CHARACTERIZATIONS OF PROBABILITY MODELS USING SOME DYNAMIC INFORMATION MEASURES

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By
LINU M. N.

Department of Statistics
Cochin University of Science and Technology
Kochi-682 022

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CERTIFICATE

Certified that the thesis entitled 'CHARACTERIZATIONS OF PROBABILITY MODELS USING SOME DYNAMIC INFORMATION MEASURES' is a bonafide record of work done by Smt. Linu M. N. under my guidance in the Department of Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

Kochi-682 022
6th July 2015

Dr. S. M. Sunoj
Professor,
Department of Statistics,
Cochin University of
Science and Technology.

DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Kochi-682 022 **_Linu M. N._**

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Contents

Chapter 1

Introduction & Literature Review

1.1 Introduction

For the last few decades there has been a tremendous growth in the volume of research in the field of information theory. As long as uncertainty rules the world there always exists a necessity to measure the amount of uncertainty that is associated with it and thus plays an important role. Study of uncertainty and information rooted in the works of Shannon (1948) who formally introduced the term entropy as a measure of uncertainty. Later on, efforts were made to generalize the Shannon entropy to represent various natural phenomena. Thus Renyi's entropy (see Renyi (1959, 1961)), R-norm entropy and other generalized measures came into picture. There are also situations in survival theory, reliability *etc.* where lifetimes are commonly truncated and the basic form of entropy measures become unsuitable and necessitates the introduction of residual, past and interval based entropy measures. They are called as the dynamic forms of these information measures as these measures are functions of time. The study on such dynamic forms can be found in Ebrahimi (1996), Ebrahimi and Pellerey (1995), Sankaran and Gupta (1999), Di Crescenzo and Longobardi (2002), Abraham and Sankaran (2005) *etc.* In parallel, another measure of information evoked that measures the distance between two populations, or two distributions is the Kullback Leibler (KL) divergence (Kullback and Leibler (1951)). It gives the information regarding how different two populations/distributions are. Similar to entropy measures, dynamic forms of KL measure were introduced to tackle truncated situations. The concept of inaccuracy evolved as a measure of missing information was proposed by Kerridge (1961). Study on its dynamic forms is done by Taneja et al. (2009) and Kumar et al. (2011). The above measures discussed are on univariate setup. There are situations where one has to deal with two-component systems when the status of one of the components is known in advance. In the present work we have made a study of the various information measures narrated above for such two-component situations.

Due to the importance and usage of certain basic reliability concepts in the study of information measures, a brief review of them is presented in this chapter. In addition, a review of information measures used in the study and other related concepts have been appended.

1.2 Basic concepts - Univariate notions

Consider a random variable (rv) X. Let $a = \inf\{x|F(x) > 0\}$ and $b = \sup\{x|F(x) < 1\}$ be such that (a, b), $-\infty \leq a < b < \infty$ is the interval support of X. Then cumulative distribution function (cdf) F, defined by $F(x) = P(X \leq x)$ is a non-decreasing continuous function satisfying $\lim_{x \to a} F(x) = 0$ and $\lim_{x \to b} F(x) = 1$. If F is differentiable, the probability density function (pdf) of X may be defined as $f(x) = \frac{dF(x)}{dx}$.

1.2.1 Survival function

The survival function or reliability function $\bar{F}(\cdot)$ is a non-increasing continuous function given by $\bar{F}(x) = P(X > x) = 1 - F(x)$ where $\bar{F}(0) = 1$ and $\lim_{x\to\infty} \bar{F}(x) = 0$. If $\bar{F}$ is differentiable, the pdf of X is given by $f(x) = -\frac{d\bar{F}(x)}{dx}$.

1.2.2 Failure rate

The failure rate (hazard rate) of a rv X, denoted by $h(\cdot)$, is defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{P[x \le X < x + \Delta x | X > x]}{\Delta x}. \tag{1.1}$$

The failure rate $h(x)$, measures the instantaneous rate of failure or death at time x, given that an individual survives at least up to time x. $h(x)\Delta x$ represents the approximate probability of failure in the interval $[x, x+\Delta x)$, given the individual survived up to time x, provided Δx is very small. Kotz and Shanbhag (1980) defined failure rate as the Radon Nikodym derivative with respect to Lebesque measure on $\{x : F(x) < 1\}$, of the hazard measure $H(B) = \int_B \frac{dF(x)}{[1-F(x)]}$ for every Borel set B of the form $(-\infty, L)$, where $L = \inf\{x : F(x) = 1\}$. If f is the pdf of X, (1.1) can be equivalently written as

$$h(x) = \frac{f(x)}{\bar{F}(x)} = -\frac{d}{dx}[\log \bar{F}(x)].$$

The above expression on integration with respect to x and applying $\lim_{x\to a} \bar{F}(x) = 1$, yields

$$\bar{F}(x) = \exp\left(-\int_a^x h(t)dt\right) = \exp(-H(x)),$$

where $H(x) = \int_a^x h(t)dt$ is known as cumulative hazard rate.

