters remain the same as in model discussed in section 3.2. In this situation the system can be modeled as follows. Let $X_1(t)$ = the number of external customers in the orbit at time t and $X_2(t)$ = the number of failed components of the k -out-of- n system, including the one getting service (if any) at time t .

$$\text{Define } S(t) = \begin{cases} 0, & \text{If the server is idle} \\ 1, & \text{If the server is busy with an external customer} \\ 2, & \text{If the server is busy with a main customer} \end{cases}$$

Now, $X(t) = (X_1(t), S(t), X_2(t))$ forms a continuous time Markov chain on the state space

$$S = \{(j_1, 0, j_2) / j_1 \geq 0, 0 \leq j_2 \leq N - 1\} \cup \{(j_1, 1, j_2) / j_1 \geq 0, 0 \leq j_2 \leq n - k + 1\} \\ \cup \{(j_1, 2, j_2) / j_1 \geq 0, 1 \leq j_2 \leq n - k + 1\}.$$

$$D_{22} = D_{11}^{(0)} - \bar{\mu}I_N,$$

$$D_{23} = \lambda c_N(N) \otimes r_{n-k+2-N}(1),$$

$$D_{33} = \lambda E_{n-k+2-N} + \lambda c_{(n-k+2-N)} \otimes r_{(n-k+2-N)}(n-k+2-N) - \bar{\mu}I_{n-k+2-N},$$

$$D_{34} = \begin{bmatrix} O_{n-k+2-N \times (N-1)} & \bar{\mu}I_{(n-k+2-N)} \end{bmatrix},$$

$$D_{44} = \lambda E_{n-k+1} + \lambda c_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1) + \mu E'_{n-k+1},$$

$$D_{41} = \mu c_{n-k+1}(1) \otimes r_N(1).$$

For $i \geq 0$ the transition from level i to $i + 1$ is represented by the matrix

$$\mathbf{A}_0 = \begin{bmatrix} 0_{N \times N} & 0 & 0 & 0 \\ 0 & \bar{\lambda}I_N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $i \geq 1$, the transition from level i to $i - 1$ is represented by the matrix

$$\mathbf{A}_{2i} = \begin{bmatrix} 0 & i\theta I_N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3.7. Steady state analysis of the retrial model

3.7.1. Stability condition. For finding the stability condition for the system study ,we apply Neuts Rao truncation by assuming $A_{1i} = A_{1M}$ and $A_{2i} = A_{2M}$ for all $i \geq M$.

Then the generator matrix of the truncated system will look like:

$$\tilde{\pi}_M(1) = (\pi_M(1, N), \dots, \pi_M(1, n - k + 1)),$$

$$\pi_M(2) = (\pi_M(2, 1), \dots, \pi_M(2, n - k + 1)).$$

be the steady state vector of the generator matrix \mathbf{A}_M . Then the relation $\pi_M \mathbf{A}_M = 0$ gives rise to the following equations:

$$\pi_M(0)D_{11}^{(M)} + \pi_M(1)D_{21} + \pi_M(2)D_{41} = 0, \quad (3.7.1)$$

$$\pi_M(0)D_{12}^{(M)} + \pi_M(1)D_{22} = 0, \quad (3.7.2)$$

$$\pi_M(1)D_{23} + \tilde{\pi}_M(1)D_{33} = 0, \quad (3.7.3)$$

$$\pi_M(0)D_{14} + \tilde{\pi}_M(1)D_{34} + \pi_M(2)D_{44} = 0. \quad (3.7.4)$$

It follows from equation (3.7.4) that

$$\pi_M(2) = -\pi_M(0)D_{14}(D_{44})^{-1} - \tilde{\pi}_M(1)D_{34}(D_{44})^{-1}. \quad (3.7.5)$$

Substituting for $\pi_M(2)$ in equation (3.7.1), we get

$$\pi_M(0)D_{11}^{(M)} + \pi_M(1)D_{21} - \pi_M(0)D_{14}(D_{44})^{-1}D_{41} - \tilde{\pi}_M(1)D_{34}(D_{44})^{-1}D_{41} = 0. \quad (3.7.6)$$

It follows from equation (3.7.3) that

$$\tilde{\pi}_M(1) = -\pi_M(1)D_{23}(D_{33}^{-1}). \quad (3.7.7)$$

Substituting for $\tilde{\pi}_M(1)$ in equation (3.7.6), we get

$$\begin{aligned} \pi_M(0)D_{11}^{(M)} + \pi_M(1)D_{21} - \pi_M(0)D_{14}(D_{44})^{-1}D_{41} \\ + \pi_M(1)D_{23}(D_{33})^{-1}D_{34}(D_{44})^{-1}D_{41} = 0. \end{aligned} \quad (3.7.8)$$

We notice that the first column of the matrix D_{41} is $-D_{44}e$ and its all other columns are zero columns. Hence the first column of the matrix $(D_{44})^{-1}D_{41}$ is $-e$ and its all other columns are zero columns. This implies that the first column of the matrix $-D_{14}(D_{44})^{-1}D_{41}$ is $D_{14}e = \lambda c_N(N)$ and its all other columns are zero columns. In other words $-D_{14}(D_{44})^{-1}D_{41} = \lambda c_N(N) \otimes r_N(1)$. Also, the first column of the matrix $D_{34}(D_{44})^{-1}D_{41}$ is $-D_{34}e$ and its all other columns are zero columns. Since $-D_{34}e = D_{33}e$, the first column of the matrix $(D_{33})^{-1}D_{34}(D_{44})^{-1}D_{41}$ is e and its all other columns are zero columns. Hence it follows that $D_{23}(D_{33})^{-1}D_{34}(D_{44})^{-1}D_{41}$ is $D_{23}e = \lambda c_N(N) \otimes r_N(1)$. Thus equation (3.7.8) becomes

$$\pi_M(0)(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1)) + \pi_M(1)(D_{21} + \lambda c_N(N) \otimes r_N(1)) = 0. \quad (3.7.9)$$

Adding equations (3.7.2) and (3.7.9), we get

$$\pi_M(0)(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1) + D_{12}^{(M)}) + \pi_M(1)(\tilde{D}_{22} + D_{21} + \lambda c_N(N) \otimes r_N(1)) = 0. \quad (3.7.10)$$

Since $D_{11}^{(M)} + D_{12}^{(M)} = \tilde{D}_{22} + D_{21} = \lambda E_N$, equation (3.7.10) reduces to

$$(\pi_M(0) + \pi_M(1))(\lambda E_N + \lambda c_N(N) \otimes r_N(1)) = 0. \quad (3.7.11)$$

which implies that $\pi_M(0) + \pi_M(1)$ is a constant multiple of the steady state vector $\frac{1}{N}e'_N$ of the generator matrix $\lambda E_N + \lambda c_N(N) \otimes r_N(1)$ and hence,

$$\pi_M(0) + \pi_M(1) = v \frac{1}{N} e'_N. \quad (3.7.12)$$

where v is a constant. Equation (3.7.2) implies that

$$\pi_M(0) = -\pi_M(1) \tilde{D}_{22} (D_{12}^{(M)})^{-1}. \quad (3.7.13)$$

Since $(D_{12}^{(M)})^{-1} = \frac{1}{(\bar{\lambda} + M\theta)} I_N$, (3.7.13) gives

$$\lim_{M \rightarrow \infty} \pi_M(0) = 0. \quad (3.7.14)$$

and hence

$$\lim_{M \rightarrow \infty} \pi_M(1) = v \frac{1}{N} e'_N, \quad (3.7.15)$$

and

$$\lim_{M \rightarrow \infty} \bar{\lambda} \pi_M(1) e = v \bar{\lambda}. \quad (3.7.16)$$

Again from (3.7.13),

$$M\theta \pi_M(0) e = -M\theta \pi_M(1) \tilde{D}_{22} (D_{12}^{(M)})^{-1} e. \quad (3.7.17)$$

Since, $\lim_{M \rightarrow \infty} M\theta (D_{12}^{(M)})^{-1} e = \lim_{M \rightarrow \infty} \frac{M\theta}{(\bar{\lambda} + M\theta)} e_N = e_N$, (3.7.17) implies that

$$\begin{aligned}
\lim_{M \rightarrow \infty} M\theta\pi_M(0)e &= -\lim_{M \rightarrow \infty} \pi_M(1)\tilde{D}_{22}e \\
&= -\nu \frac{1}{N} e'_N(-\lambda c_N(N) - \bar{\mu}e) \\
&= \nu \left(\frac{\lambda}{N} + \bar{\mu} \right).
\end{aligned} \tag{3.7.18}$$

The truncated system is stable if and only if

$$\pi_M A_0 e < \pi_M A_{2M} e, \tag{3.7.19}$$

$$\pi_M A_0 e = \bar{\lambda} \pi_M(1) e, \tag{3.7.20}$$

$$\pi_M A_{2M} e = M\theta\pi_M(0) e. \tag{3.7.21}$$

Making use of equations (3.7.16), (3.7.18), (3.7.20) and (3.7.21), the stability condition for the truncated system as $M \rightarrow \infty$ is given by

$$\nu \bar{\lambda} < \nu \left(\frac{\lambda}{N} + \bar{\mu} \right),$$

which can be re-arranged as

$$\frac{\bar{\lambda}}{\bar{\mu}} \frac{N\bar{\mu}}{(\lambda + N\bar{\mu})} < 1.$$

Hence, we conclude that the retrial problem discussed in section 3.6 has the same stability condition as the queueing problem, which was obtained in section 3.3.1.

3.7.2. Computation of Steady State Vector. We find the steady state vector of $\{X(t), t \geq 0\}$, by approximating it with the steady state vector of the truncated system. Let $\pi = (\pi_0, \pi_1, \pi_2, \dots)$ where each $\pi_i = (\pi_i(0, 0), \pi_i(0, 1), \dots, \pi_i(0, N-1), \pi_i(1, 1), \dots, \pi_i(1, n-k+1), \pi_i(2, 0), \pi_i(2, 1), \dots, \pi_i(2, n-k+1))$ be the steady state vector of the Markov chain $\{X(t), t \geq 0\}$.

Suppose $A_{1i} = A_{1M}$ and $A_{2i} = A_{2M}$ for all $i \geq M$. Let $\pi_{M+r} = \pi_{M-1}R^{r+1}$, $r \geq 0$, then from $\pi Q = 0$ we get

$$\pi_{M-1}A_0 + \pi_M A_{1M} + \pi_{M+1}A_{2M} = 0,$$

$$\pi_{M-1}A_0 + \pi_{M-1}RA_{1M} + \pi_{M-1}R^2A_{2M} = 0,$$

$$\pi_{M-1}(A_0 + RA_{1M} + R^2A_{2M}) = 0.$$

Choose R such that $A_0 + RA_{1M} + R^2A_{2M} = 0$. We call this R as R_M . Also we have

$$\pi_{M-2}A_0 + \pi_{M-1}A_{1M-1} + \pi_M A_{2M} = 0,$$

$$\pi_{M-2}A_0 + \pi_{M-1}(A_{1M-1} + R_M A_{2M}) = 0,$$

$$\pi_{M-1} = -\pi_{M-2}A_0(A_{1M-1} + R_M A_{2M})^{-1}$$

$$= \pi_{M-2}R_{M-1}.$$

where

$$R_{M-1} = -A_0(A_{1M-1} + R_M A_{2M}).$$

Next,

$$\pi_{M-3}A_0 + \pi_{M-2}A_{1M-2} + \pi_{M-1}A_{2M-1} = 0,$$

$$\pi_{M-3}A_0 + \pi_{M-2}(A_{1M-2} + \pi_{M-1}A_{2M-1}) = 0,$$

$$\begin{aligned} \pi_{M-2} &= -\pi_{M-3}A_0(A_{1M-2} + R_{M-1}(A_{2M-1}))^{-1} \\ &= \pi_{M-3}R_{M-2}. \end{aligned}$$

Where

$$R_{M-2} = -A_0(A_{1M-2} + R_{M-1}A_{2M-1})^{-1}.$$

and so on.

Finally

$$\pi_0A_{10} + \pi_1A_{21} = 0$$

becomes

$$\pi_0(A_{10} + R_1A_{21}) = 0.$$

For finding π , first we take π_0 as the steady state vector of $A_{10} + R_1A_{21}$. Then π_i for $i \geq 1$ can be found using the recursive formula, $\pi_i = \pi_{i-1}R_i$ for $1 \leq i \leq M$.

Now the steady state probability distribution of the truncated system is obtained by dividing each π_i with the normalizing constant

$$[\pi_0 + \pi_1 + \dots]e = \left[\pi_0 + \pi_1 + \dots + \pi_{N-2} + \pi_{M-1}(I - R_M)^{-1} \right]e.$$

3.7.3. Computation of the matrix R_M . Consider the matrix quadratic equation

$$A_0 + R_M A_{1M} + R_M^2 A_{2M} = 0. \quad (3.7.22)$$

which implies

$$R_M = -A_0(A_{1M} + R_M A_{2M})^{-1}. \quad (3.7.23)$$

The structure of the A_0 matrix implies that the matrix R_M has the form:

$$R_M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ R_{M1} & R_{M2} & R_{M3} & R_{M4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (3.7.24)$$

In other words, the non-zero rows of the R_M matrix are those, where the A_0 matrix has at least one nonzero entry. Now,

$$R_M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ R_{M2}R_{M1} & R_{M2}^2 & R_{M2}R_{M3} & R_{M2}R_{M4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (3.7.25)$$

Equation (3.7.22) gives rise to the following equations:

$$R_{M1}D_{11}^{(M)} + R_{M2}D_{21} + R_{M4}D_{41} = 0, \quad (3.7.26)$$

$$R_{M2}R_{M1}M\theta I_N + R_{M1}D_{12} + R_{M2}D_{22} + \bar{\lambda}I_N = 0, \quad (3.7.27)$$

$$R_{M2}D_{23} + R_{M3}D_{33} = 0, \quad (3.7.28)$$

$$R_{M1}D_{14} + R_{M3}D_{34} + R_{M4}D_{44} = 0. \quad (3.7.29)$$

From equation (3.7.28), we can write

$$R_{M3} = -R_{M2}D_{23}(D_{33})^{-1}. \quad (3.7.30)$$

From equation(3.7.29), we can write

$$R_{M4} = -R_{M1}D_{14}(D_{44})^{-1} - R_{M3}D_{34}(D_{44})^{-1}. \quad (3.7.31)$$

Substituting for R_{M3} from (3.7.30) in equation (3.7.31), we get

$$R_{M4} = -R_{M1}D_{14}(D_{44})^{-1} + R_{M2}D_{23}(D_{33})^{-1}D_{34}(D_{44})^{-1}. \quad (3.7.32)$$

Substituting for R_{M4} from (3.7.32) in equation (3.7.26), we get

$$\begin{aligned} R_{M1}D_{11}^{(M)} + R_{M2}D_{21} - R_{M1}D_{14}(D_{44})^{-1}D_{41} \\ + R_{M2}D_{23}(D_{33})^{-1}D_{34}(D_{44})^{-1}D_{41} = 0. \end{aligned} \quad (3.7.33)$$

Using the same reasoning, that lead us to equation (3.7.9), equation (3.7.33) becomes

$$R_{M1}(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1)) + R_{M2}(D_{21} + \lambda c_n(N) \otimes r_N(1)) = 0. \quad (3.7.34)$$

From (3.7.34), it follows that

$$R_{M1} = -R_{M2}(D_{21}\lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}. \quad (3.7.35)$$

Substituting for R_{M1} in (3.7.27), we get

$$\begin{aligned} & -R_{M2}^2(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}M\theta I_N \\ & -R_{M2}(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}D_{12} \\ & + R_{M2}D_{22} + \bar{\lambda}I_N = 0. \end{aligned}$$

That is

$$\begin{aligned} & R_{M2}^2 \left(-(D_{21} + \lambda c_N(N))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}M\theta I_N \right) \\ & + R_{M2} \left(-(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}D_{12} + D_{22} \right) \\ & + \bar{\lambda}I_N = 0. \quad (3.7.36) \end{aligned}$$

We notice that $-(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))e = (D_{12} + M\theta I_N)e$. and therefore

$$\begin{aligned} & -(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1}(D_{12} + M\theta I_N)e = \\ & (D_{21}\lambda c_N(N) \otimes r_N(1))e. \quad (3.7.37) \end{aligned}$$

Also,

$$D_{22}e + (D_{21} + \lambda c_N(N) \otimes r_N(1))e + \bar{\lambda}e = 0.$$

and hence

$$\begin{aligned} & \left(-(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} M \theta I_N \right) e + \\ & \left(-(D_{21} + \lambda c_N(N) \otimes r_N(1))(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} D_{12} + D_{22} \right) e \quad (3.7.38) \end{aligned}$$

$$+ \bar{\lambda} e = 0.$$

Equation (3.7.38) shows that the matrix R_{M2} is the minimal non-negative solution of the matrix quadratic equation (3.7.36). Once obtaining R_{M2} , the matrices R_{M1}, R_{M3}, R_{M2} , and R_{M4} can be found using equations (3.7.35), (3.7.30) and (3.7.31) respectively. Hence the matrix R_M can be found. From the form of the matrix $D_{11}^{(M)}$, we notice that,

$$\begin{aligned} & - \left(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1) \right) \\ & = M \theta I_N - \left(\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1) \right) \\ & = M \theta \left(I_N - \frac{1}{M \theta} (\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1)) \right). \end{aligned}$$

and hence

$$\begin{aligned} & - \left(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1) \right)^{-1} \\ & = \frac{1}{M \theta} \left(I_N - \frac{1}{M \theta} (\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1)) \right)^{-1} \\ & = \frac{1}{M \theta} \left(I_N + \frac{1}{M \theta} (\lambda E_N - \bar{\lambda} I_N + \lambda c_N(N) \otimes r_N(1)) + \dots \right). \end{aligned}$$

Therefore

$$\lim_{M \rightarrow \infty} \left(- \left(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1) \right)^{-1} M \theta I_N \right) = I_N.$$

and

$$\lim_{M \rightarrow \infty} \left(-(D_{11}^{(M)} + \lambda c_N(N) \otimes r_N(1))^{-1} D_{12} \right) = 0.$$

Hence as $M \rightarrow \infty$ equation (3.7.36) becomes

$$R_{M2}^2(D_{21} + \lambda c_N(N) \otimes r_N(1)) + R_{M2}D_{22} + \bar{\lambda}I_N = 0. \quad (3.7.39)$$

We identify $D_{21} + \lambda c_N(N) \otimes r_N(1)$ as \tilde{A}_2 , D_{22} as \tilde{A}_1 and $\bar{\lambda}I_N$ as \tilde{A}_0 , which were defined in section 3.3.2. Hence equation (3.7.39) is the same as equation (3.3.24) of section 3.3.2. That is the matrix R_M tends to the matrix R , the minimal non-negative solution of (3.3.24), as $M \rightarrow \infty$. This fact can be utilized in determining the truncation level M .

3.8. System Performance Measures

The following system performance measures were calculated numerically.

- (1) Fraction of time the system is down,

$$P_{down} = \sum_{j_1=0}^{\infty} \left(\pi_{j_1}(1, n-k+1) + \pi_{j_1}(2, n-k+1) \right).$$

- (2) System reliability, $P_{rel} = 1 - P_{down}$

$$= 1 - \sum_{j_1=0}^{\infty} \left(\pi_{j_1}(1, n-k+1) + \pi_{j_1}(2, n-k+1) \right).$$

- (3) Average number of external customers in the orbit,

$$N_{orbit} = \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=1}^{n-k+1} \pi_{j_1}(1, j_3) \right) + \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(2, j_3) \right).$$

(4) Average number of failed components in the system,

$$N_{fail} = \sum_{j_3=0}^{n-k+1} j_3 \left(\sum_{j_1=0}^{\infty} \pi_{j_1}(0, j_3) \right) + \sum_{j_3=1}^{n-k+1} \left(\sum_{j_1=0}^{\infty} \pi_{j_1}(2, j_3) \right).$$

(5) Average number of failed components waiting when server is busy with external customers

$$= \sum_{j_3=0}^{n-k+1} j_3 \left(\sum_{j_1=1}^{\infty} \pi_{j_1}(0, j_3) \right).$$

(6) Expected rate at which external customers joining the system

$$= \bar{\lambda} \left\{ \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(0, j_3) \right) + \sum_{j_3=0}^{N-1} \pi_0(0, j_3) \right\}.$$

(7) Expected number of external customers on its arrival gets service directly,

$$= \sum_{j_3=0}^{N-1} \pi_0(0, j_3).$$

(8) Fraction of time server is busy with external customers,

$$P_{ext,busy} = \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(0, j_3) \right).$$

(9) Probability that the server is found idle,

$$P_{idle} = \sum_{j_3=0}^{N-1} \pi_0(0, j_3) = N\pi_0(0, 0).$$

(10) Probability that the server is found busy,

$$P_{busy} = 1 - \sum_{j_3=0}^{N-1} \pi_0(0, j_3) = 1 - N\pi_0(0, 0).$$

(11) Expected loss rate of external customers

$$\theta_4 = \bar{\lambda} \left\{ \sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} \pi_{j_1}(1, j_3) \right) + \sum_{j_1=1}^{\infty} \left(\sum_{j_3=N}^{n-k+1} \pi_{j_1}(0, j_3) \right) \right\}.$$

(12) Expected service completion rate of external customers,

$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(0, j_3) \right).$$

(13) Expected number of external customers when server is busy with external customers

$$\theta_6 = \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=0}^{n-k+1} \pi_{j_1}(0, j_3) \right).$$

(14) Expected successful retrial rate

$$= \theta \cdot \sum_{j_1=1}^{N-1} \left(\sum_{j_3=0}^{N-1} \pi_{j_1}(0, j_3) \right).$$

3.9. Numerical study of the performance of the system

3.9.1. The effect of N policy on the server busy probability. A comparison of Tables 3.1 and 3.8 shows that the models discussed in section 3.2 and its variant where external customers are sent to the orbit, which was discussed in section 3.6 have similar behavior as far as the server busy probability is considered. Comparison of Tables 3.3 and 3.9 also points to the same for the fraction of time server remains busy with external customers. Tables 3.4 and 3.10 indicate that the two models have similar reliability.

Table 3.8. Variation in the server busy probability when external customers are allowed $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, \theta = 5$.

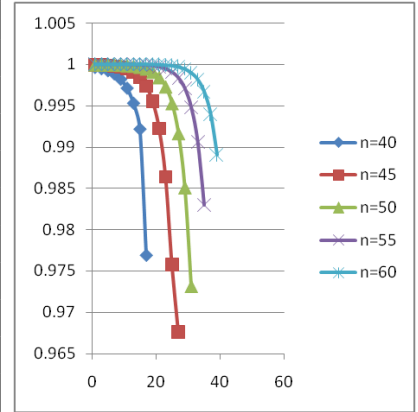
N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.82349	0.82352	0.82353	0.82353	0.82353
3	0.82995	0.82999	0.83	0.83	0.83
5	0.83222	0.83228	0.83229	0.83229	0.83229
7	0.83328	0.83336	0.83338	0.83338	0.83338
9	0.83385	0.83398	0.83401	0.83401	0.83401
11	0.83417	0.83437	0.83442	0.83442	0.83443
13	0.8343	0.83463	0.8347	0.83471	0.83472
15	0.83424	0.83479	0.8349	0.83493	0.83493
17	0.83394	0.83486	0.83505	0.83509	0.8351
19	0.83325	0.83483	0.83515	0.83521	0.83523
21	0.83192	0.83465	0.8352	0.83531	0.83533
23	0.82945	0.83424	0.83518	0.83538	0.83541

Table 3.9. Effect of the N -policy level on the fraction of time server is busy with external customers $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 3.2, \bar{\mu} = 8, \theta = 5$

N	$n=40$	$n=45$	$n=50$	$n=55$	$n=60$
1	0.09635	0.09628	0.09626	0.09626	0.09626
3	0.10287	0.10276	0.10273	0.10273	0.10273
5	0.10523	0.10506	0.10503	0.10502	0.10502
7	0.10644	0.10618	0.10612	0.10611	0.10611
9	0.10725	0.10685	0.10676	0.10675	0.10674
11	0.10798	0.10732	0.10719	0.10716	0.10716
13	0.10879	0.10772	0.1075	0.10746	0.10745
15	0.10991	0.10811	0.10775	0.10768	0.10766
17	0.11461	0.10858	0.10798	0.10786	0.10783
19	0.11983	0.10925	0.10822	0.10801	0.10797
21		0.1103	0.10851	0.10815	0.10808
23		0.11208	0.10893	0.10831	0.10818
25		0.11522	0.10959	0.1085	0.10828
27			0.1107	0.10877	0.10839
29			0.11265	0.1092	0.10852
31			0.11615	0.10991	0.1087
33				0.11116	0.10898
35				0.1134	0.10945
37					0.11026
39					0.11172
41					0.11435

Table 3.10. Variation in the system reliability with increase in N $k = 20, \lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, \theta = 5$

N	n=40	n=45	n=50	n=55	n=60
1	0.99963	0.99993	0.99998	1	1
3	0.99948	0.99989	0.99998	1	1
5	0.99924	0.99985	0.99997	0.99999	1
7	0.99885	0.99977	0.99995	0.99999	1
9	0.9982	0.99964	0.99993	0.99998	1
11	0.99712	0.99942	0.99988	0.99998	1
13	0.9953	0.99905	0.99981	0.99996	0.99999
15	0.99217	0.99843	0.99968	0.99994	0.99999
17	0.9769	0.99736	0.99947	0.99989	0.99998
19		0.9955	0.99909	0.99982	0.99996
21		0.99223	0.99844	0.99968	0.99994
23		0.98638	0.9973	0.99945	0.99989
25		0.97578	0.99528	0.99905	0.99981
27			0.99165	0.99833	0.99966
29			0.98509	0.99705	0.9994
31			0.97315	0.99475	0.99894
33				0.99058	0.99812
35				0.98297	0.99663
37					0.99393
39					0.989



3.9.2. Cost Analysis. As in the case of the queueing model discussed in section 3.2, for finding an optimal value for the N -policy level, we analyzed a cost function for the retrial model also. For defining the cost function, let C_1 be the cost per unit time incurred if the system is down, C_2 be the holding cost per unit time per external customer in the orbit, C_3 is the cost incurred for starting failed components service after accumulation of N of them, C_4 be the cost due to loss of 1 external customer, C_5 be the holding cost per unit time of one failed component, C_6 be the cost per unit time if the server is idle. We define the cost function as:

$$\text{Expected cost per unit time} = C_1 \cdot P_{down} + C_2 \cdot N_{orbit} + C_4 \cdot \theta_4 + C_5 \cdot N_{fail} + \frac{C_3}{E_{\hat{H}}} + C_6 \cdot P_{idle}.$$

where $E_{\hat{H}}$ is found exactly in the same lines as in section 3.4.1.

Our numerical study, as presented in Table 3.11, show that an optimal value for N can be found for different parameter choices and also that this optimal value happens to be a much smaller value like $N = 6$. This shows the care needed in selecting the N -policy level.

Table 3.11. Analysis of a cost function $n = 50, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, \theta = 5, C_1 = 2000, C_2 = 1000, C_3 = 800, C_4 = 1000, C_5 = 10, C_6 = 200, \theta = 5$

N	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$
1	6235.23047	6440.20947	6671.65918
2	6137.3877	6343.84668	6576.75928
3	6109.98389	6317.7207	6551.88965
4	6102.75391	6311.82178	6547.30566
5	6102.27734	6312.30322	6548.71436
6	6104.71094	6315.28613	6552.17676
7	6108.70947	6319.521	6556.51709
8	6113.67188	6324.50439	6561.33057
9	6119.2749	6329.98047	6566.44873
10	6125.32666	6335.80176	6571.76465
11	6131.69824	6341.87891	6577.22021
12	6138.31006	6348.14307	6582.78711
13	6145.10449	6354.55762	6588.43018
14	6152.04492	6361.09961	6594.13086
15	6159.104	6367.74854	6599.88428
17	6173.53564	6381.33594	6611.51611
19	6188.38672	6395.33936	6623.31689
21	6203.78809	6409.88037	6635.37354
23	6220.13477	6417.44531	6647.98535
25	6238.73828	6443.09375	6662.8042
27	6266.49854	6471.54688	6690.0752
29	6356.05566	6571.71631	6799.88672
31	7073.24658	7340.11523	7618.78223

Chapter 4

Reliability of a k -out-of- n system with a repair facility extending service to external customers – The T -policy

4.1. Introduction

In the previous chapters we concentrated elaborately as N -policy, both under pre-emptive and non pre-emptive priority basis. The pre-emptive priority to serve the failed components produced quite high reliability. There was a mild reduction in this under nonpre-emptive nature set up. We also considered the case of providing service only to the main system. Of course, under this policy the reliability can be brought to as high as .9999.... Nevertheless, the server stays idle for a long time. The utilization of this idle time is equally important. This leads us to wonder the intension of the repair facility

to external customers. As a consequence revenue could be generated without serious compromise in the main system reliability.

In this chapter, we study a k -out-of- n system where server offers service to external customers on a time-based policy, namely the T -policy. Under this policy, the server starts attending the failed components, if any present (main customers), only on the realization of a random time T . Priority is given to the main customers in the sense that, if the realization of T happens in the middle of an external customer's service, the ongoing service is preempted to start serving the main customers. Also, once the server starts attending the main customers, it continues to do so that until every component becomes operational. At the end of a cycle (the epoch at which no component of the main system is in breakdown state), a clock starts ticking. This clock has a random duration T , on realisation of which the repair facility is turned to repair of failed components, if any, of the main system. The pre-emptive rule is adopted.

The motivation for the present study comes from the real world scenarios of time-based resources sharing like those of spectrum-sharing, inventory-sharing etc.

This chapter is arranged as follows. In section 4.2 a queueing model is described for studying the problem discussed. In section 4.3, we conduct the steady state analysis of the system and give a product form solution for the steady state distribution. Several important system performance measures have been derived in section 4.4. In section 4.5 we present results from a numerical study on the behavior of the system performance measures as different system parameters are varied.

4.2. The queueing model

We consider a k -out-of- n system with a single server, offering service to external customers also. Commencement of service to failed components of the main system is governed by T -policy. ie, at the epoch the system starts with all components operational, the server starts attending external customers (if any present). The server starts the service of the failed components of the main system only at the moment of the realization of the random time T (if there is at least one failed component). If the time T is realized in the middle of an external customer's service and if there exists at least one failed component, the external customer in service is pre-empted and the server is switched on to the service of main customers. The preempted external customer goes to the queue of external customers. If there are no main customers present at the moment of realization of the time T , the server continues at his present status and the time T restarts. The random time T is assumed to follow an exponential distribution with parameter δ . The life time of a component of the k -out-of- n system follows an exponential distribution with parameter $\frac{\lambda}{i}$ when i components are operational. This assumption ensures decreasing failure rate of the entire system with increase in number of operational units. Hence the inter-arrival time of failed components follows an exponential distribution with parameter λ . Arrival of external customers has inter-occurrence time exponentially distributed with parameter $\bar{\lambda}$. External customers, arriving when the server is busy with main customers, are not allowed to join the system. Only those external customers who arrive during the service of an external customer, join the queue of such customers (of infinite capacity). An external customer, who finds the server idle on its arrival, is directly taken

for service. Service times of main customers and external customers follow exponential distributions with parameters μ and $\bar{\mu}$, respectively.

Notations. In the following sequel,

- (i) I_n denotes identity matrix of order n ;
- (ii) I denotes an identity matrix of appropriate size;
- (iii) e_n denotes a $n \times 1$ column matrix of 1's;
- (iv) e denotes a column matrix of 1's of appropriate order;
- (v) E_n denotes a square matrix of order n defined as

$$E_n(i, j) = \begin{cases} -1; & \text{if } i = j, 1 \leq i \leq n \\ 1; & \text{if } j = i + 1, 1 \leq i \leq n - 1 \\ 0; & \text{otherwise.} \end{cases}$$

- (vi) $E'_n =$ Transpose of E_n
- (vii) $r_n(i)$ denotes a $1 \times n$ row matrix whose i^{th} entry is 1 and all other entries are zeroes
- (viii) $c_n(i) =$ Transpose $r_n(i)$
- (ix) \otimes denotes Kronecker product of matrices.
- (x) O stands for zero matrix of appropriate order.

4.2.1. The Markov Chain. Let $X_1(t) =$ number of external customers in the system including the one getting service (if any) at time t .

$X_2(t) =$ number of main customers in the system including the one getting service (if any) at time t .

The transition within level 0 is represented by the matrix

$$A_{10} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

where

$$B_1 = \lambda E_{n-k+2} + \lambda c_{n-k+2}(n-k+2) \otimes r_{n-k+2}(n-k+2) - (\bar{\lambda} + \delta) I_{n-k+2} + \delta c_{n-k+2}(1) \otimes r_{n-k+2}(1)$$

$$B_2 = \begin{bmatrix} 0_{1 \times (n-k+1)} \\ \delta I_{n-k+1} \end{bmatrix}$$

B_3 is a $(n-k+1) \times (n-k+2)$ matrix whose $(1, 1)^{\text{th}}$ entry is μ and all other entries are zeroes.

$$B_4 = \lambda E_{n-k+1} + \lambda c_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1) - \mu E'_{n-k+1}$$

The transition from level i to level $i+1$, $i \geq 0$ is represented by the matrix

$$A_0 = \begin{bmatrix} \bar{\lambda} I_{(n-k+2) \times (n-k+2)} & O_{(n-k+2) \times (n-k+1)} \\ O_{(n-k+1) \times (n-k+2)} & O_{(n-k+1) \times (n-k+1)} \end{bmatrix}$$

Transition from level i to $i-1$, $i \geq 1$ is represented by the matrix

$$A_2 = \begin{bmatrix} \bar{\mu} I_{(n-k+2) \times (n-k+2)} & O_{(n-k+2) \times (n-k+1)} \\ O_{(n-k+1) \times (n-k+2)} & O_{(n-k+1) \times (n-k+1)} \end{bmatrix}$$

Transition within level i is represented by the matrix

$$A_1 = A_{10} - A_2.$$

4.3. Steady state analysis

4.3.1. Stability condition.

Consider the generator matrix $A = A_0 + A_1 + A_2$

$$\text{Then } A = \begin{bmatrix} F & B_2 \\ B_3 & B_4 \end{bmatrix}, \text{ where}$$

$$F = \lambda E_{n-k+2} + \lambda c_{n-k+2}(n-k+2) \otimes r_{n-k+2}(n-k+2) - \delta I_{n-k+2} + \delta c_{n-k+2}(1) \otimes r_{n-k+2}(1).$$

Let $\boldsymbol{\pi} = (\boldsymbol{\pi}(0), \boldsymbol{\pi}(1))$, where $\boldsymbol{\pi}(0) = (\pi(0, 0), \pi(0, 1), \dots, \pi(0, n-k+1))$, $\boldsymbol{\pi}(1) = (\pi(1, 1), \pi(1, 2), \dots, \pi(1, n-k+1))$ be the steady state vector of the generator matrix A .

The Markov chain $\{X(t), t \geq 0\}$ is stable if and only if $\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}$. It follows that $\boldsymbol{\pi} A_0 \mathbf{e} = \bar{\lambda} \boldsymbol{\pi}(0) \mathbf{e}$ and $\boldsymbol{\pi} A_2 \mathbf{e} = \bar{\mu} \boldsymbol{\pi}(0) \mathbf{e}$.

Therefore the stability condition becomes

$$\frac{\bar{\lambda}}{\bar{\mu}} < 1 \tag{4.3.1}$$

Though we have the stability condition as given by (4.3.1), for future reference, we evaluate the steady state vector $\boldsymbol{\pi}$ as follows:

The relation $\pi A = 0$ gives

$$\pi(0)F + \pi(1)B_3 = 0 \quad (4.3.2)$$

$$\pi(0)B_2 + \pi(1)B_4 = 0. \quad (4.3.3)$$

From (4.3.3), it follows that

$$\pi(1) = -\pi(0)B_2B_4^{-1}. \quad (4.3.4)$$

Substituting (4.3.4) in (4.3.2), we get

$$\begin{aligned} \pi(0)F - \pi(0)B_2B_4^{-1}B_3 &= 0 \\ \pi(0)(F - B_2B_4^{-1}B_3) &= 0. \end{aligned} \quad (4.3.5)$$

We notice that the first column of the matrix B_3 is $-B_4\mathbf{e}$ and all other columns of B_3 are columns zero. Hence the first column of the matrix is $(B_4^{-1})B_3$ which is $-\mathbf{e}$ and its all other columns are zero columns. This tells us that the first column of the matrix

$-B_2(B_4^{-1})B_3$ is $B_2\mathbf{e} = \begin{bmatrix} 0 \\ \delta e_{n-k+1} \end{bmatrix}$ and its all other columns are columns of zeros. Hence

$$F - B_2B_4^{-1}B_3 = \begin{bmatrix} -\lambda & \lambda & & & \\ \delta & -(\lambda + \delta) & \lambda & & \\ \cdot & & \cdot & & \cdot \\ \delta & & & -(\lambda + \delta) & \lambda \\ \delta & & & & -\delta \end{bmatrix}_{(n-k+2) \times (n-k+2)} \quad (4.3.6)$$

Then equation (4.3.5) gives:

$$\pi(0, i) = \left(\frac{\lambda}{\lambda + \delta} \right)^i \pi(0, 0), \quad i = 1, 2, \dots, n - k \quad (4.3.7)$$

$$\pi(0, n - k + 1) = \frac{\lambda}{\delta} \left(\frac{\lambda}{\lambda + \delta} \right)^{n-k} \pi(0, 0). \quad (4.3.8)$$

Equations (4.3.7) and (4.3.8) gives the component vector $\pi(0)$ up to a constant $\pi(0, 0)$. Hence from (4.3.4), the vector $\pi(1)$ is also obtained up to the constant $\pi(0, 0)$. The constant $\pi(0, 0)$ can be found using the normalizing condition $\boldsymbol{\pi} \mathbf{e} = 1$.

4.3.2. The steady state probability vector.

Let $\boldsymbol{\phi} = (\phi(0), \phi(1), \phi(2), \dots)$ be the steady state probability vector of the Markov chain $\{X(t), t \geq 0\}$ where

$$\boldsymbol{\phi}(i) = (\phi(i, 0, 0), \phi(i, 0, 1), \dots, \phi(i, 0, n - k + 1), \phi(i, 1, 1), \dots, \phi(i, 1, n - k + 1)), \quad i \geq 0.$$

The relation $\boldsymbol{\phi} Q = 0$ then gives rise to:

$$\boldsymbol{\phi}(0)A_{10} + \boldsymbol{\phi}(1)A_2 = 0 \quad (4.3.9)$$

$$\boldsymbol{\phi}(i - 1)A_0 + \boldsymbol{\phi}(i)A_1 + \boldsymbol{\phi}(i + 1)A_2 = 0, \quad i \geq 1. \quad (4.3.10)$$

We notice that

$$A_0 = \frac{\bar{\lambda}}{\mu} A_2 \quad (4.3.11)$$

and therefore

$$\begin{aligned}
A_0 + \frac{\bar{\lambda}}{\bar{\mu}}A_1 + \left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^2 A_2 &= \frac{\bar{\lambda}}{\bar{\mu}}A_2 + \frac{\bar{\lambda}}{\bar{\mu}}A_1 + \left(\frac{\bar{\lambda}}{\bar{\mu}}\right)A_0 \\
&= \frac{\bar{\lambda}}{\bar{\mu}}(A_2 + A_1 + A_0) \\
&= \frac{\bar{\lambda}}{\bar{\mu}}A.
\end{aligned} \tag{4.3.12}$$

Also

$$A_1 = A_{10} - A_2 \tag{4.3.13}$$

$$\begin{aligned}
\text{implies that } A_{10} + \frac{\bar{\lambda}}{\bar{\mu}}A_2 &= A_{10} + A_0 \\
&= A_1 + A_2 + A_0 \\
&= A.
\end{aligned} \tag{4.3.14}$$

Now, if we take

$$\phi(i) = \eta \left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^i \boldsymbol{\pi}, i \geq 0, \tag{4.3.15}$$

where $\boldsymbol{\pi}$ is the steady state vector of the generator matrix A , which was found in section 3.1 and η a constant, equations (4.3.14) and (4.3.15) helps us to write:

$$\begin{aligned}
\phi(0)A_{10} + \phi(1)A_2 &= \eta \boldsymbol{\pi} \left(A_{10} + \frac{\bar{\lambda}}{\bar{\mu}}A_2 \right) \\
&= \eta \boldsymbol{\pi} A \\
&= 0
\end{aligned} \tag{4.3.16}$$

$$\begin{aligned}
\phi(i-1)A_0 + \phi(i)A_1 + \phi(i+1)A_2 &= \eta \boldsymbol{\pi} \left(\frac{\bar{\lambda}}{\bar{\mu}} \right)^{i-1} \left(A_0 + \frac{\bar{\lambda}}{\bar{\mu}} A_1 + \left(\frac{\bar{\lambda}}{\bar{\mu}} \right)^2 A_2 \right) \\
&= \eta \boldsymbol{\pi} \left(\frac{\bar{\lambda}}{\bar{\mu}} \right)^i A \\
&= 0.
\end{aligned} \tag{4.3.17}$$

Hence, equations (4.3.16) and (4.3.17) show that if we take $\boldsymbol{\phi} = (\phi(0), \phi(1), \phi(2), \dots)$ as in (4.3.15), equations (4.3.9) and (4.3.10) are satisfied. Since $\boldsymbol{\pi} \mathbf{e} = 1$, it follows from the normalizing condition $\boldsymbol{\pi} \mathbf{e} = 1$ that the unknown constant $\eta = 1 - \rho$, where $\rho = \frac{\bar{\lambda}}{\bar{\mu}}$.