The concept of failure rate is widely used for characterizing lifetime distributions. For instance, failure rate constant is a characteristic property of exponential distribution (Galambos and Kotz (1978)). A large volume of literature is available on characterizations of hazard rate function and functions of hazard rate function (see, for example, Barlow et al. (1963), Nanda and Shaked (2001), Nair and Asha (2004), Nanda (2010) and references therein).

1.2.3 Reversed hazard rate

As a dual to hazard rate function, Barlow et al. (1963) proposed reversed hazard rate (RHR) function. The rv X has RHR on the interval of support (a, b), $-\infty \leq a < b < \infty$ if and only if the rv defined by $Y = -X$ has the hazard rate $h(-x)$ on $(-b, -a)$ (see Block et al.(1998)). For the rv X, reversed hazard rate denoted by $\lambda(\cdot)$ is defined as

$$\lambda(x) = \lim_{\Delta x \to 0} \frac{P[x - \Delta x < X \leq x | X \leq x]}{\Delta x}.$$

$\lambda(x)$ measures the instantaneous rate of failure of a unit at time x, given that it failed before time x. Thus, $\lambda(x)\Delta x$ gives the probability that the unit failed in an infinitesimal interval $(x - \Delta x, x]$, given that it failed before x. If the pdf f exists, the above equation can be expressed as

$$\lambda(x) = \frac{f(x)}{F(x)} = \frac{d}{dx}[\log F(x)].$$

Keilson and Sumita (1982) shown that $\lambda(x)$ determines the df through the relationship

$$F(x) = \exp\left(-\int_x^b \lambda(t)dt\right) = \exp(-\Lambda(x)),$$

where $\Lambda(x) = \int_x^b \lambda(t)dt$ is the cumulative reversed hazard rate.

Finkelstein (2002) established a relation between $\lambda(x)$ and $h(x)$ which is as follows:

$$\lambda(x) = \frac{h(x)}{\exp\left(\int\limits_a^x h(t)dt\right) - 1}.$$

For more properties and studies related to RHR one can refer to Gupta and Nanda (2001), Nanda and Shaked (2001), Finkelstein (2002), Nair and Asha (2004), Chandra and Roy (2005), Nair et al.(2005), Bartoszewicz and Skolimowska (2006), Sankaran et al. (2007) and Sunoj and Maya (2006).

1.2.4 Mean residual life function

For a rv X defined on $\mathbb{R}^+ = \{x|x \in [0,\infty)\}$ with $E(X) < \infty$, the mean residual life function (MRLF) denoted by $r(\cdot)$, defined by (Swartz(1973))

$$r(x) = E(X - x|X > x). \tag{1.2}$$

The mean residual life function $r(x)$ measures the average lifetime of a residual rv X, which has survived time x. If the df F is continuous with respect to Lebesque measure, (1.2) becomes

$$r(x) = \frac{1}{\bar{F}(x)} \int_x^\infty \bar{F}(t)dt.$$

It satisfies the following properties

(i) $0 \leq r(x) < \infty$, $x \geq 0$

(ii) $r(0) > 0$

(iii) $r(x)$ is continuous in x

(iv) $r(x) + x$ is increasing on $\mathbb{R}^+$

(v) If there exists an x_0 such that $r(x_0) = 0$ then $r(x) = 0$ for $x \geq x_0$ otherwise, there does not exist such an x_0 with $r(x_0) = 0$ then $\int_0^\infty r^{-1}(x)dx = \infty$.

Further $r(x)$ uniquely determines the underlying distribution through the relationship

$$\bar{F}(x) = \frac{r(0)}{r(x)} \exp\left[-\int_0^x \frac{1}{r(t)} dt\right].$$

Model identification can be done easily by knowing the functional form of $r(x)$. For instance, characterization of distribution using the linear form of $r(x)$ is available in Hall and Wellner (1981). MRLF is related to the failure rate by the equation

$$h(x) = \frac{1 + r'(x)}{r(x)}.$$

A large volume of literature is available on $r(x)$, for more properties one could refer to Hall and Wellner (1981), Mukherjee and Roy (1986), Nanda (2010) and references therein.

1.2.5 Reversed mean residual life function

Reversed mean residual life function is an analogous concept of MRLF but defined for the past lifetime $(t - X|X \leq t)$, denoted by $v(\cdot)$ and defined as

$$v(x) = E(x - X|X \leq x).$$

It measures the average past lifetime of a rv which failed at time x. It is also known as mean inactivity time or mean past time in reliability. If the df F is continuous with respect to Lebesgue measure, $v(x)$ can be written as

$$v(x) = \frac{1}{F(x)} \int_0^x F(x)dx.$$

Reversed mean residual life time is related to reversed hazard rate by the equation,

$$\lambda(x) = \frac{1 - v'(x)}{v(x)}.$$

Like $r(x)$, $v(x)$ also determines the underlying df through the relationship (Chandra and Roy (2001),

$$F(x) = \exp\left(-\int_x^\infty \frac{1 - v'(t)}{v(t)} dt\right).$$

For various studies related to reversed mean residual life functions we refer to Kayid and Ahmed (2004), Ahmed and Kayid (2005), Gandotra et al.(2011) and references therein.

1.2.6 Vitality function

For a rv X admitting an absolutely continuous df F, with respect to Lebesgue Stieljes measure on $\mathbb{R}$, the vitality function $m(\cdot)$ given by Kupka and Loo (1989) as

$$m(x) = E(X|X \geq x).$$

The vitality function satisfies the following properties:

(i) $m(x)$ is non-decreasing and left continuous on $[-\infty, L)$ where $L = \inf\{x : F(x) = 1\}$,

(ii) $m(x) > x$ for all $x < L$,

(iii) $\lim_{x \to L^-} m(x) = L$,

(iv) $\lim_{x \to ^-\infty} m(x) = E[X]$.

Further $m(x)$ is related to $r(x)$ through the relationship

$$m(x) = r(x) + x \text{ and } m'(x) = r(x)h(x).$$

Shanbhag (1970) shown that the relation $m(x) = x + c$, with $P(X < 0) = 0$, $E(X) < \infty$ and c is a constant, holds, when X follows exponential distribution. For certain populations $m(x)$ stands a more suitable choice for modelling than $r(x)$, for example, Nair and Sankaran (1991) characterized Pearson family of distributions by means of the relationship $m(x) = \mu + (a_0 + a_1 x + a_2 x^2)h(x)$, where $\mu = E(X)$ and a_0, a_1, a_2 are constants. As there exists a one-to-one relationship between $m(x)$ and $r(x)$, the unique determination of F is similar to $r(x)$.

1.3 Bivariate notions

Let (X_1, X_2) be a random vector defined on $\mathbb{R}_2 = (-\infty, \infty) \times (-\infty, \infty)$. Then joint (bivariate) df of (X_1, X_2) is defined as $F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$. It satisfies the following properties:

1) $\lim_{x_1 \to -\infty} \lim_{x_2 \to -\infty} F(x_1, x_2) = \lim_{x_1 \to -\infty} F(x_1, x_2) = \lim_{x_2 \to -\infty} F(x_1, x_2) = 0$,

2) $\lim_{x_1 \to \infty} \lim_{x_2 \to \infty} F(x_1, x_2) = 1$,

3) If $a < b$ and $c < d$, then $F(a, c) < F(b, d)$,

4) If $a > x_1$ and $b > x_2$, then $F(a, b) - F(a, x_2) - F(x_1, b) + F(x_1, x_2) \geq 0$.

Bivariate sf of (X_1, X_2) denoted by $\bar{F}$ is defined as $\bar{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$. $\bar{F}(x_1, x_2)$ is related to $F(x_1, x_2)$ by the equation

$$\bar{F}(x_1, x_2) = 1 - \lim_{x_2 \to \infty} F(x_1, x_2) - \lim_{x_1 \to \infty} F(x_1, x_2) + F(x_1, x_2).$$

If $F(x_1, x_2)$ is absolutely continuous and if the second order derivative exists then the joint density function f can be defined as

$$f(x_1, x_2) = \frac{\partial^2 \bar{F}(x_1, x_2)}{\partial x_1 \partial x_2} = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2}.$$

1.3.1 Bivariate failure rate

In bivariate case, the failure rate has not been defined uniquely. A straightforward extension of univariate definition of failure rate to the bivariate case is due to Basu (1971) , defined as a scalar failure rate, given by

$$\lambda(x_1, x_2) = \frac{f(x_1, x_2)}{\bar{F}(x_1, x_2)}.$$

Puri and Rubin (1974) characterized a mixture of exponential distributions by the constancy $\lambda(x_1, x_2) = c$ for $x_1 > 0$ and $x_2 > 0$. However, in general $\lambda(x_1, x_2)$ does not determine the bivariate distribution uniquely. This fact was noted by Yang and Nachlas (2001), Finkelstein (2003) and Finkelstein and Esaulova (2005).

An alternate approach is due to Johnson and Kotz (1975) who proposed a vector-valued failure rate as

$$h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2)),$$

where

$$h_i(x_1, x_2) = -\frac{\partial}{\partial x_i} \log \bar{F}(x_1, x_2), \quad i = 1, 2,$$

is the instantaneous failure rate of X_i at time x_i given that X_i was alive at time x_i and that X_{3-i} survived beyond time x_{3-i}, $i = 1, 2$. Unlike $\lambda(x_1, x_2)$, $h(x_1, x_2)$ uniquely determines the df (see Marshall and Olkin (1979) and Shanbhag and Kotz (1987)) through the expression

$$\bar{F}(x_1, x_2) = \exp\left[-\int_0^{x_1} h_1(u, 0)du - \int_0^{x_2} h_2(x_1, u)du\right]$$

or

$$\bar{F}(x_1, x_2) = \exp\left[-\int_0^{x_1} h_1(u, x_2)du - \int_0^{x_2} h_2(0, u)du\right].$$

Some of the characterization results using the hazard gradient $h(x_1, x_2)$ can be found in Navarro and Ruiz (2004), Kotz et al. (2007), and Navarro et al. (2007).