We state the above discussion in the following theorem.

Theorem 4.3.1. *The steady state probability vector $\boldsymbol{\phi} = (\phi(0), \phi(1), \phi(2), \dots)$ of the Markov chain $\{X(t), t \geq 0\}$ is given in product form as:*

$$\phi(i) = (1 - \rho) \left(\frac{\bar{\lambda}}{\bar{\mu}} \right)^i \boldsymbol{\pi}, i \geq 0,$$

where $\boldsymbol{\pi}$ is the steady state probability vector of the generator matrix $A = A_0 + A_1 + A_2$.

4.4. Performance measures

4.4.1. Busy period of the server with the failed components of the main system.

Let T_m denote the server busy period with failed components of the main system which starts with m failed components and i external customers in the system. We notice that external customers have no influence on T_m since our repair policy is pre-emptive.

For analyzing T_m , let $Y(t)$ be the number of failed components of the k -out-of- n system. When T_m starts, $Y(t) = m$; then $Y(t)$ may increase by 1 at the rate λ and may decrease by 1 at the rate μ .

When $Y(t) = 0$, T_m gets realized. $\{Y(t), t \geq 0\}$ is a Markov chain with state space $\{0, 1, 2, \dots, n - k + 1\}$, where 0 is an absorbing state. The infinitesimal generator matrix of $\{Y(t)\}$ is given by

$$S_h = \begin{bmatrix} 0 & 0 \\ S^0 & S \end{bmatrix}, \quad \text{where } S = \lambda E_{n-k+1} + \lambda c_{n-k+1}(n - k + 1)$$

$$\otimes r_{n-k+1}(n - k + 1) + \mu E'_{n-k+1} \text{ and } S^0 = -S \mathbf{e}.$$

Busy period T_m is the time until absorption in the Markov chain $\{Y(t)\}$, assuming that it starts at the state m . Hence T_m has a phase type distribution with representation (α, S) where $\alpha = r_{n-k+1}(m)$. The expected value of T_m is therefore given by

$$ET_m = -(\alpha S^{-1} \mathbf{e})$$

$$ET_m = \frac{1}{\mu} \left(m \sum_{j=0}^{n-k-m+1} \left(\frac{\lambda}{\mu} \right)^j + \sum_{j=n-k-m+2}^{n-k} (n - k + 1 - j) \left(\frac{\lambda}{\mu} \right)^j \right).$$

We recall that the busy period of failed components starts when at least 1 failed component is present at the realization epoch of the random time T . The state of the Markov chain $\{X(t), t \geq 0\}$ just before start of busy period T_m is $(i, 0, j)$, $i \geq 0, 1 \leq j \leq n - k + 1$. We take the probability of finding m failed components just before the start of a busy

period T_m with an arbitrary number of external customers as $\phi(m) = \sum_{i=0}^{\infty} \phi(i, 0, m)$. The expected value of busy period with failed components, which start with an arbitrary number of failed components and an arbitrary number of external customers, is then given by

$$E_S = \frac{\sum_{m=1}^{n-k+1} \phi(m)ET_m}{\sum_{m=1}^{n-k+1} \phi(m)}.$$

4.4.2. Probability that the main system goes to the down state before the random time T is materialized.

Here we derive $P_T(i)$ the probability of finding i , $0 \leq i \leq n - k + 1$ failed components at the realization epoch of the random time T . For this purpose, we consider the Markov chain $\{\tilde{Y}(t), t \geq 0\}$, where $\tilde{Y}(t)$ represents the number of failed components. Besides the states $0, 1, \dots, n - k + 1$, we consider $n - k + 2$ absorbing states for $\tilde{Y}(t)$ denoted by $\Delta_0, \Delta_1, \dots, \Delta_{n-k+1}$, where absorption to the state Δ_i means that at the realization epoch of T , there were i failed components in the system. Hence the state space of $\tilde{Y}(t)$ is given by $\{\Delta_0, \Delta_1, \dots, \Delta_{n-k+1}, 0, 1, \dots, n - k + 1\}$. Let ∇ denote the collection of non-absorbing states $\{0, 1, \dots, n - k + 1\}$. The infinitesimal generator matrix of $\tilde{Y}(t)$ is given by

$$\tilde{U} = \begin{matrix} & \Delta_0 & \cdots & \Delta_{n-k+1} & \nabla \\ \Delta_0 & \left(\begin{array}{cccc} 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ U_0 & \cdots & U_{n-k+1} & U \end{array} \right) \\ \vdots & & & & \\ \Delta_{n-k+1} & & & & \\ \nabla & & & & \end{matrix},$$

where $U_i = \delta c_{n-k+2}(i+1)$, $0 \leq i \leq n-k+1$ is the column matrix, which governs absorption from ∇ to Δ_i and $U = \lambda E_{n-k+2} + \lambda c_{n-k+2}(n-k+2) \otimes r_{n-k+2}(n-k+2) - \delta I_{n-k+2}$ is the matrix, which governs transitions between the various states in ∇ . Now $P_T(i)$ is the probability that absorption occurs to the state Δ_i in $\tilde{Y}(t)$ and hence, $P_T(i) = -\beta U^{-1} U_i$, where $\beta = r_{n-k+2}(1)$. Therefore, $P_T(i)$ is the first entry of the column matrix $-U^{-1} U_i = Z = (z_1, z_2, \dots, z_{n-k+2})'$. That is $P_T(i) = z_1$. To compute this for $1 \leq i \leq n-k$, we notice that $UZ = -U_i = -\delta c_{n-k+2}(i+1)$, which gives rise to the following equations:

$$-(\lambda + \delta)z_j + \lambda z_{j+1} = 0, 1 \leq j \leq i \quad (4.4.1)$$

$$-(\lambda + \delta)z_{i+1} + \lambda z_{i+2} = -\delta \quad (4.4.2)$$

$$-(\lambda + \delta)z_j + \lambda z_{j+1} = 0, i+2 \leq j \leq n-k+1$$

$$-\delta z_{n-k+2} = 0. \quad (4.4.3)$$

It follows from equations (4.4.3) that $z_j = 0$ for $i+2 \leq j \leq n-k+2$ and equation (4.4.2)

gives

$$z_{i+1} = \frac{\delta}{(\lambda + \delta)}. \quad (4.4.4)$$

Iterating backwards, equation (4.4.1) gives

$$P_T(i) = z_1 = \left(\frac{\lambda}{\lambda + \delta} \right)^i \frac{\delta}{(\lambda + \delta)}, 1 \leq i \leq n-k.$$

A similar computation gives

$$P_T(0) = \frac{\delta}{(\lambda + \delta)} \quad \text{and} \quad P_T(n - k + 1) = \left(\frac{\lambda}{\lambda + \delta} \right)^{n-k+1}.$$

4.4.3. Other performance measures.

The measures that are described below refer to system condition in a cycle.

(1) Fraction of time the system is down,

$$P_{down} = \sum_{j_1=0}^{\infty} \phi(j_1, 0, n - k + 1) + \sum_{j_1=0}^{\infty} \phi(j_1, 1, n - k + 1)$$

(2) System reliability,

$$= 1 - P_{down}$$

(3) Average number of external customers waiting in the queue,

$$N_q = \sum_{j_1=2}^{\infty} (j_1 - 1) \sum_{j_3=0}^{n-k+1} \phi(j_1, 0, j_3) + \sum_{j_1=0}^{\infty} j_1 \sum_{j_3=1}^{n-k+1} \phi(j_1, 1, j_3).$$

(4) Average number of failed components of the main system,

$$N_{fail} = \sum_{j_3=0}^{n-k+1} J_3 \sum_{j_1=0}^{\infty} \phi(j_1, 0, j_3) + \sum_{j_3=1}^{n-k+1} j_1 \sum_{j_1=0}^{\infty} \phi(j_1, 1, j_3)$$

(5) Average number of failed components waiting when the server is busy with external customers

$$= \sum_{j_3=0}^{n-k+1} j_3 \sum_{j_1=1}^{\infty} \phi(j_1, 0, j_3)$$

(6) Expected number of external customers joining the system,

$$\theta_3 = \bar{\lambda} \sum_{j_1=0}^{\infty} \sum_{j_3=0}^{n-k+1} \phi(j_1, 0, j_3).$$

(7) Expected number of external customers on its arrival gets service directly

$$= \bar{\mu} \sum_{j_3=0}^{n-k+1} \phi(0, 0, j_3)$$

(8) Fraction of time the server is busy with external customers,

$$P_{ex.busy} = \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \phi(j_1, 0, j_3) \right)$$

(9) Probability that the server is idle,

$$P_{idle} = \sum_{j_3=0}^{n-k+1} \phi(0, 0, n - k + 1)$$

(10) Probability that the server is busy,

$$P_{busy} = 1 - p_{idle}$$

(11) Expected loss rate of external customers,

$$\theta_4 = \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} \phi(j_1, 1, j_3) \right)$$

(12) Expected service completion rate of external customers,

$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=0}^{n-k+1} \phi(j_1, 0, j_3) \right)$$

(13) Expected number of external customers when server is busy with external customers,

$$\theta_6 = \sum_{j_1=1}^{\infty} j_1 \left(\sum_{j_3=0}^{n-k+1} \phi(j_1, 0, j_3) \right)$$

4.5. Numerical study of the performance of the system

We notice that under the T -policy discussed in this chapter, the priority of failed components begins only on the realization of the random time T . If T is not realized, there is possibility of system being found in the down state. Table 4.1 shows that as δ , the realization rate of the random time T decreases, the reliability of the system also decreases. Due to the preemptive nature of the service to external customers, allowing them doesn't affect the reliability further. This fact also follows from the nature of the steady state distribution given in Theorem 4.3.1, where $\boldsymbol{\pi}$ is the steady state probability vector of a k -out-of- n system with T -policy and no external customers. In a k -out-of- n system with T -policy, the server remains idle if the random time T is not realized. Table 4.2 shows that the server idle probability is 0.27 even when $\delta = 2$. Hence rendering service to external customers during this idle period might be a good idea for generating additional income to the system. Table 4.3 justifies this intuition. If the realization rate is small, for example $\delta = 0.005$, Table 4.2 shows that server idle probability is 0.93 (when $n = 45$), in a system where external customers are not allowed; however when

external customers are allowed, it follows from Table 4.3 that the server idle probability is reduced to 0.56. At the same time Table 4.1 show that the system reliability is just 0.1, when $\delta = 0.005$. Hence finding an optimal value for δ seems to be an interesting problem. For the same, we constructed and analyzed a cost function as follows:

Let C_1 be the cost per unit time incurred if the system is down, C_2 be the holding cost per unit time per external customer in the queue, C_3 is the cost incurred for starting failed components service, C_4 be the cost due to loss of 1 external customer, C_5 be the holding cost per unit time of one failed component, C_6 be the cost per unit time if the server is idle. Now, consider the cost function,

$$\textbf{Expected cost per unit time} = C_1 \cdot P_{down} + C_2 \cdot N_q + C_4 \cdot \theta_4 + \frac{C_3}{E_S} + C_5 \cdot N_{fail} + C_6 \cdot p_{idle}.$$

Table 4.4 shows that an optimal value for δ can be obtained for different component failure rates $\lambda = 4, 4.5, 5, 6$.

Table 4.1. Effect of T -policy on the system reliability

$$\lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$$

δ	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
2	0.9999279	0.9999905	0.9999988	1	1
1.9	0.9999885	0.9999839	0.9999976	0.9999994	1
1.7	0.99997264	0.9999535	0.9999923	0.9999988	1
1.5	0.99993074	0.9998593	0.9999714	0.9999994	1
1.3	0.99981839	0.9999556	0.9998915	0.9999732	0.9999934
1.1	0.99950525	0.99985361	0.99995661	0.99998713	0.99999102
0.9	0.99859357	0.99949306	0.9998166	0.9999336	0.99997592
0.7	0.99579561	0.99814415	0.99917561	0.99963278	0.99983621
0.5	0.98655808	0.99271452	0.99600875	0.99780077	0.99878436
0.3	0.95170438	0.96792483	0.9783597	0.98524916	0.98987627
0.1	0.75673604	0.8010478	0.83504582	0.86177111	0.88318461
0.09	0.72990203	0.77637798	0.81244963	0.84109616	0.86426902
0.07	0.66210544	0.71273649	0.75308931	0.78589147	0.81298745
0.05	0.56635225	0.61985165	0.66396767	0.7009027	0.73222154
0.03	0.42258197	0.47405708	0.51860726	0.55751562	0.59176707
0.01	0.18571174	0.21704859	0.24641669	0.27399105	0.29992962
0.009	0.16983104	0.19904071	0.22656441	0.25254005	0.27709335
0.007	0.1364851	0.1608994	0.18416959	0.20637238	0.22757864
0.005	0.1008448	0.11963266	0.1377635	0.15526974	0.17218304

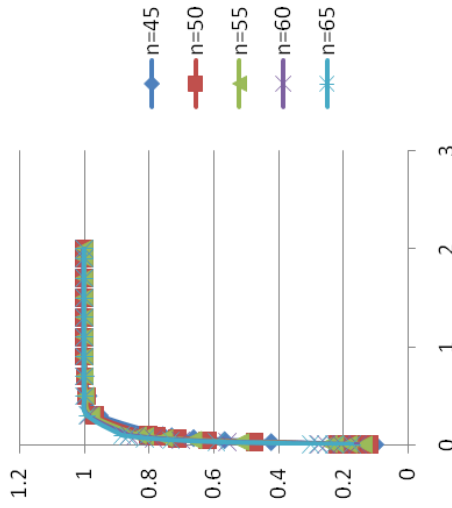


Table 4.2. Server idle probability when external customers are not allowed

$$\lambda = 4, \mu = 5.5$$

δ	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
2	0.272913	0.272767	0.272736	0.27273	0.272729
1.9	0.272936	0.272772	0.272738	0.27273	0.272729
1.7	0.273006	0.272789	0.272741	0.272731	0.272729
1.5	0.273136	0.272825	0.272751	0.272734	0.272729
1.3	0.27339	0.272903	0.272774	0.27274	0.272732
1.1	0.273938	0.273099	0.272841	0.272762	0.272739
0.9	0.275209	0.273634	0.273058	0.272848	0.272772
0.7	0.278417	0.27525	0.27385	0.273228	0.272951
0.5	0.287366	0.280672	0.277082	0.275128	0.274055
0.3	0.31662	0.301886	0.292402	0.286139	0.281933
0.1	0.462284	0.42776	0.401269	0.380445	0.363758
0.09	0.481702	0.445748	0.41784	0.395673	0.377745
0.07	0.530501	0.49188	0.461096	0.436072	0.415402
0.05	0.598988	0.558741	0.525551	0.497763	0.474201
0.03	0.701245	0.66305	0.62999	0.601116	0.575699
0.01	0.868924	0.845983	0.824482	0.804295	0.785306
0.009	0.880143	0.858773	0.838636	0.819633	0.801669
0.007	0.903695	0.885858	0.868857	0.852636	0.837142
0.005	0.928859	0.915152	0.901924	0.889152	0.876812

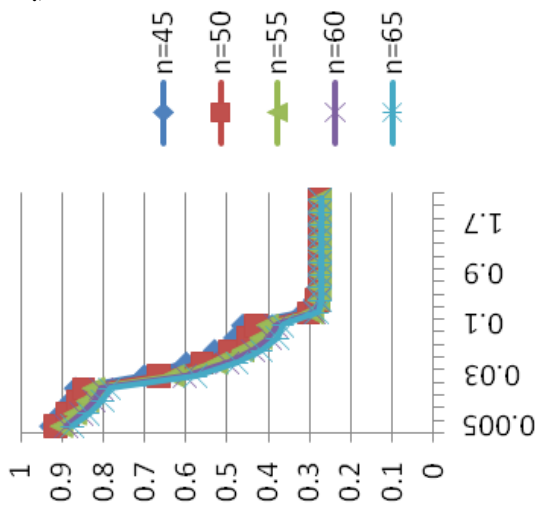


Table 4.3. Server busy probability $\lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8$

δ	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
2	0.83625174	0.83633959	0.83635795	0.83636183	0.8363626
1.9	0.83623791	0.83633637	0.83635724	0.83636159	0.83636248
1.7	0.83619607	0.83632624	0.83635497	0.83636123	0.83636248
1.5	0.83611846	0.83630478	0.83634913	0.83635932	0.8363626
1.3	0.83596551	0.83625788	0.83633566	0.83635581	0.83636087
1.1	0.83563685	0.83614051	0.83629549	0.83634257	0.83635658
0.9	0.83487439	0.83581924	0.83616483	0.8362909	0.83633661
0.7	0.83294982	0.83485001	0.83568978	0.83606309	0.83622938
0.5	0.82757992	0.83159673	0.83375055	0.83492297	0.83556664
0.3	0.81002748	0.81886792	0.82455844	0.82831609	0.83084011
0.1	0.72262919	0.74334335	0.75923824	0.77173293	0.78174466
0.09	0.71097857	0.73255086	0.74929571	0.76259595	0.7733531
0.07	0.6816991	0.70487189	0.72334206	0.73835635	0.75075841
0.05	0.6406064	0.66475487	0.68466902	0.70134163	0.7154789
0.03	0.57925224	0.60216975	0.62200582	0.63932979	0.65458047
0.01	0.47864473	0.49240989	0.50531	0.5174222	0.52881599
0.009	0.4719137	0.48473579	0.49681765	0.50822002	0.51899797
0.007	0.45778227	0.46848452	0.47868532	0.4884181	0.49771422
0.005	0.44268376	0.45090812	0.45884484	0.46650821	0.47391206

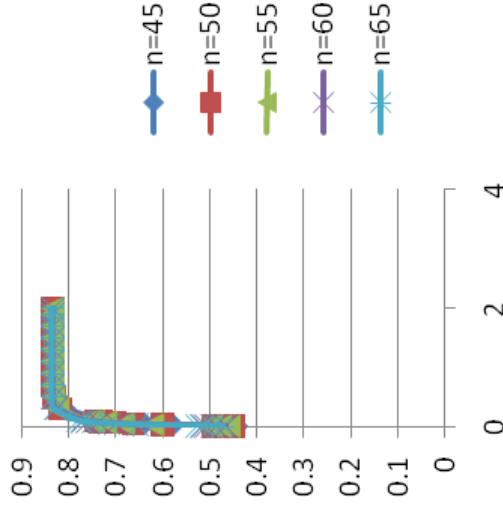
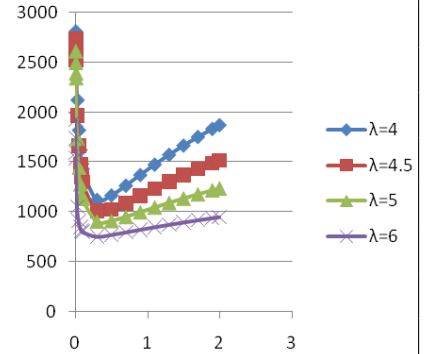


Table 4.4. Variation in cost $n = 45, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, C_1 = 2000, C_2 = 1000, C_3 = 1600, C_4 = 1000, C_5 = 500, C_6 = 100, k = 20$

δ	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$	$\lambda = 6$
0.005	2812.05054	2734.50244	2608.92358	1932.0752
0.007	2735.31641	2643.51587	2492.33667	1740.06384
0.009	2663.59082	2559.8562	2388.50049	1597.91943
0.01	2629.45825	2520.50146	2340.71875	1539.83276
0.03	2123.09912	1969.05518	1735.47485	1051.33716
0.05	1820.06433	1665.29358	1446.2063	912.222778
0.07	1621.95947	1475.9408	1279.27649	848.368347
0.09	1484.99597	1348.74854	1172.36658	812.769226
0.1	1431.97754	1300.20605	1132.58997	800.651184
0.3	1122.22144	1012.10541	905.285889	751.959412
0.5	1166.1283	1028.01672	911.207886	770.260803
0.7	1264.1792	1090.52588	950.656372	795.817017
0.9	1371.0564	1162.78784	997.501831	822.161316
1.1	1475.37793	1235.22021	1045.06689	847.852661
1.3	1574.04211	1304.87305	1091.26477	872.51886
1.5	1666.43896	1370.8844	1135.39673	895.990112
1.7	1752.74353	1433.12891	1177.30688	918.267517
1.9	1833.38123	1491.73572	1217.02124	939.585693
2	1871.72095	1519.74512	1236.07495	949.794006



Chapter 5

Reliability analysis of a k -out-of- n system with repair facility extending service to external customers in a pool of infinite capacity

5.1. The queueing model

We consider a k -out-of- n system with a single server, rendering service to external customers also. It has a finite buffer of capacity $n - k + 1$ in which the failed components of the main system wait for service in the order of their arrival. Also it has a pool of external customers with infinite capacity.