Some other versions of failure rate in bivariate setup are also available in literature, for example Cox (1972), Marshall (1975), Shaked and Shanthikumar (1987), Basu and Sun (1997), Finkelstein (2003) and references therein.

1.3.2 Bivariate mean residual life function

As a direct extension of the definition in the univariate MRLF, Buchanan and Singpurwalla (1977) introduced a bivariate MRLF as

$$m(x_1, x_2) = \frac{\int_0^\infty \int_0^\infty P[X_1 > x_1 + t_1, X_2 > x_2 + t_2]}{\bar{F}(x_1, x_2)} dt_1 dt_2, \quad x_i > 0, \ i = 1, 2.$$

Even if $m(x_1, x_2)$ is a direct extension, however, it does not uniquely determine the underlying distribution, a limitation of $m(x_1, x_2)$.

An alternate definition to bivariate MRLF is provided by Shanbhag and Kotz (1987) and Arnold and Zahedi (1988) and is defined as follows. Let (X_1, X_2) be a random vector on $\mathbb{R}_2^+ = \{(x_1, x_2)|x_i > 0, i = 1, 2\}$ with joint df F and let (L_1, L_2) be the vector of extended real numbers such that $L_i = \inf\{x|F_i(x_i) = 1\}$ where F_i is the df of X_i. Further let $E(X_i) < \infty$, for $i = 1, 2$. The vector-valued Borel measurable function r on $\mathbb{R}_2^+$ is given by

$$\begin{aligned} r(x_1, x_2) &= (r_1(x_1, x_2), r_2(x_1, x_2)) \\ &= (E(X_1 - x_1|X_1 > x_1, X_2 > x_2), E(X_2 - x_2|X_1 > x_1, X_2 > x_2)), \end{aligned}$$

for all $(X_1, X_2) \in \mathbb{R}_2^+$, $x_i < L_i$, $i = 1, 2$ is called the bivariate mean residual life function. When (X_1, X_2) is continuous and non-negative, the components of bivariate MRLF are given by

$$r_1(x_1, x_2) = E(X_1 - x_1|X_1 > x_1, X_2 > x_2) = \frac{1}{\bar{F}(x_1, x_2)} \int_{x_1}^{\infty} \bar{F}(t, x_2)dt$$

and

$$r_2(x_1, x_2) = E(X_2 - x_2|X_1 > x_1, X_2 > x_2) = \frac{1}{\bar{F}(x_1, x_2)} \int_{x_2}^{\infty} \bar{F}(x_1, t)dt.$$

Unlike $m(x_1, x_2)$ due to Buchanan and Singpurwalla (1977), the bivariate MRLF $r(x_1, x_2)$ due to Arnold and Zahedi (1988) uniquely determines the distribution through the following identities (Nair and Nair (1988))

$$\bar{F}(x_1, x_2) = \frac{r_1(0, 0)r_2(x_1, 0)}{r_1(x_1, 0)r_2(x_1, x_2)} \exp\left[-\int_0^{x_1} \frac{dt}{r_1(t, 0)} - \int_0^{x_2} \frac{dt}{r_2(x_1, t)}\right]$$

or

$$\bar{F}(x_1, x_2) = \frac{r_1(0, x_2)r_2(0, 0)}{r_1(x_1, x_2)r_2(0, x_2)} \exp\left[-\int_0^{x_2} \frac{dt}{r_2(0, t)} - \int_0^{x_1} \frac{dt}{r_1(t, x_2)}\right].$$

Similar to the relationship between failure rate and MRLF in the univariate case, the bivariate MRLF is related to bivariate failure rate by

$$h_i(x_1, x_2) = \frac{1 + \frac{\partial}{\partial x_i} r_i(x_1, x_2)}{r_i(x_1, x_2)}, \quad i = 1, 2.$$

1.3.3 Bivariate reversed mean residual life function

Bivariate (vector-valued) reversed mean residual life function is proposed by Nair and Asha (2008), the definition to which is as follows: Let (X_1, X_2) be a random vector defined on $\mathbb{R}_2$ with joint df F and marginal df F_i, $i = 1, 2$, $E(X_1, X_2) < \infty$ and let (a_1, a_2) and (b_1, b_2) be vectors of real numbers such that $a_i = \inf(x|F_i(x) > 0)$ and $b_i = \sup(x|F_i(x) < 1)$ then bivariate reversed mean residual life function is defined as a Borel measurable function

$$v(x_1, x_2) = (v_1(x_1, x_2), v_2(x_1, x_2)),$$

where

$$v_1(x_1, x_2) = E(x_1 - X_1 | X_1 \leq x_1, X_2 \leq x_2) = \frac{1}{F(x_1, x_2)} \int_{a_1}^{x_1} F(t, x_2) dt$$

and

$$v_2(x_1, x_2) = E(x_2 - X_2 | X_1 \leq x_1, X_2 \leq x_2) = \frac{1}{F(x_1, x_2)} \int_{a_2}^{x_2} F(x_1, t) dt.$$

Bivariate reversed mean residual life function determines the underlying bivariate distribution through the expressions

$$F(x_1, x_2) = \frac{v_1(b_1, b_2) v_2(x_1, b_2)}{v_1(x_1, b_2) v_2(x_1, x_2)} \exp\left(- \int_{x_1}^{b_1} \frac{du}{v_1(u, b_2)} - \int_{x_2}^{b_2} \frac{du}{v_2(x_1, u)} \right)$$

and

$$F(x_1, x_2) = \frac{v_1(b_1, x_2)v_2(b_1, b_2)}{v_1(x_1, x_2)v_2(b_1, x_2)} \exp\left(-\int_{x_1}^{b_1} \frac{du}{v_1(u, x_2)} - \int_{x_2}^{b_2} \frac{du}{v_2(b_1, u)}\right).$$

Further, bivariate reversed mean residual life function is related to bivariate reversed hazard rate by the expression

$$\lambda_i(x_1, x_2) = \frac{1 - \frac{\partial}{\partial x_i} v_i(x_1, x_2)}{v_i(x_1, x_2)}, \quad i = 1, 2.$$

1.4 Weighted distributions

The concept of weighted distributions can be traced back to the study of the effects of methods of ascertainment upon estimation of frequencies by Fisher (1934). In extending the basic ideas of Fisher, Rao (1965) identified the need for a unifying concept and studied various sampling situations that can be modeled by what he called weighted distributions. These situations occur where the recorded observations cannot be considered as a random sample from the original distributions, such as non observability of some events or damage caused to the original observation resulting in reduced value, or adoption of a sampling procedure which gives unequal chances to the units in the original. A formal definition of a weighted distribution is obtained by considering a probability space $(\Omega, \Im, P)$ and a rv $X : \Omega \to H$, where $H = (a, b)$ is an interval on the real line. If f is the pdf of X and $w(\cdot)$, a non-negative function satisfying $\mu^w = E(w(X)) < \infty$, then the rv X^w with pdf

$$f^w(x) = \frac{w(x)}{\mu^w} f(x), \quad a < x < b,$$

is said to have weighted distribution, corresponding to the distribution of X.

Depending on the selection of weight function $w(\cdot)$, we obtain different weighted models. When $w(x) = x$, then X^w is called the length-biased rv denoted by X^L and its pdf is given by

$$f^L(x) = \frac{x}{\mu} f(x), \quad a < x < b,$$

where $\mu = E(X) < \infty$. Length-biased sampling is usually adopted when a proper method of selection of sampling is absent. In such situations items are sampled at a rate proportional to their length, so that larger values of the quantity being measured are sampled with higher probabilities. When dealing with the problem of sampling and selection from a length-biased distribution, the possible bias due to the nature of data-collection process can be utilized to connect the population parameters to that of the sampling distribution. That is, biased sampling is not always harmful to the process of inference on population parameters. Inference based on a biased sample of a certain size may yield more information than that given by an unbiased sample of the same size, provided that the choice mechanism behind the biased sample is known. Length-biased sampling and the associated model are very popular in literature and have been found many applications in various topics such as reliability theory, survival analysis, population studies and clinical trials. For a detailed survey on various aspects of length-biased sampling one can refer to Fisher (1934), Rao (1965), Neel and Schull (1966), Eberhardt (1968), Zelen (1971), Zelen (1974), Cook and Martin (1974), Patil and Rao (1977), Patil and Rao (1978), Eberhardt (1978), Sankaran and Nair (1993b), Sen and Khattree (1996), Oluyede (1999), Oluyede (2000), Van et al. (2000), Sunoj (2004) and Bar-Lev and Schouten (2004). More generally, when sampling of units is made with probability proportional to some measure of unit size *i.e.* when $w(x) = x^\alpha$, $\alpha > 0$, then the resulting distribution is called size-biased (see Blumenthal (1967) and

Scheaffer (1972)). Size-biased rv denoted by X^s of order α is specified by the density

$$f^s(x) = \frac{x^\alpha}{\mu^s} f(x), \quad a < x < b, \tag{1.3}$$

where $\mu^s = E(X^\alpha) < \infty$. When $\alpha = 1$, (1.3) reduces to the pdf of X^L.

For works on weighted distribution one can refer to Rao (1965, 1985), Blumenthal (1967), Scheaffer (1972), Patil and Ord (1976), Patil and Rao (1978), Gupta (1984), Sankaran and Nair (1993b), Oluyede (1999, 2000), Sunoj (2000), Navarro et al. (2001), Sunoj (2004), Di Crescenzo and Longobardi (2006), Sunoj and Maya (2006) and references therein.