When no external customers are present, the system behaves like a $M/PH/1/n-k+1$ queue. At the end of a service if there are external customers in the pool, the system operates as follows: (i) if the buffer is empty an external customer from the pool is

transferred to the buffer with probability 1 and immediately starts its service (ii) if the queue size in the buffer is less than L ($1 \leq L \leq n - k + 1$), a pre-assigned number called the transition level, an external customer from the pool is transferred to the head of the queue in the buffer with probability ‘ p ’ and immediately enters for service (iii) if there are between L and $n - k + 1$ failed components in the buffer, the customer at the head of the queue in the buffer enters in to the service process. We assume that an external customer who on arrival finds the server busy with main customers, joins the pool with probability γ , $0 \leq \gamma \leq 1$.

Failure time of components of the main system is assumed to follow an exponential distribution with parameter $\frac{\lambda}{i}$ when i components are operational. External customers arrive according to a Poisson process with parameter $\bar{\lambda}$. The service process of main customers and external customers has the same phase type distribution with representation (S, α) of order m .

In the sequel, \mathbf{e} denote a column vector of 1’s of appropriate order, I_n denotes an identity matrix of order n , \otimes stands for Kronecker product of matrices and S^0 is given by $S^0 = -S \mathbf{e}$.

5.1.1. The Markov Chain.

Let $J_1(t)$ = number of external customers in the pool including the one getting service (if any) at time t ,

$J_2(t)$ = number of main customers in the buffer including the one getting service (if any) at time t ,

$$S(t) = \begin{cases} 0, & \text{if the server is idle or the server is busy with external customers} \\ 1, & \text{if the server is busy with main customers} \end{cases}$$

$J_3(t)$ = phase of the service process at time t .

Then $X(t) = (J_1(t), S(t), J_2(t), J_3(t))$ is a continuous time Markov chain on the state space

$$\bigcup_{i=0}^{\infty} l(i) \text{ where } l(0) = \{(0, 0, 0)\} \cup \{(0, 1, j_2, j_3) / 1 \leq j_2 \leq n - k + 1, 1 \leq j_3 \leq m\} \text{ and for } i \geq 1$$

$$l(i) = \{(i, 0, j_2, j_3) / 0 \leq j_2 \leq n - k + 1, 1 \leq j_3 \leq m\} \cup \{(i, 1, j_2, j_3) / 1 \leq j_2 \leq n - k + 1, 1 \leq j_3 \leq m\}.$$

The infinitesimal generator of this process,

$$Q = \begin{bmatrix} B_0 & B_1 & & & & & & & \\ B_2 & A_1 & A_0 & & & & & & \\ & A_2 & A_1 & A_0 & & & & & \\ & & A_2 & A_1 & A_0 & & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & \\ & & & & & & \cdot & \cdot & \cdot \end{bmatrix}$$

where the matrix B_0 is a square matrix of order $1 + m(n - k + 1)$; the matrices B_1 and B_2 are of orders $(1 + m(n - k + 1)) \times (m(2n - 2k + 3))$ and $(m(2n - 2k + 3)) \times (1 + m(n - k + 1))$ respectively; A_0, A_1 and A_2 are square matrices of order $(m(2n - 2k + 3)) \times (m(2n - 2k + 3))$

Transition from level from 0 to 1 is represented by the matrix

$$B_1 = \begin{bmatrix} \bar{\lambda}\alpha & 0_{1 \times (n-k+1)m} & 0 \\ 0 & 0 & I_{n-k+1} \otimes \gamma \bar{\lambda} I_m \end{bmatrix}$$

The transition from level i to $i + 1, i \geq 1$, is represented by

$$A_0 = \begin{bmatrix} \bar{\lambda} I_{(n-k+2)m} & 0 \\ 0 & \gamma \bar{\lambda} I_{(n-k+1)m} \end{bmatrix}$$

Transition from level 1 to 0 is represented by

$$B_2 = \begin{bmatrix} S^0 & 0 \\ 0 & I_{n-k+1} \otimes S^0 \alpha \\ 0_{(n-k+1)m \times 1} & 0_{(n-k+1)m \times (n-k+1)m} \end{bmatrix}$$

The transition from level i to $i - 1, i \geq 2$ is represented by

$$A_2 = \begin{bmatrix} S^0 \alpha & 0 & 0 & 0 & 0 \\ 0 & I_{L-1} \otimes p S^0 \alpha & 0 & I_{L-1} \otimes (1-p) S^0 \alpha & 0 \\ 0 & 0 & 0_{(n-k+2-L)m \times (n-k+2-L)m} & 0 & I_{(n-k+2-L)} \otimes S^0 \alpha \\ 0_{(n-k+1)m \times m} & 0_{(n-k+1)m \times (L-1)m} & 0_{(n-k+1)m \times (n-k+2-L)m} & 0_{(n-k+1)m \times (L-1)m} & 0_{(n-k+1)m \times (n-k+2-L)m} \end{bmatrix}$$

The transition from level i to i , $i \geq 1$ is represented by the matrix A_1 .

For easy understanding, the structures of the matrix A_1 for $n = 10$, $k = 6$ and $L = 3$ is

$$A_1 = \begin{array}{ccccccc} S - (\lambda + \bar{\lambda})I_m & \lambda I_m & & & & & \\ & S - (\lambda + \bar{\lambda})I_m & \lambda I_m & & & & \\ & & S - (\lambda + \bar{\lambda})I_m & \lambda I_m & & & \\ & & & S - (\lambda + \bar{\lambda})I_m & \lambda I_m & & \\ & & & & S - \bar{\lambda}I_m & 0 & \\ S^0\alpha & & & & S - (\lambda + \gamma\bar{\lambda})I_m & \lambda I_m & \\ & pS^0\alpha & & & (1-p)S^0\alpha & S - (\lambda + \gamma\bar{\lambda})I_m & \lambda I_m \\ & & & & & (1-p)S^0\alpha & S - (\lambda + \gamma\bar{\lambda})I_m & \lambda I_m \\ & & & & & & S^0\alpha & S - (\lambda + \gamma\bar{\lambda})I_m & \lambda I_m \\ & & & & & & & S^0\alpha & S - \gamma \end{array}$$

5.2. Steady state analysis

5.2.1. Stability condition. Consider the generator matrix $A = A_1 + A_2 + A_0$.

For $n = 10$, $k = 6$ and $L = 3$, the structure of the matrix A is

$$A = \left[\begin{array}{cccccccc}
 S + S^0\alpha - \lambda M_m & \lambda M_m & & & & & 0 & \\
 S + pS^0\alpha - \lambda M_m & \lambda M_m & & & & & (1-p)S^0\alpha & \\
 S + pS^0\alpha - \lambda M_m & \lambda M_m & & & & & (1-p)S^0\alpha & \\
 S - \lambda M_m & \lambda M_m & & & & & S^0\alpha & \\
 S - \lambda M_m & \lambda M_m & & & & & S^0\alpha & \\
 S & & & & & & S & \\
 S^0\alpha & & & & & & S - \lambda M_m & \lambda M_m \\
 pS^0\alpha & & & & & & (1-p)S^0\alpha & S - \lambda M_m & \lambda M_m \\
 & & & & & & pS^0\alpha & (1-p)S^0\alpha & S - \lambda M_m & \lambda M_m \\
 & & & & & & & S^0\alpha & S - \lambda M_m & \lambda M_m \\
 & & & & & & & S^0\alpha & S^0\alpha & S
 \end{array} \right]$$

Let $\boldsymbol{\pi} = (\pi_{(0)}, \pi_{(1)}, \dots, \pi_{(n-k+1)}, \tilde{\pi}_{(1)}, \tilde{\pi}_{(2)}, \dots, \tilde{\pi}_{(n-k+1)})$ be the steady state vector of the generator matrix A , where $\pi_{(i)} = (\pi_{(i,1)}, \pi_{(i,2)}, \dots, \pi_{(i,m)})$, $i = 0, 1, \dots, n-k+1$ and $\tilde{\pi}_{(i)} = (\tilde{\pi}_{(i,1)}, \tilde{\pi}_{(i,2)}, \dots, \tilde{\pi}_{(i,m)})$, $i = 1, 2, \dots, n-k+1$, then the equations $\boldsymbol{\pi}A = 0$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$ gives the equations

$$\begin{aligned} \pi_{(0)}(S + S^0\alpha - \lambda I_m) + \tilde{\pi}_{(1)}S^0\alpha &= 0 \\ \pi_{(i)}\lambda I_m + \pi_{(i+1)}(S + pS^0\alpha - \lambda I_m) + \tilde{\pi}_{(i+2)}pS^0\alpha &= 0, 0 \leq i \leq L-2 \\ \pi_{(i)}\lambda I_m + \pi_{(i+1)}(S - \lambda I_m) &= 0, L-1 \leq i \leq n-k+1 \\ \pi_{(n-k)}\lambda I_m + \pi_{(n-k+1)}S &= 0. \end{aligned} \tag{5.2.1}$$

$$\begin{aligned} \pi_{(i)}(1-p)S^0\alpha + \tilde{\pi}_{(1)}(S - \lambda I_m) + \tilde{\pi}_{(2)}(1-p)S^0\alpha &= 0 \\ \pi_{(i)}(1-p)S^0\alpha + \tilde{\pi}_{(i-1)}\lambda I_m + \tilde{\pi}_{(i)}(S - \lambda I_m) + \tilde{\pi}_{(i+1)}(1-p)S^0\alpha &= 0, 2 \leq i \leq L-1 \\ \pi_{(i)}S^0\alpha + \tilde{\pi}_{(i-1)}\lambda I_m + \tilde{\pi}_{(i)}(S - \lambda I_m) + \tilde{\pi}_{(i+1)}S^0\alpha &= 0, L \leq i \leq n-k \\ \pi_{(n-k+1)}S^0\alpha + \tilde{\pi}_{(n-k)}\lambda I_m + \tilde{\pi}_{(n-k+1)}S &= 0. \end{aligned} \tag{5.2.2}$$

On simplification, we can express the equations represented by equ (5.2.1) as

$$\begin{aligned} \pi_{(0)}(S + S^0\alpha - \lambda I_m + \lambda e\alpha) &= 0 \\ \pi_{(i)}\lambda(I_m - e\alpha) + \pi_{(i+1)}(S + S^0\alpha - \lambda I_m + \lambda e\alpha) &= 0, 0 \leq i \leq n-k-1 \\ \pi_{(n-k)}\lambda(I_m - e\alpha) + \pi_{(n-k+1)}(S + S^0\alpha) &= 0. \end{aligned} \tag{5.2.3}$$

Adding these equations we get

$$(\pi_{(0)} + \pi_{(1)} + \dots + \pi_{(n-k+1)})(S + S^0\alpha) = 0. \tag{5.2.4}$$

This shows that the vector $\pi_{(0)} + \pi_{(1)} + \dots + \pi_{(n-k+1)}$ is a constant multiple of the steady state vector η of the generator matrix $S + S^0\alpha$. Let

$$\pi_{(0)} + \pi_{(1)} + \dots + \pi_{(n-k+1)} = \delta\eta. \quad (5.2.5)$$

Similarly, on simplification of the equations represented by (5.2.2), we obtain $(\tilde{\pi}_{(1)} + \tilde{\pi}_{(2)} + \dots + \tilde{\pi}_{(n-k+1)})(S + S^0\alpha) = 0$. This implies that $\tilde{\pi}_{(1)} + \tilde{\pi}_{(2)} + \dots + \tilde{\pi}_{(n-k+1)}$ is a constant multiple of the steady state vector η . Since $\pi\mathbf{e} = 1$, we have

$$\tilde{\pi}_{(1)} + \tilde{\pi}_{(2)} + \dots + \tilde{\pi}_{(n-k+1)} = (1 - \delta)\eta. \quad (5.2.6)$$

The stability condition $\pi A_0\mathbf{e} < \pi A_2\mathbf{e}$, that is

$$\begin{aligned} (\pi_{(0)} + \pi_{(1)} + \dots + \pi_{(n-k+1)})\bar{\lambda}\mathbf{e} + (\tilde{\pi}_{(1)} + \tilde{\pi}_{(2)} + \dots + \tilde{\pi}_{(n-k+1)}) \\ \gamma\bar{\lambda}\mathbf{e} < (\pi_{(0)} + \pi_{(1)} + \dots + \pi_{(n-k+1)})S^0, \end{aligned}$$

thus becomes

$$\delta\bar{\lambda} + (1 - \delta)\gamma\bar{\lambda} < \delta\eta S^0. \quad (5.2.7)$$

If $\gamma = 0$, that is if the arrival of external customers is blocked while the server is busy with main customers, the stability condition (5.2.7) becomes

$$\bar{\lambda} < \eta S^0. \quad (5.2.8)$$

5.2.2. The steady state probability vector. Let $\boldsymbol{\pi} = (\boldsymbol{\pi}(0), \boldsymbol{\pi}(1), \boldsymbol{\pi}(2), \dots)$ be the steady state vector of the Markov chain $\{X(t), t \geq 0\}$ where

$$\boldsymbol{\pi}(0) = (\pi_{(0,0)}, \tilde{\pi}_{(0,1)}, \tilde{\pi}_{(0,2)}, \dots, \tilde{\pi}_{(0,n-k+1)}), \text{ and}$$

$$\boldsymbol{\pi}(i) = (\pi_{(i,0)}, \pi_{(i,1)}, \pi_{(i,2)}, \dots, \pi_{(i,n-k+1)}, \tilde{\pi}_{(0,1)}, \tilde{\pi}_{(0,2)}, \dots, \tilde{\pi}_{(0,n-k+1)}), \text{ here}$$

$$\boldsymbol{\pi}_{(i,j)} = (\pi_{(i,j,1)}, \pi_{(i,j,2)}, \dots, \pi_{(i,j,m)}), i = 1, 2, \dots \text{ and } j = 0, 1, \dots, n - k + 1$$

and $\tilde{\pi}_{(i,j)} = (\tilde{\pi}_{(i,j,1)}, \tilde{\pi}_{(i,j,2)}, \dots, \tilde{\pi}_{(i,j,m)}), i = 0, 1, 2, \dots$ and $j = 1, 2, \dots, n - k + 1$.

Let

$$\boldsymbol{\pi}(i+1) = \boldsymbol{\pi}(1)R^i, i \geq 1, \dots$$

Then from $\boldsymbol{\pi}Q = 0$, we get

$$\boldsymbol{\pi}(0)A_0 + \boldsymbol{\pi}(1)A_1 + \boldsymbol{\pi}(2)A_2 = 0, \quad (5.2.9)$$

$$\boldsymbol{\pi}(0)(A_0 + RA_1 + R^2A_2) = 0.$$

Choose R as the minimal non negative solution of $A_0 + RA_1 + R^2A_2 = 0$. Then from (5.2.9), we have

$$\boldsymbol{\pi}(1) = -\boldsymbol{\pi}(0)B_1(A_1 + RA_2)^{-1}$$

$$\boldsymbol{\pi}(1) = \boldsymbol{\pi}(0)\omega \quad (5.2.10)$$

where $\omega = -B_1(A_1 + RA_2)^{-1}$.

Also

$$\boldsymbol{\pi}(\mathbf{0})B_0 + \boldsymbol{\pi}(\mathbf{1})B_2 = 0$$

$$\boldsymbol{\pi}(\mathbf{0})(B_0 + \omega B_2) = 0.$$

First take $\boldsymbol{\pi}(\mathbf{0})$ as the steady state vector of $B_0 + \omega B_2$, then $\boldsymbol{\pi}(\mathbf{1})$ can be obtained using (5.2.10). $i \geq 2$, $\boldsymbol{\pi}(\mathbf{i})$ can be found using the recursive formula $\boldsymbol{\pi}(\mathbf{i} + \mathbf{1}) = \boldsymbol{\pi}(\mathbf{1})R^i$, $i \geq 1$.

The steady state probability distribution of the system is obtained by dividing each $\pi(i)$ with the normalising constant $(\boldsymbol{\pi}(\mathbf{0}) + \boldsymbol{\pi}(\mathbf{1}) + \dots) \mathbf{e} = \boldsymbol{\pi}(\mathbf{0})\mathbf{e} + \boldsymbol{\pi}(\mathbf{1})(I - R)^{-1} \mathbf{e}$.

5.3. Performance measures

(1) Fraction of time the system is down,

$$P_{down} = \sum_{j_1=0}^{\infty} \sum_{j_4=1}^m \pi(j_1, n - k + 1, 1, j_4) + \sum_{j_1=0}^{\infty} \sum_{j_4=1}^m \pi(j_1, n - k + 1, 0, j_4)$$

(2) System reliability,

$$P_{rel} = 1 - P_{down}$$

(3) Average number of external customers waiting in the pool,

$$N_q = \sum_{j_1=2}^{\infty} (j_1 - 1) \left(\sum_{j_2=0}^{n-k+1} \sum_{j_4=1}^m \pi(j_1, j_2, 0, j_4) \right) + \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_2=1}^{n-k+1} \sum_{j_4=1}^m \pi(j_1, j_2, 1, j_4) \right).$$

(4) Average number of failed components in the main system,

$$N_{fail} = \sum_{j_2=0}^{n-k+1} J_2 \left(\sum_{j_1=1}^{\infty} \sum_{j_4=1}^m \pi(j_1, j_2, 0, j_4) \right) + \sum_{j_2=1}^{n-k+1} j_2 \left(\sum_{j_1=0}^{\infty} \sum_{j_4=1}^m \pi(j_1, j_2, 1, j_4) \right).$$

(5) Average number of failed components waiting when the server is busy with external customers

$$= \sum_{j_2=0}^{n-k+1} j_2 \left(\sum_{j_1=1}^{\infty} \sum_{j_4=1}^m \pi(j_1, j_2, 0, j_4) \right)$$

(6) Fraction of time the server is busy with external customers,

$$P_{ex.busy} = \sum_{j_1=1}^{\infty} \sum_{j_2=0}^{n-k+1} \sum_{j_4=1}^m \pi(j_1, j_2, 0, j_4)$$

(7) Probability that the server is found idle,

$$P_{idle} = \pi(0, 0, 0)$$

(8) Probability that the server is busy,

$$P_{busy} = 1 - p_{idle}$$

(9) Expected loss rate of external customers,

$$\theta_4 = \bar{\lambda} \sum_{j_1=1}^{\infty} \sum_{j_2=0}^{n-k+1} \sum_{j_4=1}^m \pi(j_1, j_2, 1, j_4)$$

(10) Expected number of external customers in the system when server is busy with external customers,

$$\theta_6 = \sum_{j_1=1}^{\infty} j_1 \left(\sum_{j_2=0}^{n-k+1} \sum_{j_4=1}^m \pi(j_1, j_2, 0, j_4) \right)$$

5.4. Numerical study of the performance of the system

Here, since the service to external customers is of non-preemptive nature, there is a possibility of system going to the down status while external customers are getting service. Hence, we studied the effect of the transition level L on the reliability of the system. However, Table 5.1 shows that a very high reliability is maintained in the system. The decrease in reliability as L increases is expected, since the increase in L leads to more external customers being selected for service. However, Table 5.1 shows that the rate of decrease in reliability is very slow. We have compared the reliability of the current system with that of a system, where external customers are not allowed and had found that they agree up to first 7 decimal places for different values of n . The server busy probability was found to be 0.4, for a system where no external customers are allowed. Table 5.2 shows that the server busy probability is above 0.57, when external customers are allowed. Table 5.3 shows that the fraction of time the server remains busy with external customers $P_{ext.busy}$ is > 0.24 . The increase in $P_{ext.busy}$ as L increases, as shown by Table 5.3 is expected, since as L increases, external customers obtain service more frequently. The same reasoning can be attributed to a decrease in the server busy probability with main customers, which is reflected in the decrease in p_{busy} with an increase in L , is noticed in Table 5.2. Though the entire system reliability may be satisfactory, with external customers getting more frequent service, while L increases, the possible dissatisfaction caused to the main customers forced us to investigate a cost function in hope of finding an optimal value for L .