1.4.1 Equilibrium distributions

Equilibrium distributions arise naturally in renewal theory (see Cox (1962), Blumenthal (1967), Despande et al. (1986), Singh (1989), Nair and Hitha (1990)). It is the distribution of the backward and the forward recurrence time in the limiting case. That is, if we have a set of components with continuous, independent and identically distributed lifetimes $L_1, L_2, L_3, ...$ having pdf f, such that $f(x) \to 0$ as $x \to \infty$, sf $\bar{F}$ and finite mean μ and that the first component is replaced upon failure by second, second by third and so on. Then the sequence of points $S_n = L_1 + L_2 + ... + L_n$ constitute a renewal process. At some fixed time $t > 0$, if $N(t) = \sup\{n : S_n \leq t\}$, then the rvs $U_t = t - S_{N(t)}$ and $V_t = S_{N(t)+1} - t$ are called the age (the backward recurrence time) and the residual life (the forward recurrence time) of the component working at the time t. When $t \to \infty$, both the age U_t and the residual life V_t of the component in use at the time t, have the same asymptotic distribution with pdf

$$g(x) = \frac{\bar{F}(x)}{\mu}, \quad x \geq 0.$$

This distribution is called the equilibrium distribution. Equilibrium distribution can be viewed as a special case of weighted distribution with weight function $w(x) = \frac{1}{h(x)}$, where $h(x)$ is the hazard rate function. One can refer to Gupta and Sankaran (1998), Gupta (2007), Nair and Preeth (2008), Navarro and Sarabia (2010) for works on equilibrium distributions.

1.4.2 Bivariate weighted distribution

The wide applicability of weighted distributions in the univariate case has motivated many researchers to extend the concept of weighted distribution to higher dimensions. Let (X_1, X_2) be a bivariate random vector in the support of $(a_1, b_1) \times (a_2, b_2)$, $b_i > a_i$, $i = 1, 2$ where (a_i, b_i) is an interval on the real line with absolutely continuous df F, and pdf f. By defining $w(\cdot, \cdot)$ as a non-negative weight function satisfying $E(w(X_1, X_2)) < \infty$, Mahfoud and Patil (1982) defined bivariate weighted distribution as the distribution of the random vector (X_1^w, X_2^w) with pdf

$$f^w(x_1, x_2) = \frac{w(x_1, x_2)}{E(w(X_1, X_2))} f(x_1, x_2), \quad a_i < x_i < b_i, \ i = 1, 2.$$

For more properties of bivariate weighted distributions one can refer to Nair and Sunoj (2003), Sunoj and Sankaran (2005), Navarro et al. (2006) and references therein.

1.5 Truncation

Statistical problems of truncation arise when a standard statistical model is appropriate for analysis except that values of the rv falling below or above some value are not measured at all. For example, in a study of particle size, particles below the resolving power of observational equipment will not be seen at all. If values below a certain lower limit, a, are not observed at all, the distribution is said to be truncated on the left. If values larger than an upper limit, b, are not observed, the distribution is said to be truncated on the right. If only values lying between a and b are observed, the distribution is said to be doubly truncated.

1.6 Proportional hazard rate model

Proportional hazard rate model, well known as Cox proportional hazards model was proposed by Cox (1972). Let X and Y be two rvs with pdfs f and g, sfs $\bar{F}$ and $\bar{G}$ and hazard rates h_X and h_Y respectively, then X and Y are said to satisfy proportional hazard rate (PHR) model if they satisfy the relationship

$$h_Y(x) = \theta h_X(x) \quad \text{or equivalently} \quad \bar{G}(x) = (\bar{F}(x))^{\theta},$$

where $\theta > 0$, is a constant, with the pdf $g(x) = \theta(\bar{F}(x))^{\theta-1} f(x)$. Proportional hazards model has been used to model failure time data in reliability and survival analysis. Studies related to PHR model could be found in Clayton and Cuzik (1985), Ebrahimi and Kirmani (1996a), Kundu and Gupta (2004), Nair and Gupta (2007), Sankaran and Sreeja (2007), Dewan and Sudheesh (2009) and references therein.

1.7 Proportional reversed hazard rate model

In contrast to Cox's proportional hazard rate model, Gupta et al.(1998) proposed the proportional reversed hazard rate model (also known as Lehman family of alternatives). Two rvs X and Y with dfs F and G and reversed hazard rates λ_X and λ_Y satisfy proportional reversed hazard rate (PRHR) model with proportionality constant $\theta > 0$ if they satisfy the relationship

$$\lambda_Y(x) = \theta\lambda_X(x) \quad \text{or} \quad G(x) = (F(x))^{\theta},$$

with pdf $g(x) = \theta(F(x))^{\theta-1}f(x)$. For more details on PRHR, we refer to Sengupta et al. (1999), Di Crescenzo (2000), Gupta and Gupta (2007), Sankaran and Gleeja (2008), Nanda (2010) and references therein.