Let C_1 be the cost per unit time incurred if the system is down, C_2 be the holding cost per unit time per external customer in the pool, C_3 is the cost incurred for starting

service of failed components, C_4 be the cost due to loss of 1 external customer, C_5 be the holding cost per unit time of one failed component, C_6 be the cost per unit time when the server is idle.

Now, consider the cost function,

$$\text{Expected Cost per unit time} = C_1 \cdot P_{down} + C_2 \cdot N_q + C_4 \cdot \theta_4 + \frac{C_3}{E_S} + C_5 \cdot N_{fail} + C_6 \cdot idle.$$

Table 5.4 shows that an optimal value for L can be obtained for different component failure rates $\lambda = 4, 4.5, 5$.

Table 5.1. Effect of the Transition level L on the system reliability $\lambda < \mu$ case

$$\lambda = 4, \mu = 5.5, \bar{\lambda} = 3.2, \bar{\mu} = 8, \gamma = 0.55, m = 3$$

L	$n = 45$	$n = 50$	$n = 55$
1	1	1	1
3	1	1	1
5	1	1	1
7	1	1	1
9	1	1	1
11	1	1	1
13	1	1	1
15	1	1	1
17	1	1	1
19	1	1	1
21	1	1	1
23	0.99999994	1	1
25	0.99999994	1	1
27		1	1
29		1	1
31		1	1
33			1
35			1

Table 5.2. Effect of L on the server busy probability

$$\lambda = 4, \mu = 5.5, \bar{\lambda} = 3.2, \bar{\mu} = 8, \gamma = 0.55, m = 3$$

L	$n = 45$	$n = 50$	$n = 55$
1	0.60810256	0.60810256	0.60810256
3	0.586476922	0.586476922	0.586476922
5	0.576753855	0.576753855	0.576753855
7	0.573665261	0.573665261	0.573665261
9	0.572723567	0.572723567	0.572723567
11	0.57243067	0.572430611	0.57243067
13	0.572336793	0.572336733	0.572336793
15	0.572305799	0.572305799	0.572305799
17	0.572295308	0.572295308	0.572295308
19	0.572291672	0.572291672	0.572291672
21	0.572290421	0.572290421	0.572290421
23	0.572290003	0.572290003	0.572290003
25	0.572289705	0.572289824	0.572289824
27		0.572289705	0.572289765
29		0.572289705	0.572289705
31		0.572289467	0.572289705
33			0.572289705
35			0.572289646

Table 5.3. Effect of L on the probability that server is busy with external customers

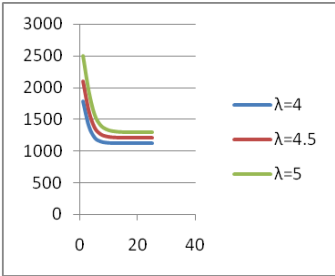
$$\lambda = 4, \mu = 5.5, \bar{\lambda} = 3.2, \bar{\mu} = 8, \gamma = 0.55, m = 3$$

L	$n = 45$	$n = 50$	$n = 55$
1	0.244336635	0.244336635	0.244336635
3	0.250016034	0.250016123	0.250016034
5	0.252569616	0.252569586	0.252569616
7	0.253380775	0.253380775	0.253380775
9	0.253628808	0.253628045	0.253628075
11	0.253704965	0.253704935	0.253704965
13	0.253729612	0.253729612	0.253729613
15	0.253737718	0.253737718	0.253737718
17	0.253740489	0.253740489	0.253740489
19	0.253741443	0.253741443	0.253741443
21	0.253741801	0.253741771	0.253741801
23	0.25374189	0.25374186	0.25374189
25	0.25374189	0.25374192	0.25374192
27		0.25374195	0.25374195
29		0.25374195	0.25374195
31		0.253741831	0.25374195
33			0.25374198
35			0.25374192

Table 5.4. Cost analysis

$n = 45, k = 20, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, \gamma = 0.55, m = 3, C_1 = 900000,$
 $C_2 = 1000, C_3 = 2000, C_4 = 200, C_5 = 500, C_6 = 100$

L	$\lambda=4$	$\lambda=4.5$	$\lambda=5$
1	1781.65845	2097.99414	2493.99146
3	1398.34412	1631.46753	1935.72974
5	1217.66528	1371.07092	1572.30933
7	1157.48804	1269.76624	1409.78735
9	1138.29138	1232.53027	1342.26868
11	1132.05823	1218.68481	1314.16382
13	1129.97559	1213.39905	1302.21362
15	1129.26135	1211.33142	1297.0354
17	1129.01233	1210.51465	1294.7998
19	1128.92676	1210.20691	1293.92249
21	1128.90308	1210.13	1293.76453
23	1128.90906	1210.19128	1294.10291
25	1128.9364	1210.3645	1294.83313



Chapter 6

Reliability of a k -out-of- n System with a repair facility rendering service to external customers in a retrial set up and orbital search under N -policy

6.1. The queueing model

We consider a k -out-of- n system with a single server extending service also to external customers according to N -policy. An external customer, who finds an idle server on its arrival, is immediately taken for service and who finds the server busy with another external customer, joins an orbit of external customers with infinite capacity and from there retries for service. The service to failed components starts only on the epoch

of accumulation of N of them. If such an epoch happens in the middle of an external customer's service, the external customer in service will get pre-empted and the server will be switched over to the service of the failed components. The external customer whose service got preempted is sent back to the orbit. For decreasing the waiting of the external customers in the orbit and also for effectively utilizing the server idle time, we apply a search mechanism for selecting customers from the orbit. This works as follows: at the epoch of service completion of an external customer or at the epoch of service completion of the last main customer, the server makes a search with probability p and selects a customer (if any) randomly from the orbit for the next service. The search time is assumed to be negligible. Also we assumed that the arrival of external customers is completely blocked while serving main customers. Arrival of main and external customers has inter-occurrence times exponentially distributed with parameters λ and $\bar{\lambda}$ respectively. Service times of main customers and external customers are independent exponentially distributed with parameters μ and $\bar{\mu}$ respectively. The inter-retrial times are independent exponentially distributed random variables with parameter θ .

6.1.1. The Markov Chain.

Let $X_1(t)$ = number of external customers in the orbit including the one getting service (if any) at time t .

$X_2(t)$ = number of main customers in the system including the one getting service (if any) at time t .

If $X_1(t) = X_2(t) = 0$, then an arriving external customer is taken for service.

In the sequel,

- (i) I_n denotes identity matrix of order n ;
- (ii) I denotes an identity matrix of appropriate size;
- (iii) e_n denotes a $n \times 1$ column matrix of 1's;
- (iv) e denotes a column matrix of 1's of appropriate order;
- (v) E_n denotes a square matrix of order n defined as

$$E_n(i, j) = \begin{cases} -1; & \text{if } i = j, 1 \leq i \leq n \\ 1; & \text{if } j = i + 1, 1 \leq i \leq n - 1 \\ 0; & \text{otherwise.} \end{cases}$$

- (vi) $E'_n =$ Transpose of E_n
- (vii) $r_n(i)$ denotes a $1 \times n$ row matrix whose i^{th} entry is 1 and all other entries are zeroes
- (viii) $c_n(i) =$ Transpose of $r_n(i)$
- (ix) \otimes denotes Kronecker product of matrices.

The structures of these matrices for $n = 10, k = 6$ and $N = 3$ are as follows.

The transition from level '0' to level '0' is represented by the matrix

$$A_{00} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & 0 \\ H_{31} & 0 & H_{33} \end{bmatrix}$$

$$H_{11} = \lambda E_N - \bar{\lambda} I_N$$

$$H_{12} = \lambda c_N(N) \otimes r_{n-k+1}(N)$$

$$H_{13} = \bar{\lambda}I_N$$

$$H_{21} = \mu c_{n-k+1}(1) \otimes r_N(1)$$

$$H_{22} = \lambda E_{n-k+1} + \lambda c_{n-k+1}(n-k+1) \otimes r_{n-k+1}(n-k+1) + \mu E'_{n-k+1},$$

$$H_{31} = \bar{\mu}I_N$$

$$H_{33} = \lambda E_N - (\bar{\lambda} + \bar{\mu})I_N.$$

The transition level ' i ' to level ' $i+1$ ', $i \geq 0$ is represented by the matrix

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda c_N(N) \otimes r_{n-k+1}(N) & \bar{\lambda}I_N \end{bmatrix}$$

The transition from level ' i ' to level ' $i-1$ ', $i \geq 1$ is represented by the matrix

$$A_{i2} = \begin{bmatrix} 0 & 0 & i\theta I_N \\ 0 & 0 & p\mu c_{n-k+1}(1) \otimes r_N(1) \\ 0 & 0 & p\bar{\mu}I_N \end{bmatrix}.$$

The transition within level i , $i \geq 1$, is represented by the matrix

$$A_{i1} = \begin{bmatrix} H_{11}^{(i)} & H_{12} & H_{13} \\ \tilde{H}_{21} & H_{22} & 0 \\ \tilde{H}_{31} & 0 & H_{33} \end{bmatrix}$$

Define $A_m = A_0 + A_{m1} + A_{m2}$.

$$A_m = \begin{bmatrix} H_{11}^{(m)} & H_{12} & H_{13}^{(m)} \\ \tilde{H}_{21} & H_{22} & \tilde{H}_{23} \\ \tilde{H}_{31} & H_{32} & \tilde{H}_{33} \end{bmatrix}$$

$$H_{13}^{(m)} = (\bar{\lambda} + m\theta)I_N$$

$$\tilde{H}_{23} = p\mu c_{n-k+1}(1) \otimes r_N(1)$$

$$H_{32} = \lambda c_N(N) \otimes r_{n-k+1}(N)$$

$$\tilde{H}_{33} = \lambda E_N - (1-p)\bar{\mu}I_N.$$

Let $\pi_m = (\pi_m(0), \pi_m(1), \pi_m(2))$, where

$$\pi_m(0) = (\pi_m(0, 0), \pi_m(0, 1), \dots, \pi_m(0, N-1))$$

$$\pi_m(1) = (\pi_m(1, 1), \dots, \pi_m(1, n-k+1))$$

$$\pi_m(2) = (\pi_m(2, 0), \pi_m(2, 0), \dots, \pi_m(2, N-1))$$

be the steady state vector of the generator matrix A_m . Then the relation $\pi_m A_m = 0$ implies:

$$\pi_m(0)H_{11}^{(m)} + \pi_m(1)\tilde{H}_{21} + \pi_m(2)\tilde{H}_{31} = 0 \quad (6.2.1)$$

$$\pi_m(0)H_{12} + \pi_m(1)H_{22} + \pi_m(2)H_{32} = 0 \quad (6.2.2)$$

$$\pi_m(0)H_{13}^{(m)} + \pi_m(1)\tilde{H}_{23} + \pi_m(2)\tilde{H}_{33} = 0. \quad (6.2.3)$$

From (6.2.2), it follows that,

$$\pi_m(1) = -\pi_m(0)H_{12}(H_{22})^{-1} - \pi_m(2)H_{32}(H_{22})^{-1}. \quad (6.2.4)$$

Substituting for $\pi_m(1)$ from (6.2.4) in (6.2.1), we obtain,

$$\pi_m(0)H_{11}^{(m)} - \pi_m(0)H_{12}(H_{22})^{-1}\tilde{H}_{21} - \pi_m(2)H_{32}(H_{22})^{-1}\tilde{H}_{21} + \pi_m(2)\tilde{H}_{31} = 0. \quad (6.2.5)$$

We notice that the first column of the matrix \tilde{H}_{21} is $-(1-p)H_{22}\mathbf{e}$ and all other columns are zero columns. Hence the first column of the matrix $(H_{22})^{-1}\tilde{H}_{21}$ is $-(1-p)\mathbf{e}$ and all other columns are zero columns. This in turn tells that the first column of the matrix $-H_{12}(H_{22})^{-1}\tilde{H}_{21}$ is $(1-p)\lambda_{c_N}(N)$ and all other columns are zero columns. In other words,

$$-H_{12}(H_{22})^{-1}\tilde{H}_{21} = (1-p)\lambda_{c_N}(N) \otimes r_N(1). \quad (6.2.6)$$

Since $H_{32} = H_{12}$, it follows that

$$-H_{32}(H_{22})^{-1}\tilde{H}_{21} = (1-p)\lambda_{c_N}(N) \otimes r_N(1). \quad (6.2.7)$$

In the light of equations (6.2.6) and (6.2.7), equation (6.2.5) becomes

$$\pi_m(0)\left(H_{11}^{(m)} + (1-p)\lambda_{c_N}(N) \otimes r_N(1)\right) + \pi_m(2)\left(\tilde{H}_{31} + (1-p)\lambda_{c_N}(N) \otimes r_N(1)\right) = 0. \quad (6.2.8)$$

Substituting for $\pi_m(1)$ from (6.2.4) in (6.2.3) and noticing that the first column of the matrix \widetilde{H}_{23} is $-pH_{22}\mathbf{e}$, reasoning as for equation (6.2.8), we can write

$$\pi_m(0)\left(H_{13}^{(m)} + p\lambda c_N(N) \otimes r_N(1)\right) + \pi_m(2)\left(\widetilde{H}_{33} + p\lambda c_N(N) \otimes r_N(1)\right) = 0. \quad (6.2.9)$$

We notice that $H_{11}^{(m)} + H_{13}^{(m)} = \widetilde{H}_{33} + \widetilde{H}_{31} = \lambda E_N$. Hence adding equations (6.2.8) and (6.2.9), we get

$$(\pi_m(0) + \pi_m(2))(\lambda E_N + \lambda c_N(N) \otimes r_N(1)) = 0. \quad (6.2.10)$$

Equation (6.2.10) implies that the vector $\pi_m(0) + \pi_m(2)$ is a constant multiple of the steady state vector $\frac{1}{N}e'_N$ of the generator matrix $\lambda E_N + \lambda c_N(N) \otimes r_N(1)$ and hence,

$$\pi_m(0) + \pi_m(2) = \eta \frac{1}{N}e'_N \quad (6.2.11)$$

where η is a constant.

Since $H_{32} = H_{12}$, it follows from equation (6.2.2) that,

$$(\pi_m(0) + \pi_m(2))H_{12} + \pi_m(1)H_{22} = 0. \quad (6.2.12)$$

Post multiplying equation (6.2.12) with the column vector \mathbf{e} , we get

$$(\pi_m(0) + \pi_m(2))H_{12}\mathbf{e} + \pi_m(1)H_{22}\mathbf{e} = 0. \quad (6.2.13)$$

We notice that

$$H_{12}\mathbf{e} = \lambda c_N(N) \quad \text{and} \quad H_{22}\mathbf{e} = -\mu c_{n-k+1}(1). \quad (6.2.14)$$

In the light of equations (6.2.11) and (6.2.14), equation (6.2.13) becomes

$$\frac{\eta\lambda}{N} = \pi_m(1, 1)\mu. \quad (6.2.15)$$

Now, from equations (6.2.11) and (6.2.12), it follows that

$$\pi_m(1) = -\frac{\eta}{N}e'_N H_{12} (H_{22})^{-1}. \quad (6.2.16)$$

Post multiplying with the column matrix \mathbf{e} , equation (6.2.16) gives

$$\pi_m(1)\mathbf{e} = -\frac{\eta}{N}e'_N H_{12} (H_{22})^{-1} \mathbf{e}. \quad (6.2.17)$$

Since $H_{12} = \lambda c_N(N) \otimes r_{n-k+1}(N)$, we get $\frac{\eta}{N}e'_N H_{12} = \frac{\eta\lambda}{N}r_{n-k+1}(N)$. Now,

$$-r_{n-k+1}(N)(H_{22})^{-1} \mathbf{e} = \frac{1}{\mu} \left(N \sum_{j=0}^{n-k-N+1} \left(\frac{\lambda}{\mu}\right)^j + \sum_{j=n-k-N+2}^{n-k} (n-k+1-j) \left(\frac{\lambda}{\mu}\right)^j \right). \quad (6.2.18)$$

For details on the derivation of equation (6.2.18), one may refer to Krishnamoorthy, see section 2.4.3 of chapter 2.

Thus equation (6.2.17) becomes

$$\pi_m(1)\mathbf{e} = \frac{\eta\lambda}{N} \frac{1}{\mu} \left(N \sum_{j=0}^{n-k-N+1} \left(\frac{\lambda}{\mu}\right)^j + \sum_{j=n-k-N+2}^{n-k} (n-k+1-j) \left(\frac{\lambda}{\mu}\right)^j \right). \quad (6.2.19)$$

Now, from the normalizing condition $\pi_m \mathbf{e} = 1$, we can write

$$(\pi_m(0) + \pi_m(2))\mathbf{e} + \pi_m(1)\mathbf{e} = 1$$

that is

$$\eta + \frac{\eta\lambda}{N\mu} \left(N \sum_{j=0}^{n-k-N+1} \left(\frac{\lambda}{\mu}\right)^j + \sum_{j=n-k-N+2}^{n-k} (n-k+1-j) \left(\frac{\lambda}{\mu}\right)^j \right) = 1 \quad (6.2.20)$$

which gives the constant η as

$$\eta = \left(1 + \frac{\lambda}{N\mu} \left(N \sum_{j=0}^{n-k-N+1} \left(\frac{\lambda}{\mu}\right)^j + \sum_{j=n-k-N+2}^{n-k} (n-k+1-j) \left(\frac{\lambda}{\mu}\right)^j \right) \right)^{-1}. \quad (6.2.21)$$

Equation (6.2.21) shows that the constant η is independent of the retrial rate θ .

Now, from equation (6.2.9) it follows that,

$$\pi_m(0) = -\pi_n(2) \left(\widetilde{H}_{33} + p\lambda c_N(N) \otimes r_N(1) \right) \left(H_{13}^{(m)} + p\lambda c_N(N) \otimes r_N(1) \right)^{-1}. \quad (6.2.22)$$

From the structure of the matrix $H_{13}^{(m)} + p\lambda c_N(N) \otimes r_N(1)$, it follows that the non zero entries of its inverse are given by

$$\begin{aligned} \left(H_{13}^{(m)} + p\lambda c_N(N) \otimes r_N(1) \right)_{ii}^{-1} &= \frac{1}{\bar{\lambda} + m\theta}, \quad 1 \leq i \leq N \\ \left(H_{13}^{(m)} + p\lambda c_N(N) \otimes r_N(1) \right)_{N1}^{-1} &= -\frac{p\lambda}{(\bar{\lambda} + m\theta)^2}. \end{aligned} \quad (6.2.23)$$

It then follows from (6.2.23) that as $m \rightarrow \infty$, the matrix $\left(H_{13}^{(m)} + p\lambda c_N(N) \otimes r_N(1) \right)^{-1}$ tends to the zero matrix and the matrix $m\theta \left(H_{13}^{(m)} + p\lambda c_N(N) \otimes r_N(1) \right)^{-1}$ tends to the identity matrix I_N . Hence equation (6.2.22) gives

$$\lim_{m \rightarrow \infty} \pi_m(0) = 0 \quad (6.2.24)$$

and hence equation (6.2.11) implies that

$$\lim_{m \rightarrow \infty} \pi_m(2) = \eta \frac{1}{N} e'_N. \quad (6.2.25)$$

Therefore,

$$\lim_{m \rightarrow \infty} m\theta\pi_m(0) = -\eta \frac{1}{N} e'_N \left(\widetilde{H}_{33} + p\lambda c_N(N) \otimes r_N(1) \right). \quad (6.2.26)$$

Since $\left(\widetilde{H}_{33} + p\lambda c_N(N) \otimes r_N(1) \right) \mathbf{e} = -(1-p)\bar{\mu}\mathbf{e} - (1-p)\lambda c_N(N)$, it follows from (6.2.26)

that,

$$\lim_{m \rightarrow \infty} m\theta\pi_m(0)\mathbf{e} = \eta(1-p)\bar{\mu} + \frac{\eta(1-p)\lambda}{N}. \quad (6.2.27)$$

Now from the structure of the A_0 and A_{m2} matrices, it follows that

$$\pi_m A_0 \mathbf{e} = \pi_m(2)(\lambda c_N(N) + \bar{\lambda}\mathbf{e}) \quad (6.2.28)$$

$$\pi_m A_{m2} \mathbf{e} = m\theta\pi_m(0)\mathbf{e} + p\mu\pi_m(1)c_{n-k+1}(1) + p\bar{\mu}\pi_m(2)\mathbf{e}. \quad (6.2.29)$$

Hence the stability condition $\pi_m A_0 \mathbf{e} < \pi_m A_{m2} \mathbf{e}$ for the truncated system becomes

$$\pi_m(2)(\lambda c_N(N) + \bar{\lambda}\mathbf{e}) < m\theta\pi_m(0)\mathbf{e} + p\mu\pi_m(1)c_{n-k+1}(1) + p\bar{\mu}\pi_m(2)\mathbf{e}. \quad (6.2.30)$$

As $m \rightarrow \infty$, equations (6.2.25), (6.2.27), (6.2.15), implies that inequality (6.2.30) becomes

$$\frac{\eta\lambda}{N} + \eta\bar{\lambda} < \eta(1-p)\bar{\mu} + \frac{\eta(1-p)\lambda}{N} + \frac{\eta p\lambda}{N} + \eta p\bar{\mu}. \quad (6.2.31)$$

On simplification, inequality (6.2.31) reduces to

$$\bar{\lambda} < \bar{\mu} \quad (6.2.32)$$

which leads to the following theorem

Theorem 6.2.1. *The Markov chain $\{X(t), t \geq 0\}$ is stable, if and only if $\bar{\lambda} < \bar{\mu}$.*

6.2.2. The steady state vector.

We find the steady state vector of $\{X(t), t \geq 0\}$, by approximating it with the steady state vector of the truncated system. Let $\pi = (\pi_0, \pi_1, \pi_2, \dots)$ where each

$$\pi_i = (\pi_i(0, 0), \pi_i(0, 1), \dots, \pi_i(0, N-1), \pi_i(1, 1), \dots, \\ \pi_i(1, n-k+1), \pi_i(2, 0), \pi_i(2, 1), \dots, \pi_i(2, N-1))$$

be the steady state vector of the $\{X(t), t \geq 0\}$.

Suppose $A_{i1} = A_{m1}$ and $A_{i2} = A_{m2}$ for all $i \geq m$.

Let $\pi_{m+r} = \pi_{m-1}R^{r+1}$, $r \geq 0$, then from $\pi Q = 0$ we get

$$\pi_{m-1}A_0 + \pi_m A_{m1} + \pi_{m+1}A_{m2} = 0 \\ \pi_{m-1}(A_0 + RA_{m1} + R^2A_{m2}) = 0.$$

Choose R as the minimal non negative solution of $A_0 + RA_{m1} + R^2A_{m2} = 0$. We call this R as R_m .

Also we have

$$\pi_{m-2}A_0 + \pi_{m-1}A_{m-11} + \pi_m A_{m2} = 0 \\ \pi_{m-1} = -\pi_{m-2}A_0(A_{m-11} + R_m A_{m2})^{-1} \\ = \pi_{m-2}R_{m-1},$$

where $R_{m-1} = -A_0(A_{m-11} + R_m A_{m2})^{-1}$.

Further

$$\pi_{m-3}A_0 + \pi_{m-2}A_{m-21} + \pi_{m-1}A_{m-12} = 0$$

$$\pi_{m-2} = -\pi_{m-3}A_0(A_{m-21} + R_{m-1}A_{m-12})^{-1}$$

$$= \pi_{m-3}R_{m-2},$$

$$\text{where } R_{m-2} = -A_0(A_{m-21} + R_{m-1}A_{m-12})^{-1}.$$

And so on. Finally $\pi_0 A_{00} + \pi_1 A_{12} = 0 \Rightarrow \pi_0(A_{00} + R_1 A_{12}) = 0$.

First we take π_0 as the steady state vector of $(A_{00} + R_1 A_{12})$. Then π_i for $i \geq 1$ can be found using the recursive formula, $\pi_i = \pi_{i-1} R^i$ for $1 \leq i \leq m-1$.

Now the steady state probability distribution of the truncated system is obtained by dividing each π_i with the normalizing constant

$$[\pi_0 + \pi_1 + \dots] \mathbf{e} = [\pi_0 + \pi_1 + \dots + \pi_{m-2} + \pi_{m-1}(I - R_m)^{-1}] \mathbf{e}.$$

6.2.3. Computation of the matrix R_m .

Consider the matrix quadratic equation

$$A_0 + R_m A_{m1} + R_m^2 A_{m2} = 0, \tag{6.2.33}$$

which implies

$$R_m = -A_0(A_{m1} + R_m A_{m2})^{-1}. \tag{6.2.34}$$

The structure of the A_0 matrix implies that the matrix R_m has the form:

$$R_m = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R_{m_1} & R_{m_2} & R_{m_3} \end{bmatrix}. \quad (6.2.35)$$

In other words, the non-zero rows of the R_m matrix are those, where the A_0 matrix has at least one nonzero entry. Now

$$R_m^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R_{m_2}R_{m_1} & R_{m_3}R_{m_2} & R_{m_3}^2 \end{bmatrix}. \quad (6.2.36)$$

Equation (6.2.33) gives rise to the following equations:

$$R_{m_1}H_{11}^{(m)} + R_{m_2}\tilde{H}_{21} + R_{m_3}\tilde{H}_{31} = 0. \quad (6.2.37)$$

$$R_{m_1}H_{12} + R_{m_2}H_{22} + \lambda c_N(N) \otimes r_{n-k+1}(N) = 0. \quad (6.2.38)$$

$$R_{m_3}R_{m_1}m\theta I_N + R_{m_3}R_{m_2}\tilde{H}_{23} + R_{m_3}^2 p\bar{\mu}I_N + R_{m_1}H_{13} + R_{m_3}H_{33} + \bar{\lambda}I_N = 0. \quad (6.2.39)$$

From equation (6.2.38), we can write

$$R_{m_2} = -R_{m_1}H_{12}(H_{22})^{-1} - \lambda c_N(N) \otimes r_{n-k+1}(N)(H_{22})^{-1}. \quad (6.2.40)$$

Substituting for R_{m_2} in equation (6.2.37), we get

$$R_{m_1}H_{11}^{(m)} - R_{m_1}H_{12}(H_{22})^{-1}\tilde{H}_{21} - \lambda c_N(N) \otimes r_{n-k+1}(N)(H_{22})^{-1}\tilde{H}_{21} + R_{m_3}\tilde{H}_{31} = 0. \quad (6.2.41)$$

From the discussion that has lead us to equations (6.2.6) and (6.2.7), it follows that

$$-\lambda_{c_N}(N) \otimes r_{n-k+1}(N) (H_{22})^{-1} \widetilde{H}_{21} = (1-p)\lambda_{c_N}(N) \otimes r_N(1). \quad (6.2.42)$$

Equations (6.2.6), (6.2.7) and (6.2.42) transform equation (6.2.41) as

$$R_{m_1} \left(H_{11}^{(m)} + (1-p)\lambda_{c_N}(N) \otimes r_N(1) \right) + R_{m_3} \left(\widetilde{H}_{31} \right) + (1-p)\lambda_{c_N}(N) \otimes r_N(1) = 0. \quad (6.2.43)$$

Denoting the matrix $\left(H_{11}^{(m)} + (1-p)\lambda_{c_N}(N) \otimes r_N(1) \right)^{-1}$ as W_m , and $\lambda_{c_N}(N) \otimes r_N(1)$ as W^0 from equation (6.2.43), it follows that

$$R_{m_1} = -R_{m_3} \left(\widetilde{H}_{31} \right) W_m - (1-p)W^0 W_m. \quad (6.2.44)$$

Using equation (6.2.40), it follows that

$$R_{m_2} \widetilde{H}_{23} = -R_{m_1} H_{12} (H_{22})^{-1} \widetilde{H}_{23} - \lambda_{c_N}(N) \otimes r_{n-k+1}(N) (H_{22})^{-1} \widetilde{H}_{23}. \quad (6.2.45)$$

We notice that $\widetilde{H}_{21} = (1-p)H_{21}$, where as $\widetilde{H}_{23} = pH_{21}$. Hence replacing $1-p$ by p in equations (6.2.6), (6.2.7) and (6.2.42), we can write the equations

$$-H_{12}(H_{22})^{-1}\widetilde{H}_{23} = p\lambda_{c_N}(N) \otimes r_N(1). \quad (6.2.46)$$

$$-H_{32}(H_{22})^{-1}\widetilde{H}_{23} = p\lambda_{c_N}(N) \otimes r_N(1). \quad (6.2.47)$$

$$-\lambda_{c_N}(N) \otimes r_{n-k+1}(N) (H_{22})^{-1} \widetilde{H}_{23} = p\lambda_{c_N}(N) \otimes r_N(1). \quad (6.2.48)$$

Equations (6.2.46) to (6.2.48) transform equation (6.2.45) as

$$R_{m_2} \widetilde{H}_{23} = R_{m_1} p\lambda_{c_N}(N) \otimes r_N(1) + p\lambda_{c_N}(N) \otimes r_N(1). \quad (6.2.49)$$

Substituting for R_{m_1} from equation (6.2.44), the above equation becomes

$$R_{m_2} \tilde{H}_{23} = \left(-R_{m_3} \left(\tilde{H}_{31} \right) W_m - (1-p)W^0 W_m \right) pW^0 + pW^0. \quad (6.2.50)$$

Substituting for R_{m_1} from (6.2.44), for $R_{m_2} \tilde{H}_{23}$ from (6.2.50), in equation (6.2.39), it reduces to,

$$\begin{aligned} R_{m_3} \left(-R_{m_3} \left(\tilde{H}_{31} \right) W_m - (1-p)W^0 W_m \right) m\theta I_N \\ + R_{m_3} \left(\left(-R_{m_3} \left(\tilde{H}_{31} \right) W_m - (1-p)W^0 W_m \right) pW^0 + pW^0 \right) \\ + R_{m_3}^2 p\bar{\mu} I_N + \left(-R_{m_3} \left(\tilde{H}_{31} \right) W_m - (1-p)W^0 W_m \right) H_{13} \\ + R_{m_3} H_{33} + \bar{\lambda} I_N = 0 \end{aligned} \quad (6.2.51)$$

$$\begin{aligned} \text{that is } R_{m_3}^2 \left(-\left(\tilde{H}_{31} \right) W_m m\theta I_N - \left(\tilde{H}_{31} \right) W_m pW^0 + p\bar{\mu} I_N \right) \\ + R_{m_3} \left(-(1-p)W^0 W_m m\theta I_N - (1-p)W^0 W_m pW^0 + pW^0 - \left(\tilde{H}_{31} \right) W_m H_{13} + H_{33} \right) \\ + \left(-(1-p)W^0 W_m H_{13} + \bar{\lambda} I_N \right) = 0 \end{aligned} \quad (6.2.52)$$

Which is a matrix quadratic equation of the form

$$R_{m_3}^2 \tilde{A}_{m_2} + R_{m_3} \tilde{A}_{m_1} + \tilde{A}_{m_0} = 0, \quad (6.2.53)$$

which can be solved for obtaining R_{m_3} . The matrix R_{m_1} can then be obtained from (6.2.44) and R_{m_2} from (6.2.40).

We notice that

$$\begin{aligned} -\lim_{m \rightarrow \infty} W_m &= -\lim_{m \rightarrow \infty} \left(H_{11}^{(m)} + (1-p)\lambda c_N(N) \otimes r_N(1) \right)^{-1} = 0. \\ -\lim_{m \rightarrow \infty} W_m m\theta I_N &= -\lim_{m \rightarrow \infty} \left(H_{11}^{(m)} + (1-p)\lambda c_N(N) \otimes r_N(1) \right)^{-1} m\theta I_N = I_N \end{aligned}$$

$$\begin{aligned}
\text{and hence } \widehat{A}_2 &= \lim_{m \rightarrow \infty} \widetilde{A}_{m_2} = \lim_{m \rightarrow \infty} \left(-(\widetilde{H}_{31}) W_m m \theta I_N - (\widetilde{H}_{31}) W_m p W^0 + p \bar{\mu} I_N \right) \\
&= \widetilde{H}_{31} + p \bar{\mu} I_N \\
&= (1 - p) \bar{\mu} I_N + p \bar{\mu} I_N \\
&= \bar{\mu} I_N. \tag{6.2.54}
\end{aligned}$$

$$\begin{aligned}
\widehat{A}_1 &= \lim_{m \rightarrow \infty} \widetilde{A}_{m_1} = \lim_{m \rightarrow \infty} \left(-(1 - p) W^0 W_m m \theta I_N - (1 - p) W^0 W_m p W^0 \right. \\
&\quad \left. + p W^0 - \widetilde{H}_{31} W_m H_{13} + H_{33} \right) \\
&= (1 - p) W^0 + p W^0 + H_{33} \\
&= H_{33} + W^0. \tag{6.2.55}
\end{aligned}$$

$$\begin{aligned}
\widehat{A}_0 &= \lim_{m \rightarrow \infty} \widetilde{A}_{m_0} = \lim_{m \rightarrow \infty} \left(-(1 - p) W^0 W_m H_{13} + \bar{\lambda} I_N \right) \\
&= \bar{\lambda} I_N. \tag{6.2.56}
\end{aligned}$$

Hence as $m \rightarrow \infty$, equation (6.2.53) becomes $R^2 \widehat{A}_2 + R \widehat{A}_1 + \widehat{A}_0 = 0$, whose minimal non-negative solution R satisfies the relation

$$\lim_{m \rightarrow \infty} R_{m_3} = R. \tag{6.2.57}$$

The relation (6.2.57) can be made use of selecting the truncation level m .

6.3. Performance measures

We now turn to deriving a few important characteristics of the system

(1) Fraction of time the system is down:

$$P_{down} = \sum_{j_1=0}^{\infty} \pi_{j_1}(1, n - k + 1).$$

(2) System reliability defined as the probability that at least k components are operational: $P_{rel} = 1 - P_{down}$.

(3) Average number of external units in the orbit is given by:

$$N_O = \sum_{j_1=0}^{\infty} j_1 \sum_{j_3=1}^{n-k+1} \pi_{j_1}(1, j_3) + \sum_{j_1=2}^{\infty} j_1 \left\{ \sum_{j_3=0}^{N-1} \pi_{j_1}(0, j_3) + \sum_{j_3=0}^{N-1} \pi_{j_1}(2, j_3) \right\}.$$

(4) Average number of failed components of the main system:

$$N_{fail} = \sum_{j_3=0}^{N-1} j_3 \left(\sum_{j_1=0}^{\infty} \pi_{j_1}(0, j_3) + \sum_{j_1=0}^{\infty} \pi_{j_1}(2, j_3) \right) + \sum_{j_3=1}^{n-k+1} j_3 \left(\sum_{j_1=0}^{\infty} \pi_{j_1}(1, j_3) \right).$$

(5) Average number of failed components waiting when the server is busy with external customers:

$$EFSBE = \sum_{j_3=0}^{N-1} j_3 \left(\sum_{j_1=1}^{\infty} \pi_{j_1}(2, j_3) \right).$$

(6) Expected number of external customers joining the system:

$$\theta_3 = \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=0}^{N-1} \pi_{j_1}(0, j_3) + \sum_{j_3=0}^{N-1} \pi_{j_1}(2, j_3) \right).$$

(7) Expected number of external customers on arrival getting service directly:

$$= \bar{\mu} \sum_{j_1=0}^{\infty} \sum_{j_3=0}^{N-1} \pi_{j_1}(0, j_3).$$

(8) Fraction of time the server is busy with external customers:

$$P_{ex.busy} = \sum_{j_1=1}^{\infty} \left(\sum_{j_3=0}^{N-1} \pi_{j_1}(2, j_3) \right).$$

(9) Probability that the server is found idle:

$$P_{idle} = \sum_{j_1=0}^{\infty} \sum_{j_3=0}^{N-1} \pi_{j_1}(0, j_3).$$

(10) Probability that the server is found busy:

$$P_{busy} = 1 - P_{idle} = 1 - \sum_{j_1=0}^{\infty} \sum_{j_3=0}^{N-1} \pi_{j_1}(0, j_3).$$

(11) Expected loss rate of external customers:

$$\theta_4 = \bar{\lambda} \sum_{j_1=0}^{\infty} \left(\sum_{j_3=1}^{n-k+1} \pi_{j_1}(1, j_3) \right).$$

(12) Expected service completion rate of external customers:

$$\theta_5 = \bar{\mu} \sum_{j_1=0}^{\infty} \sum_{j_3=0}^{N-1} \pi_{j_1}(2, j_3).$$

(13) Expected number of external customers when the server is busy with external customers:

$$\theta_6 = \sum_{j_1=0}^{\infty} j_1 \left(\sum_{j_3=0}^{N-1} \pi_{j_1}(2, j_3) \right).$$

(14) Expected number of successful retrials:

$$\text{ESR} = \theta \cdot \sum_{j_1=1}^{\infty} \sum_{j_3=0}^{N-1} \pi_{j_1}(0, j_3).$$

(15) The effective search rate is given by:

$$\text{EFSR} = p\bar{\mu} \sum_{j_1=0}^{\infty} \sum_{j_3=0}^{N-1} \pi_{j_1}(2, j_3) + p\mu \sum_{j_1=0}^{\infty} \pi_{j_1}(1, 1).$$

6.4. Numerical study of the performance of the system

According to the N -policy considered here, at the epoch when the number of failed components in the main system reaches N , external customer's (if any) service is pre-empted in order to attend the failed components. Due to the pre-emptive nature of service to external customers, allowing them does not affect the reliability of the system further. Table 6.2 shows that, system reliability decreases as the value of N increases. We want to notice that this is not due to the presence of the external customers; rather this is because as N increases, it gets late for the server to start attending the failed components and also it takes more time for the server to repair all the failed components accumulated, and in the mean time the system can reach the down status. We have compared the server busy probability of the system discussed here with that of a system where external customers are not allowed. Table 6.4 shows that the server busy probability is between 0.71 and 0.73, for a system where no external customers are not allowed; where as the same is between 0.84 and 0.86 when external customers are allowed as can be found in Table 6.3. Table 6.1 shows that the fraction of the time the server remains busy with external customers is $P_{ext.busy}$ greater than 0.094 and it is increases as the value of N increases. This is because, as the value of N increases, the external customers gets more attention from the server. In table 6.3, it can be seen that the server busy probability increases initially as N increases and then it begins to decrease after N exceed some

value. For explaining this behavior, we notice that the server busy probability P_{busy} is the sum of two probabilities namely the server busy probability with external customers $P_{ext.busy}$ and the server busy probability with failed components $P_{m.busy}$. Among these, $P_{ext.busy}$ increases as N increases, while $P_{m.busy}$ decreases as N increases. As N exceeds some value, which depends on the choice of the other parameters also, the magnitude of decrease in $P_{m.busy}$ exceeding the magnitude of increase in $P_{ext.busy}$ could be the reason behind the decrease of P_{busy} . This points to while N increases, even though the system reliability maintained as very high with external customers getting frequent service, a possible dissatisfaction of the main customers forced us to find an optimal value for N . For this we constructed a cost function as follows

Let C_1 be the cost per unit time incurred if the system is down, C_2 , be the holding cost per unit time per external customer in the orbit, C_3 is the cost incurred towards set up (instantaneous) of the server to serve main customers, C_4 be the cost due to loss of an external customer, C_5 be the holding cost per unit time of one failed component, C_6 be the cost per unit idle time.

$$\text{Expected cost per unit time} = C_1 \cdot P_{down} + C_2 \cdot N_0 + C_4 \cdot \theta_4 + C_5 \cdot N_{fail} + \left(\frac{C_3}{E_B} \right) + C_6 \cdot P_{idle}.$$

Table 6.5 gives the variation of the cost function as the N -policy level increases. According to the cost values and the other parameters assumed for Table 6.5, an optimum value for N happens to be a much smaller value $N = 3$, which points to the care needed for selecting N .