1.8 Characterization

A basic problem in reliability analysis, when the data on lifetimes are the only input, is to identify the underlying distribution that is supposed to generate the observations. In general, it is not easy to isolate all the physical causes that contribute individually or collectively to the life mechanism and to mathematically account for each and hence the task of identifying the suitable model representing the data becomes very difficult. In many situations, the information content on the ageing pattern available from the data is not specific enough to enable the analyst to narrow down his consideration to a particular model. When the data are the only criteria for selecting the model it is customary to start with a general system of distributions and then to select an appropriate member from the system that fits the data. The problem one has to face here is that most of the models used in this connection have different right tail behaviour and the sample size may not be large enough to notice such differences. A standard practice adopted in such modelling situations is to ascertain the physical properties of the process generating the observations, express them by means of equations or inequalities and then solve them to obtain the model. In reliability, some basic concepts such as failure rate, mean residual life, vitality *etc.* have been developed by analysts through which the physical characteristics of the life mechanism can be adequately described and therefore these concepts form the basis of specifying a probability distribution of lifetimes. The only exact method of determining a probability distribution is to use a characterization theorem, which in general terms say that under certain conditions a family of distributions $\boldsymbol{F}$ is the only one possessing a designated property $\boldsymbol{P}$. Thus if one can translate the characteristics of the life mechanism in terms of the failure rate, mean residual life or any ageing criteria and if there exists a probability distribution characterised by such a property, the problem of model identification is satisfactorily resolved.

1.9 Ordering of random variables

Stochastic orders and inequalities have been in use during the last few decades, at an accelerated rate, in many diverse areas of probability and statistics such as reliability theory, queuing theory, survival analysis, biology, economics, insurance, actuarial science, operations research, and management science. The simplest way to compare two distribution functions is through their means (if they exist) or their variances (if the means are equal). However, such comparisons usually are not informative, because they are based on only one or two specific characteristics. Ordering of rvs is an effective tool used in such situations (see Marshall and Olkin (1979), Ross (1983), Shaked and Shanthikumar (2007)). There are several ways in which one can assert that a rv X

(or equivalently its df F_X) is 'greater than' another rv Y (or equivalently its df F_Y). Stochastic ordering, hazard rate ordering and likelihood ratio ordering are among the various notions of ordering between rvs.

A rv X is said to be stochastically greater than a rv Y, written as $X \geq_{ST} Y$ if

$$\bar{F}_X(t) \geq \bar{F}_Y(t) \quad \forall t.$$

We say a rv X is greater than another rv Y in hazard rate ordering, written as $X \geq_{HR} Y$ if

$$h_X(t) \leq h_Y(t) \quad \forall\, t \geq 0.$$

X is larger than Y in likelihood ratio, written as $X \geq_{LR} Y$ if $\frac{f_X(t)}{f_Y(t)}$ is non-decreasing in t. Here f_X, $\bar{F}_X$ and h_X denote the pdf, the sf and the hazard rate of X and f_Y, $\bar{F}_Y$ and h_Y denote the respective functions of the rv Y. It is well known that $X \geq_{LR} Y \Rightarrow X \geq_{HR} Y \Rightarrow X \geq_{ST} Y$ (Ross (1983)) *i.e.* likelihood ratio ordering implies the other two.

1.10 Conditionally specified models

It is inherently difficult to visualise bivariate distributions. Conditional densities can be easily visualised unlike marginal or joint densities. For example, in some human population it is reasonable to visualise the unimodal distribution of heights for a given weight with the mode of the conditional distribution varying monotonically with the weight. In a similar way a unimodal distribution of weights for a given height can be easily visualised with the mode varying monotonically with the height. But it is not so easy to visualise the appropriate joint distributions without certain assertion. A variety of transformation are being used to characterize the joint df. Joint characteristic function, joint moment generating function, and joint hazard function are some among them. They are well defined and will determine the joint df uniquely.

It is known that to determine the joint df, the knowledge of the marginals is inadequate. But if we incorporate conditional specification instead of marginal specification or together with marginal specification then the picture brightens. Sometimes one could characterize joint distribution in this way, *i.e.* the knowledge of one marginal density say f_X and the conditional density of Y given X will completely specify the joint density function f_{XY} of a bivariate rv. Alternatively one may specify the distribution solely in terms of the features of two families of conditional densities. This approach is called conditional specification of the joint distribution. For works on conditionally specified models one can refer to Arnold (1991), Arnold et al. (1992), Arnold et al. (1993), Arnold (1996), Arnold et al. (1998) and references therein.

1.11 Conditional survival models

In conditionally specified bivariate distribution, joint density f_{XY} has been referred with all conditionals of X given $Y = y$ belonging to a particular parametric family and all conditionals of Y given $X = x$, belonging to another parametric family. In the case of bivariate survival models, component survival *i.e.* on events such as $\{X > x\}$ and $\{Y > y\}$ have been conditioned. For works on conditional survival models we refer to Arnold (1995).