Table 6.1. Effect of N -policy on server busy probability with external customers

$$\lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, p = 0.6$$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.09514951	0.095134187	0.095131874	0.095131271	0.095131151
3	0.09431535	0.095429464	0.09429042	0.094289564	0.094289385
5	0.094264291	0.094233871	0.094227701	0.094226435	0.094226182
7	0.094269626	0.094223641	0.094214283	0.094212391	0.094211996
9	0.094294891	0.094223373	0.094208852	0.094205901	0.094205275
11	0.094344407	0.094230562	0.094207436	0.094202712	0.094201751
13	0.094431661	0.094246693	0.094209157	0.094201528	0.094200015
15	0.09458229	0.094276384	0.09421435	0.094201744	0.094199158
17	0.094842531	0.094328441	0.094224438	0.09420336	0.094199084
19	0.095295467	0.09441901	0.094242677	0.094206855	0.094199538
21	0.096093215	0.094577178	0.094274521	0.09421321	0.094200686
23	0.097520299	0.094854973	0.094330438	0.094224364	0.094202787
25	0.100133777	0.095347486	0.094429068	0.094244376	0.09420681
27		0.096229687	0.094603822	0.094279647	0.094213978
29		0.097834721	0.094915159	0.094342321	0.094226636
31		0.100827694	0.095474646	0.094454207	0.094249271
33			0.096490197	0.094654784	0.094289944
35			0.098364212	0.095016219	0.094362795
37				0.095672056	0.094493702
39				0.096875757	0.094730146
41				0.099126294	0.095160082
43					0.095947176
45					0.097406671

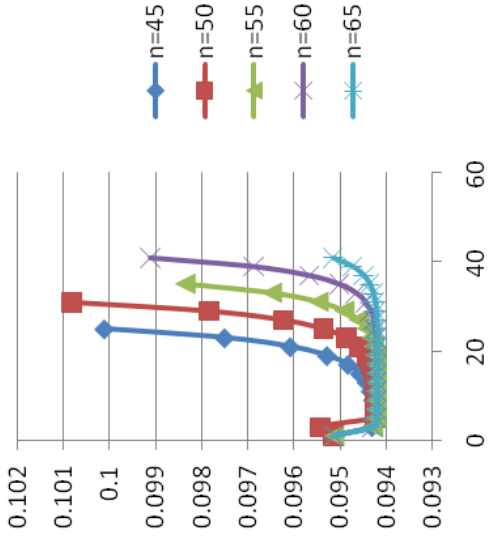


Table 6.2. Effect of N -policy on system reliability

$$\lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, p = 0.6$$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.999927512	0.999985278	0.99999702	0.999999404	0.999999881
3	0.999896646	0.99997896	0.999995708	0.999999106	0.999999821
5	0.999848127	0.999969125	0.999993742	0.999998748	0.999999762
7	0.999770224	0.99995327	0.999990463	0.999998093	0.999999583
9	0.999642789	0.999927402	0.999985218	0.99999702	0.999999404
11	0.999431372	0.999884427	0.999976516	0.999995232	0.999999046
13	0.999076188	0.999812424	0.999961853	0.999992251	0.99999845
15	0.998472273	0.999690175	0.999936998	0.999987185	0.999997377
17	0.997434139	0.999480784	0.9998945	0.999978542	0.999995649
19	0.995629668	0.999118984	0.999821067	0.999963582	0.999992609
21	0.992453277	0.998488724	0.999693513	0.999937654	0.999987304
23	0.986771345	0.997382045	0.999470294	0.999892354	0.999978125
25	0.976366401	0.99542129	0.99907738	0.999812663	0.999961913
27		0.991909087	0.998381615	0.999671817	0.999933302
29		0.98551929	0.9971416	0.999421954	0.999882519
31		0.973603964	0.994914472	0.99897635	0.999792159
33			0.990871668	0.998178065	0.999630749
35			0.98341167	0.996739745	0.999341249
37				0.994129002	0.998820186
39				0.989336848	0.997878492
41				0.980377853	0.996167362
43					0.99303767
45					0.987223387

Table 6.3. Effect of N -policy on server busy probability

$$\lambda = 4, \bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, p = 0.6$$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.857275724	0.857297719	0.857302189	0.857303083	0.857303262
3	0.858526945	0.858557999	0.858564317	0.858565569	0.858565807
5	0.858603597	0.858649135	0.858658373	0.858660281	0.85860638
7	0.858595431	0.858664453	0.85867846	0.858682321	0.858681917
9	0.858557642	0.85866493	0.858686686	0.858691096	0.85869205
11	0.85843315	0.858654141	0.858688831	0.858695865	0.858697295
13	0.858352482	0.858629882	0.85868609	0.858697653	0.858699918
15	0.85812664	0.858585477	0.858678579	0.858697355	0.858701289
17	0.85773623	0.858507335	0.858663321	0.858694911	0.858701229
19	0.857056856	0.858371615	0.858636022	0.858689725	0.858700752
21	0.855860233	0.85813427	0.858588219	0.858680248	0.858698964
23	0.853719652	0.857717574	0.858504355	0.85866344	0.858695865
25	0.849799335	0.856978774	0.858356416	0.858633459	0.858689785
27		0.855655491	0.858094394	0.858580709	0.858679175
29		0.85324846	0.857627392	0.858486652	0.858660221
31		0.848758399	0.856788158	0.858318806	0.858626127
33			0.855264723	0.858017802	0.858565032
35			0.852453589	0.857475638	0.858455837
37				0.856491923	0.85825938
39				0.85468632	0.857904792
41				0.851310493	0.85725981
43					0.856079221
45					0.853889942

Table 6.4. Variation in the server busy probability when external customers are not allowed

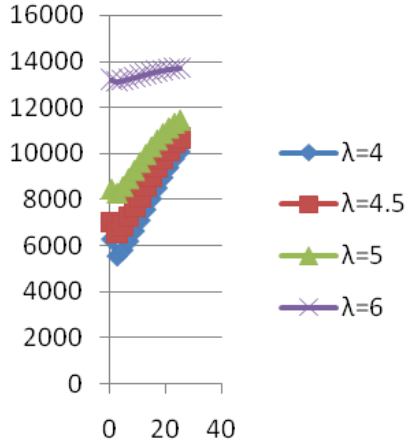
$$k = 20, \lambda = 4, \mu = 5.5$$

N	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
1	0.72722	0.72726	0.72727	0.72727	0.72727
3	0.7272	0.72726	0.72727	0.72727	0.72727
5	0.72717	0.72725	0.72727	0.72727	0.72727
7	0.72711	0.72724	0.72727	0.72727	0.72727
9	0.72703	0.72722	0.72726	0.72727	0.72727
11	0.72688	0.72719	0.72726	0.72727	0.72727
13	0.72663	0.72714	0.72725	0.72727	0.72727
15	0.72622	0.72706	0.72723	0.72726	0.72727
17	0.7255	0.72691	0.7272	0.72726	0.72727
19	0.72425	0.72666	0.72715	0.72725	0.72727
21	0.72206	0.72623	0.72706	0.72723	0.72726
23	0.71814	0.72546	0.72691	0.7272	0.72726

Table 6.5. Variation in cost

$\bar{\lambda} = 3.2, \mu = 5.5, \bar{\mu} = 8, p = 0.6, C_1 = 2000, C_2 = 1000,$
 $C_3 = 1600, C_4 = 1000, C_5 = 500, C_6 = 100$

N	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$	$\lambda = 6$
1	6294.77881	7014.58057	8451.76855	13187.4873
3	5568.93115	6558.06396	8271.66699	13118.4756
5	5818.25	6848.77002	8555.66797	13162.373
7	6210.08398	7246.75537	8901.53516	13227.1445
9	6648.28174	7676.01807	9259.15332	13297.1064
11	7105.86084	8114.13281	9613.13281	13367.0576
13	7571.48877	8550.64063	9955.93164	13434.373
15	8038.32568	8978.31543	10282.1611	13497.3008
17	8500.26953	9390.24902	10586.7256	13554.3633
19	8949.75098	9778.28027	10863.8584	13604.1982
21	9375.36719	10131.4531	11106.3301	13645.4541
23	9758.08301	10433.8193	11304.4141	13676.7705
25	10063.5977	10660.3818	11444.3047	13696.7246



Chapter 7

Reliability of a k -out-of- n system with a repair facility- Essential and Inessential services

7.1. Introduction

We consider a k -out-of- n system with a single server repair facility. At the epoch the system starts, all components are in operational state. Service to failed components is in the order of their arrival. When a component is selected for repair, we assume that, the server may select it for a service that turns out to be different from what is exactly needed for it. In other words, each failed component may get selected for an unwanted service, which we call the inessential service with probability p and with probability $(1 - p)$, it is taken for desired service, called the essential service. Once the inessential service process starts, the customer either completes the service there and moves for the essential service or leaves the system before completing the service in the first part. A

random clock is assumed to start ticking the moment the inessential service starts, which decides the event to follow: if the clock realises first (still the inessential service is going on) the customer leaves the system immediately without going for the essential service. On the other hand if the inessential service gets completed before the realisation of the random clock, then the component moves for the essential service immediately.

The arrival process of the failed components has inter-arrival times exponentially distributed with parameter λ . The essential service time of a failed component is exponentially distributed with parameter μ and the service time of failed components in inessential service has a phase type distribution with representation (α, S) of order m . We assume that $S^0 = -S \mathbf{e}$. S be a square matrix of order m with entries μ_{ij} , where μ_{ij} is the parameter of the exponentially distributed sojourn time in state i when it moves from j to i . The random clock time is assumed to be exponentially distributed with parameter δ .

7.2. The Markov Chain

Let $N(t) =$ at time t number of failed components in the system.

$$J(t) = \begin{cases} 0, & \text{if the failed component getting essential service} \\ 1, & \text{if a failed component getting } i^{\text{th}} \text{ phase of inessential service,} \\ & \text{where } i = 1, 2, \dots, m \end{cases}$$

Then $\{X(t), t \geq 0\}$ where $X(t) = (N(t), J(t))$ is a continuous time Markov chain with state space $\{(0, 0)\} \cup \{1, 2, \dots, n - k + 1\} \times \{0, 1, 2, \dots, m\}$.

since this system is finite, it is stable. Let

$$\boldsymbol{\pi} = (\pi(0), \pi(1), \dots, \pi(n - k + 1))$$

with

$$\pi(i) = (\pi(i, 0), \pi(i, 1), \pi(i, 2), \dots, \pi(i, m)), \quad 1 \leq i \leq n - k + 1$$

be the steady state probability vector of the system $\{X(t), t \geq 0\}$. Then it satisfies the equations $\boldsymbol{\pi}Q = 0$ and $\boldsymbol{\pi} \mathbf{e} = 1$.

The equation $\boldsymbol{\pi}Q = 0$ gives rise to

$$\pi(0)A_{00} + \pi(1)B_1 = 0 \tag{7.2.1}$$

$$\pi(0)B_0 + \pi(1)A_1 + \pi(2)A_2 = 0 \tag{7.2.2}$$

$$\pi(i - 1)A_0 + \pi(i)A_1 + \pi(i + 1)A_2 = 0, \quad 2 \leq i \leq n - k \tag{7.2.3}$$

$$\pi(n - k)A_0 + \pi(n - k + 1)\tilde{A}_1 = 0. \tag{7.2.4}$$

Since $A_{00} = [-\lambda]$ and $B_1 = A_2 \mathbf{e}$, from (7.2.1) it follows that

$$\lambda\pi(0) = \pi(1)A_2 \mathbf{e}. \tag{7.2.5}$$

Since $B_0 = \lambda\beta$, equation (7.2.2) becomes

$$\pi(0)\lambda\beta + \pi(1)A_1 + \pi(2)A_2 = 0. \tag{7.2.6}$$

Using (7.2.5) we can write this equation as

$$\pi(1)B_1\beta + \pi(1)A_1 + \pi(2)A_2 = 0. \quad (7.2.7)$$

We notice that $B_1\beta = A_2$ and hence equation (7.2.7) becomes

$$\pi(1)(A_1 + A_2) + \pi(2)A_2 = 0. \quad (7.2.8)$$

Post multiplying equation (7.2.8) with \mathbf{e} , we get

$$\pi(1)(A_1 + A_2)\mathbf{e} + \pi(2)A_2\mathbf{e} = 0 \quad (7.2.9)$$

but $(A_1 + A_2)\mathbf{e} = -A_0\mathbf{e} = -\lambda\mathbf{e}$. Hence (7.2.9) becomes

$$\pi(1)\lambda\mathbf{e} = \pi(2)A_2\mathbf{e}. \quad (7.2.10)$$

We notice that $A_2 = A_2\mathbf{e}\beta$, which transforms equation (7.2.8) in to

$$\pi(1)(A_1 + A_2) + \pi(2)A_2\mathbf{e}\beta = 0. \quad (7.2.11)$$

Substituting for $\pi(2)A_2\mathbf{e}$ from (7.2.10) in (7.2.11), we get

$$\pi(1)(A_1 + A_2) + \pi(1)\lambda\mathbf{e}\beta = 0.$$

That is

$$\pi(1)(A_1 + A_2 + \lambda\mathbf{e}\beta) = 0. \quad (7.2.12)$$

Equation (7.2.12) shows that $\pi(1)$ is a constant multiple of the steady state vector $\boldsymbol{\varphi}$ of the generator matrix $A_1 + A_2 + \lambda \mathbf{e}\boldsymbol{\beta}$. That is

$$\pi(1) = \eta \boldsymbol{\varphi} \quad (7.2.13)$$

where η is a constant.

Equation (7.2.3) for $i = 2$ gives

$$\pi(1)A_0 + \pi(2)A_1 + \pi(3)A_2 = 0. \quad (7.2.14)$$

Since $A_2 = A_2 \mathbf{e}\boldsymbol{\beta}$, equation (7.2.14) becomes

$$\pi(1)A_0 + \pi(2)A_1 + \pi(3)A_2 \mathbf{e}\boldsymbol{\beta} = 0. \quad (7.2.15)$$

Post multiplying with \mathbf{e} , we get

$$\pi(1)\lambda \mathbf{e} + \pi(2)A_1 \mathbf{e} + \pi(3)A_2 \mathbf{e} = 0. \quad (7.2.16)$$

Using (7.2.10) the above equation can be written as

$$\pi(2)A_2 \mathbf{e} + \pi(2)A_1 \mathbf{e} + \pi(3)A_2 \mathbf{e} = 0$$

$$\text{i.e., } \pi(2)(A_1 + A_2) \mathbf{e} = -\pi(3)A_2 \mathbf{e}$$

$$\text{i.e., } \pi(2)\lambda \mathbf{e} = \pi(3)A_2 \mathbf{e}. \quad (7.2.17)$$

In the light of equation (7.2.17), equation (7.2.15) becomes,

$$\pi(1)A_0 + \pi(2)A_1 + \pi(2)\lambda \mathbf{e}\beta = 0$$

$$\text{i.e., } \pi(1)A_0 + \pi(2)(A_1 + \lambda \mathbf{e}\beta) = 0$$

which implies that

$$\pi(2) = -\pi(1)A_0(A_1 + \lambda \mathbf{e}\beta)^{-1}.$$

That is

$$\pi(2) = -\eta\varphi A_0(A_1 + \lambda \mathbf{e}\beta)^{-1}. \quad (7.2.18)$$

Post-multiplying equation (7.2.3) with \mathbf{e} and proceeding in the same lines as we derived equation (7.2.17), we can derive that

$$\pi(i+1)A_2\mathbf{e} = \pi(i)\lambda \mathbf{e}, \text{ for } 3 \leq i \leq n-k. \quad (7.2.19)$$

Equation (7.2.19) then transforms equation (7.2.3) as

$$\pi(i-1)A_0 + \pi(i)A_1 + \pi(i)\lambda \mathbf{e}\beta = 0, 3 \leq i \leq n-k,$$

which implies that

$$\pi(i) = -\pi(i-1)A_0(A_1 + \lambda \mathbf{e}\beta)^{-1}, 2 \leq i \leq n-k \quad (7.2.20)$$

which in turn gives

$$\pi(i) = (-1)^{i-1} \eta\varphi(A_0(A_1 + \lambda \mathbf{e}\beta)^{-1})^{i-1}, 2 \leq i \leq n-k. \quad (7.2.21)$$

We notice that $\tilde{A}_1 \mathbf{e} = -A_2 \mathbf{e}$; post-multiplying equation (7.2.4) with \mathbf{e} , we get

$$\pi(n-k)\lambda \mathbf{e} = \pi(n-k+1)A_2 \mathbf{e}. \quad (7.2.22)$$

From equation (7.2.4), we can also write

$$\pi(n-k+1) = -\pi(n-k)A_0(\tilde{A}_1)^{-1}. \quad (7.2.23)$$

Using (7.2.21) for $i = n-k$, (7.2.23) becomes

$$\pi(n-k+1) = (-1)^{n-k} \eta \boldsymbol{\varphi} (A_0(A_1 + \lambda \mathbf{e}\boldsymbol{\beta})^{-1})^{n-k+1} A_0(\tilde{A}_1)^{-1}. \quad (7.2.24)$$

Hence, we have the following theorem for the steady state distribution:

Theorem 7.2.1. *The steady state distribution $\boldsymbol{\pi} = (\pi(0), \pi(1), \dots, \pi(n-k+1))$ of the Markov chain $\{X(t), t \geq 0\}$ is given by*

$$\pi(0) = \frac{1}{\lambda} \eta \boldsymbol{\varphi} B_1$$

$$\pi(1) = \eta \boldsymbol{\varphi}$$

$$\pi(i) = (-1)^{i-1} \eta \boldsymbol{\varphi} (A_0(A_1 + \lambda \mathbf{e}\boldsymbol{\beta})^{-1})^{i-1}, \quad 2 \leq i \leq n-k$$

$$\pi(n-k+1) = (-1)^{n-k} \eta \boldsymbol{\varphi} (A_0(A_1 + \lambda \mathbf{e}\boldsymbol{\beta})^{-1})^{n-k-1} A_0(\tilde{A}_1)^{-1},$$

where $\boldsymbol{\varphi}$ is the steady state vector of the generator matrix $A_1 + A_2 + \lambda \mathbf{e}\boldsymbol{\beta}$ and η is a constant, which can be found from the normalizing condition $\boldsymbol{\pi} \mathbf{e} = 1$.

7.3. System performance measures

(1) Fraction of time the system is down,

$$P_{down} = \sum_{j=0}^m \pi(n - k + 1, j)$$

(2) System reliability,

$$P_{rel} = 1 - P_{down} = 1 - \sum_{j=0}^m \pi(n - k + 1, j)$$

(3) Average number of failed components in the system,

$$N_{fail} = \sum_{i=0}^{n-k+1} i \left(\sum_{j=0}^m \pi(i, j) \right).$$

(4) Expected rate at which failed components are taken for essential service:

$$E_{es} = (1 - p)\lambda\pi(0) + \sum_{i=2}^{n-k+1} (1 - p)\mu\pi(i, 0) + \sum_{i=2}^{n-k+1} (1 - p)\delta \left(\sum_1^m \pi(i, j) \right).$$

(5) Expected rate at which failed components are taken for inessential service

$$E_{ines} = p\lambda\pi(0) + \sum_{i=2}^{n-k+1} p\mu\pi(i, 0) + \sum_{i=2}^{n-k+1} p\delta \left(\sum_{j=1}^m \pi(i, j) \right).$$

(6) Expected rate at which new components were bought:

$$E_{C.R} = \sum_{i=1}^{n-k+1} \delta \left(\sum_{j=1}^m \pi(i, j) \right).$$

(7) Expected rate at which failed components that start with inessential service subsequently moves to essential service before clock realisation :

$$E_{INE} = \sum_{i=1}^{n-k+1} \sum_{j=2}^{m+1} \pi(i, j) S^0(j-1, 1).$$

(8) Fraction of time server is idle:

$$P_{idle} = \pi(0).$$

(9) Fraction of time server is busy:

$$P_{busy} = 1 - \pi(0).$$

Numerical study of the system performance measures

Notice that if a component is selected for inessential service, it is either replaced by a new component (if the random clock realises before completion of the inessential service) or is got repaired (if the inessential service completes before the random clock realises). Hence a component getting selected for inessential service according to probability p affects the system reliability only through an increase in the repair time by a random amount of time (minimum of inessential service time and random clock time). Table 7.1 shows that very high reliability is maintained in the system, which decreases slightly as the probability p that a failed component receives an undesired service initially, increases. The decrease in the average rate at which components directly receive essential service with an increase in p , as seen in Table 7.2, was expected. So is the

increase in the rate at which components receive inessential service initially as seen in Table 7.3, with an increase in p . According to the modelling assumption, if the random clock expires during an inessential service, the component receiving the inessential service is replaced with a new component. Hence, as the probability p increases, more components will get selected for inessential service, which leads to an increase in the replacement rate as seen in Table 7.4.

Since the inessential service is not helping the system in any way whatsoever, one would expect the optimal value for the probability p as to be zero. However in a situation where the possibility for inessential service can't be avoided, one would like to know its harm through some number. For this purpose, we have constructed a cost function as follows:

Let C_1 be the cost per unit time incurred if the system is down, C_2 , be the repair cost per unit time for essential service per failed component, C_3 is the cost incurred towards the time loss due to wrong diagnosis with failed components and consequent realisation of random clock before inessential service completion. C_4 is the extra cost incurred on failed components that start with inessential service subsequently moves to essential service before clock realisation, C_5 be the repair cost per unit time for inessential service

$$\text{Expected cost per unit time} = C_1 \cdot P_{down} + C_2 \cdot E_{es} + C_3 \cdot E_{C.R} + C_4 \cdot E_{INE} + C_5 \cdot E_{ines}.$$

Table 7.5 presents the variation in cost function as the probability p increases for different component failure rates. In all the cases studied, the optimum value of p was

found zero as was expected. The table also shows that as the component failure rate increases, the cost function also increases.

Table 7.1. Variation in system reliability, $\lambda = 4, \mu = 3.8, \delta = 5, S^0 = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}$

p	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
0.001	0.999985933	0.999985933	0.999997139	0.999999404	0.999999881
0.003	0.999985874	0.999985874	0.999997139	0.999999404	0.999999881
0.005	0.999985874	0.999985874	0.999997139	0.999999404	0.999999881
0.007	0.999985814	0.999985814	0.999997139	0.999999404	0.999999881
0.009	0.999985814	0.999985814	0.999997139	0.999999404	0.999999881
0.01	0.999985814	0.999985814	0.999997079	0.999999404	0.999999881
0.03	0.999985576	0.999985576	0.99999702	0.999999404	0.999999881
0.05	0.999985278	0.999985278	0.99999696	0.999999404	0.999999881
0.07	0.999985039	0.999985039	0.999996901	0.999999344	0.999999881
0.09	0.999984741	0.999984741	0.999996841	0.999999344	0.999999881
0.1	0.999984622	0.999984622	0.999996841	0.999999344	0.999999881
0.3	0.999981642	0.999981642	0.999996066	0.999999166	0.999999821
0.5	0.999978125	0.999978125	0.999995172	0.999998927	0.999999762
0.7	0.999973893	0.999973893	0.99999404	0.999998629	0.999999702
0.9	0.999968886	0.999968886	0.999992669	0.999998271	0.999999583
0.99	0.999966323	0.999966323	0.999991954	0.999998093	0.999999523

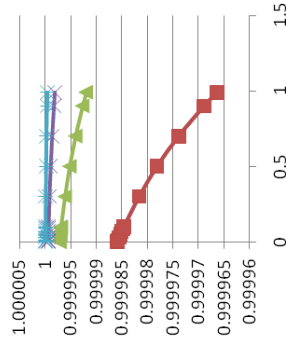


Table 7.2. Average rate at which components taken for essential service $\lambda = 4, \mu = 3.8, \delta = 5, S^0 = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}$

p	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
0.001	3.99649739	3.99649739	3.99654222	3.99655128	3.99655718
0.003	3.98938107	3.9896009	3.98964596	3.98965478	3.98965669
0.005	3.98248005	3.98269963	3.98274446	3.98275352	3.98275566
0.007	3.97557425	3.97579312	3.97583771	3.97584677	3.97584867
0.009	3.96866465	3.96888351	3.9689281	3.9689374	3.9689393
0.01	3.96520758	3.96542716	3.96547127	3.96548033	3.965482
0.03	3.89581704	3.89603281	3.89607692	3.89608598	3.89608765
0.05	3.82595038	3.82616258	3.82620668	3.82621574	3.82621717
0.07	3.75561023	3.7558198	3.75586271	3.75587177	3.75587368
0.09	3.68479681	3.6850028	3.68504548	3.68505406	3.684505597
0.1	3.64921474	3.64941883	3.64946103	3.64946985	3.649472
0.3	2.91331673	2.91348505	2.91352081	2.91352844	2.91353035
0.5	2.13270593	2.13283181	2.13285947	2.13286543	2.13286662
0.7	1.30956876	1.30964661	1.30966437	1.30966842	1.30966926
0.9	0.44612866	0.446155071	0.44616127	0.446162701	0.446163058
0.99	0.045032669	0.04503531	0.045035943	0.045036085	0.045036126

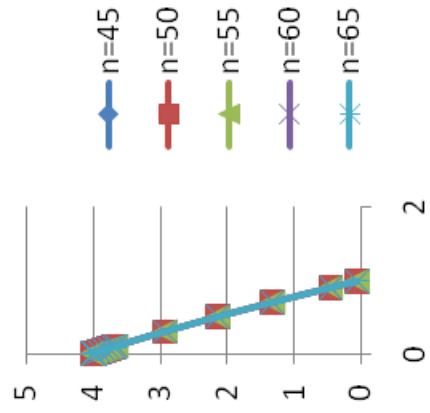


Table 7.3. Average rate at which components taken for essential service $\lambda = 4, \mu = 3.2, \delta = 5, S^0 = 8$

P	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
0.001	0.001677582	0.001677335	0.001677285	0.001677275	0.001677273
0.003	0.005032186	0.005031444	0.005031293	0.005031263	0.005031257
0.005	0.008386041	0.008384803	0.008384562	0.008384501	0.008384449
0.007	0.011391506	0.011737416	0.011737063	0.01173699	0.011736976
0.009	0.015091507	0.015089272	0.015088818	0.015088724	0.015088706
0.01	0.016767405	0.01676492	0.016764414	0.016764313	0.016764291
0.03	0.050246038	0.050238468	0.050236922	0.050236605	0.050236542
0.05	0.083649568	0.083636761	0.083634131	0.083633594	0.083633349
0.07	0.116977736	0.11695952	0.116955772	0.116955005	0.116954848
0.09	0.150230378	0.150206596	0.150201693	0.15020068	0.150200471
0.1	0.16682826	0.166801631	0.166796118	0.166794986	0.166794747
0.3	0.494757891	0.494664699	0.494644821	0.494640589	0.494639665
0.5	0.814846277	0.814665198	0.814625323	0.814616561	0.814614654
0.7	1.12684083	1.12654567	1.12647855	1.12646341	1.12645984
0.9	1.43048191	1.43004024	1.429936605	1.42991233	1.42990661
0.99	1.5643245	1.56380451	1.56368077	1.56365132	1.56364441

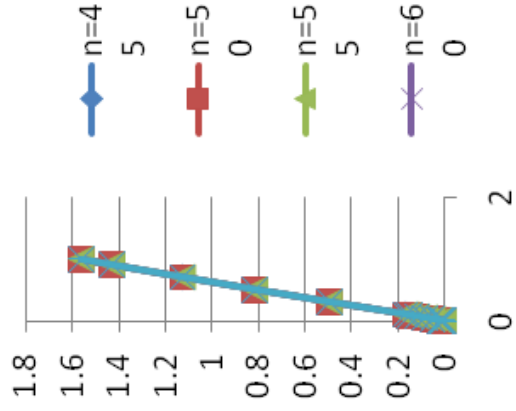
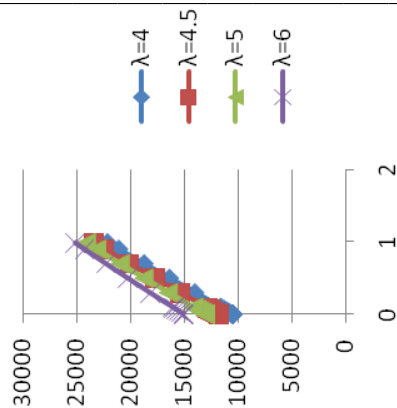


Table 7.4. Average rate at which components were bought $\lambda = 4, \mu = 3.2, \delta = 5, S^0 =$

p	$n = 45$	$n = 50$	$n = 55$	$n = 60$	$n = 65$
0.001	0.001538355	0.00153844	0.001538457	0.001538461	0.001538462
0.003	0.004615065	0.004615319	0.004615371	0.004615382	0.004615384
0.005	0.007691774	0.007692199	0.007692286	0.007692303	0.007692307
0.007	0.010768481	0.010769077	0.010769199	0.010769224	0.010769229
0.009	0.01384519	0.013845956	0.013846113	0.013846145	0.013846132
0.01	0.015383544	0.015384398	0.015384572	0.015384607	0.015384615
0.03	0.046150584	0.04615318	0.046153713	0.046153817	0.046153843
0.05	0.076917566	0.076921947	0.076922849	0.076923035	0.076923072
0.07	0.107684486	0.107690714	0.107691996	0.107692257	0.107692316
0.09	0.138451293	0.138459414	0.138461098	0.138461441	0.138461515
0.1	0.153834701	0.15384379	0.153845653	0.15384604	0.15384613
0.3	0.461498737	0.46153	0.461536676	0.461538808	0.461538374
0.5	0.769154251	0.769213974	0.769227087	0.769230008	0.769230664
0.7	1.07679927	1.076895	1.07691669	1.0769217	1.07692277
0.9	1.38443172	1.38457251	1.38460553	1.38461328	1.38461506
0.99	1.52286148	1.52302587	1.52306497	1.52307427	1.52307653

Table 7.5. Variation in cost $C_1 = 9500, C_2 = 2600, C_3 = 4000, C_4 = 1600, C_5 = 3000$

P	$\lambda = 4$	$\lambda = 4.5$	$\lambda = 5$	$\lambda = 6$
0.001	10411.8662	11709.9268	12982.9131	15048.7344
0.003	10435.6221	11734.1328	13006.5225	15070.1104
0.005	10459.3789	11758.334	13030.1328	15091.4854
0.007	10483.1328	11782.5352	13053.7373	15112.8564
0.009	10506.8916	11806.7393	13077.3428	15134.2402
0.01	10518.7715	11818.8389	13089.1475	15144.9258
0.03	10756.3047	12060.7119	13324.9258	15358.3809
0.05	10993.7998	12302.373	13560.29	15571.4082
0.07	11231.2598	12543.8252	13795.2471	15784.0264
0.09	11468.6729	12789.0537	14029.7676	15996.2021
0.1	11587.3662	12905.5889	14146.8809	16102.1484
0.3	13958.7285	15304.4629	16466.709	18199.4043
0.5	16324.3867	17679.4102	18742.8711	20256.9434
0.7	18682.9082	20028.3828	20973.9453	22276.6172
0.9	21032.8359	22349.3281	23158.7715	24260.2168
0.99	22087.1016	23384.0664	24126.6035	25141.5078



Conclusion

In this thesis, we studied different k -out-of- n systems where the server, besides repairing failed components, renders service to external customers also. Rendering service to external customers could be an effective way for utilizing the server idle time and there by earn more profit to the system. However, in the case of a system, where a minimum number of working components is necessary for its operation, the external service should be carefully managed so that it does not affect the system reliability seriously.

In chapter 2, we adopted an N -policy for managing the external service. Precisely, we assume that the server starts attending failed components of the main system only on accumulation of N of them. During this idle period, the server renders service to external customers (if there is any). This scenario was modeled using a continuous time Markov chain. Further we make the reasonable assumption that the external service is pre-emptive in nature on accumulation of N failed components and also that the external arrivals which finds the server busy with failed system components are blocked from entering the system. These assumptions lead us to a product form solution for the system steady state distribution, when the underlying distributions are all assumed to be exponential; and for obtaining the same, we used a novel matrix decomposition approach. Our numerical study of the system performance measures reveals that by introducing N -policy, we can optimize the system revenue, by rendering service to external customers, still maintaining high system reliability. Analysis of a cost function has helped us in finding an optimal value for the N -policy level.

In chapter 3, we extended the model in chapter 2 by considering a non-preemptive service for external customers thereby making their service more attractive. We analyzed two models: one in which the external customers joins a queue and another in which they move to an orbit of infinite capacity. Our numerical study showed that rendering non-preemptive service to external customers has not affected the system reliability much, thereby re-asserting that the same could be an effective idea for utilizing the server idle time and there by earning more profit to the system. Here also we analyzed a cost function, which helped us in finding an optimal value for the N -policy level.

In chapter 4, we replaced the N -policy for the service of failed components with a T -policy. That is at the epoch the system starts with all components operational, the server starts attending the external customers (if there is any). The server starts the service of the failed components only at the moment of the realization of the random time T (if there is at least one failed component). If the time T is realized in the middle of an external customer's service and if there exists at least one failed component, the external customer in service is pre-empted and the server is switched over to the service of main customers. The preempted external customer goes to the queue of external customers. Our numerical study showed that the realization rate of the random time T should be chosen very carefully since it may severely affect the reliability of the k -out-of- n system. More precisely if T takes large values with positive probability, reliability is very small and at the same time the server busy probability is not very high. We have therefore constructed a cost function for selecting an optimal value for the realization rate of T . As in the case of classical queue, the performance of N -policy excels that of T -policy.

In chapter 5, it was assumed that the server selects an external customer from the pool of external customers for service with probability ' p ', if the number of failed components is less than ' L ', a pre-assigned number called the transition level. We notice that in the case of an N -policy (assumed in chapters 2 and 3), the server starts attending the failed components only on the accumulation of N of them and in the case of T -policy (assumed in chapter 4) it happens on the realization of time T . In contrast to these, according to the policy adopted in this chapter, even if there is only one failed component found at an external customers service completion epoch, its repair is started with probability $1 - p$. Hence this policy helps to maintain very high system reliability and at the same time gives much attention for external customers. Optimal value for L was found based on a cost function.

Chapter 6 differs from the preceding chapters that it assumes the external customers are sent to an orbit instead of a queue. We assume an N -policy for starting the service of failed components and the service of an external customer is preempted and it is sent back to the orbit at the epoch of accumulation of N failed components. Because of the assumption of the orbit of external customers, the server goes idle after each service completion of an external customer. In order to reduce the server idle probability, an orbital search of external customers was applied. An optimal value for N was found using a cost function.

Chapter 7 does not assume any external customers in the system; instead here the reliability of a k -out-of- n system is studied in a setup where a failed component may get selected for an undesirable service initially, which may be due to some wrong diagnosis

of the reason for its failure. Each failed component may complete the different stages of the undesirable service to finally receive the essential service or may get replaced with a new component. This decision was done on the basis of the elapse of a random clock T . More precisely, if T realizes before the completion of the unwanted service, the failed component is replaced with a new component. A cost function was studied for selecting an optimal value for the probability p with which an external customer is selected for the unwanted service and it was found that zero is its optimal value.

There are several extensions to the work reported in this thesis. For example external arrivals, wherever considered could be assumed to follow an Markovian Arrival Process with appropriate representation. D -policy as a control policy could be examined. Here it is the accumulated work load (D) that is to be considered. Yet another direction of extension is a multi server system. The extension of the results reported to the case of more than one essential service is worth examining. This has applications in medicine, biology and several other fields of activity.

Bibliography

- [1] C.B. Ajayakumar; *Queues with postponed work under N-policy*, Ph.D. Thesis, Cochin University of science & Technology, 2011.
- [2] J.R. Artalejo; A unified cost function for $M/G/1$ queueing systems with removable server, *Trabajos De Investigation Operativa*, 7 (1992): 95–104, No.1.
- [3] J.R. Artalejo; Accessible bibliography on retrial queues: Progress in 2000–2009, *Mathematical and Computer Modelling*, Volume 51, Issues 9-10, May 2010, Pages 1071–1081.
- [4] J.R. Artalejo, V.C. Joshua and A. Krishnamoorthy; *An M/G/1 Retrial Queue with Orbital Search by the Server*, Advances in Stochastic Modelling, Notable Publications Inc., NJ, (2002), pp. 41–54.
- [5] P.P. Bocharov, O.I. Pavlova and V.A. Puzikova; $M/G/1/r$ retrial queueing systems with priority of primary customers, *TOP journal*, 30(3-4), (1999), pages 89–98.
- [6] S. R. Chakravarthy, A. Krishnamoorthy and P. V. Ushakumari; A k -out-of- n reliability system with an unreliable server and Phase type repairs and services: The (N, T) policy, *Journal of Applied Mathematics and Stochastic Analysis*; 14 (4), (2001), 361–380.

- [7] P. Chitra Devi, B. Sivakumar and A. Krishnamoorthy; Optimal Control Policy of an Inventory System with Postponed Demand, *RAIRO-Oper.Res.*, 50 (2016), 145–155.
- [8] T.G. Deepak, V.C. Joshua and A. Krishnamoorthy; Queues with postponed work, *TOP – Spanish Journal of Statistics and Operational Research*, Vol. 12, 375–398.
- [9] A.N. Dudin, A. Krishnamoorthy and C. N. Viswanath; Reliability of a k -out-of- n -system through Retrial Queues, *Trans. of XXIV Int. Seminar on Stability Problems for Stochastic Models*, Jurmala, Latvia, September, (2004), 10–17.
Also appeared in the *Journal of Mathematical Sciences*, Vol.191, No.4, June 2013, 506–517.
- [10] S. Karlin and H. J. Taylor; *A First Course in Stochastic Processes*, Academic Press, 1975.
- [11] A. Krishnamoorthy and A. Rekha; k -out-of- n system with repair: T -policy, *Korean Journal of Computational & Applied Mathematics*, January 2001, Volume8, Issue 1, pp 199–212.
- [12] A. Krishnamoorthy, C. N. Viswanath and T. G. Deepak; Reliability of a k -out-of- n -system with Repair by a Service Station Attending a Queue with Postponed Work; Advanced Reliability Modelling: *Proceedings of the 2004 Asian International Workshop (AIWARM 2004)*, Hiroshima, Japan, 26-27 August,2004, pp.293-300. Also published in *Journal of Applied Mathematics & Computing (JAMC)*, pp.389–405, Volume 25, Numbers 1-2/September 2007, Springer.

- [13] A. Krishnamoorthy, C. N. Viswanath and T. G. Deepak; Maximizing of Reliability of a k -out-of- n -system with Repair by a facility attending external customers in a Retrial Queue; *Proceedings of V International Workshop on Retrial Queues*, Ed. Bong Dae Choi, September, 2004, TMRC, Korea University, Seoul, pp.31-38. Also published in *International Journal of Reliability, Quality and Safety Engineering (IJRQSE)*, Volume 14, No. 4, 2007, pp. 379–398.
- [14] A. Krishnamoorthy and P. V. Ushakumari; Reliability of k -out-of- n system with repair and retrial of failed units, *TOP*, 7(2), (1999), 293–304.
- [15] A. Krishnamoorthy, P.V. Ushakumari and B. Lakshmi; k -out-of- n system with the repair: the N -policy, *Asia pacific Journal of Operations Research* , 19, (2002), 47–61.
- [16] G. Latouche and V. Ramaswami; Introduction to Matrix Analytic Methods in Stochastic Modeling, *SIAM*, (1999).
- [17] M.F. Neuts; *Matrix Geometric solutions in stochastic processes-An Algorithmic Approach*, The John Hopkins University Press (1981).
- [18] M.F. Neuts and M.F. Ramalhoto; A service model in which the server is required to search for customers, *Journal of Applied Probability*, 21, (1984), 157-166.
- [19] M.F. Neuts and B.M. Rao; Numerical investigation of a multiserver retrial model, *Queueing Systems*, Volume 7, Number 2, 169–189, June 1990.
- [20] T. Phung Duc; Retrial Queue for Cloud Systems with Separated Processing and Storage Units Chapter Queueing Theory and Network Applications Volume 383 of

the series *Advances in Intelligent Systems and Computing*, pp 143–151 Date: 24 July 2015.

- [21] P.V. Ushakumari and A. Krishnamoorthy; Reliability of a k -out-of- n system with general repair, the N -policy, In Janson and Limnios, editors, *Proceedings of 11th International Conference on Semi-Markov Processes and application*, Cedex, France, Gordon and Breach, 1998.
- [22] P.V. Ushakumari and A. Krishnamoorthy; k -out-of- n system with repair: the $x(N, T)$ -policy, *Performance Evaluation*, 57: (2004), 221–234.