1.12 Shannon's entropy

The development of the idea of entropy by Claude Shannon (Shannon (1948)) provided the beginning of information theory. The entropy of a probability distribution is not only a measure of uncertainty but also a measure of information. In fact, the amount of information which we obtain when observing an experiment (depending on chance) can be taken numerically equal to the amount of uncertainty concerning the outcome of the experiment before carrying it out. Shannon (1948) defined entropy in discrete case.

If X is a discrete rv taking values $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$ then Shannon entropy is defined as

$$I(\boldsymbol{p}) = I(p_1, p_2, ..., p_n) = -\sum_{i=1}^{n} p_i \log p_i,$$

When the rv X takes the value x_i with probability $p_i = 1$ for some i, then $I(\boldsymbol{p}) = 0$. Then there is no uncertainty about the predictability of X by the probability mass function $\boldsymbol{p}$. When X follows discrete uniform distribution, then $p_i = 1/n$ for all i. This is the most uncertain situation as the outcome of such an experiment is the hardest to be predicted.

If X is a rv having an absolutely continuous df F with pdf f, then Shannon's entropy of X is defined as

$$I(f) = I_X = -\int_0^{\infty} f(x) \log f(x) dx = -E(\log f(X)). \tag{1.4}$$

When $I(f_1) > I(f_2)$, it is more difficult to predict the outcome of f_1 as compared to predict outcome of f_2. In various life testing experiments one has information only about the current age of the system under consideration and thus (1.4) is not a suitable measure in such situations and should be modified to take the current age into account. Accordingly, Ebrahimi (1996) introduced a measure of uncertainty known as residual entropy, defined as

$$I_X(t) = -\int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx. \tag{1.5}$$

Clearly (1.5) is the Shannon entropy corresponding to the residual rv $(X - t | X > t)$.

$I_X(t)$ can be equivalently written as

$$\begin{aligned} I_X(t) &= \log \bar{F}(t) - \frac{1}{\bar{F}(t)} \int_t^{\infty} f(x) \log f(x) dx, \\ &= 1 - \frac{1}{\bar{F}(t)} \int_t^{\infty} f(x) \log(h(x)) dx, \end{aligned}$$

where $h(x)$ is the hazard rate of the rv X.

For further study on residual entropy one can refer to Ebrahimi and Pellerey (1995), Nair and Rajesh (1998), Rajesh and Nair (1998), Sankaran and Gupta (1999), Asadi and Ebrahimi (2000), Rajesh (2001), Belzunce et al. (2004), Sunoj and Sankaran (2012) and references therein.

In many realistic situations uncertainty is not necessarily related to future but can also be associated with past. Based on this idea, Di Crescenzo and Longobardi (2002) proposed the past entropy using the past life rv $(t - X|X \leq t)$ given by

$$\bar{I}_X(t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx. \tag{1.6}$$

Similar to (1.5), a more useful expression in terms of the reversed failure rate is given by

$$\begin{aligned} \bar{I}_X(t) &= \log F(t) - \frac{1}{F(t)} \int_0^t f(x) \log f(x) dx, \\ &= 1 - \frac{1}{F(t)} \int_0^t f(x) \log(\lambda(x)) dx, \end{aligned}$$

where $\lambda(x)$ is the reversed hazard rate of X.

For properties and applications we refer to Nanda and Paul (2006), Di Crescenzo and

Longobardi (2006), Kundu et al. (2010) and Thapliyal and Taneja (2012).

1.13 Renyi's entropy

The notion of Shannon entropy can be generalized to provide additional information about the importance of specific events, for example outliers or rare events. A generalized measure of uncertainty, called Renyi's entropy of order α, was proposed and studied by Renyi (see Renyi (1959, 1961)) which is given by

$$I_R(\alpha) = \frac{1}{1-\alpha}\log\left(\int_0^{\infty} f^{\alpha}(x)dx\right) \text{ for } \alpha > 0,\ \alpha \neq 1.$$

It possesses the same properties of Shannon entropy, but it contains an additional parameter α which can be used to make it more or less sensitive to the shape of probability distribution. It plays a vital role as a measure of complexity and uncertainty in different areas such as Physics, Electronics and Engineering to describe various chaotic systems (Kurths et al. (1995)). It can be seen that $\lim_{\alpha\to 1} I_R(\alpha) = I_X$, so that Shannon entropy is a limiting case of Renyi's entropy. As a function of the parameter α, $I_R(\alpha)$ is known as the spectrum of Renyi's information.

Abraham and Sankaran (2005) introduced Renyi's entropy for residual lifetime rv, defined by

$$I_R(\alpha, t) = \frac{1}{1-\alpha}\log\left(\int_t^{\infty} \frac{f^{\alpha}(x)}{\bar{F}^{\alpha}(t)}dx\right) \quad \text{for } \alpha > 0,\ \alpha \neq 1. \tag{1.7}$$

Abraham and Sankaran (2005) have shown that (1.7) determines the distribution uniquely. Unlike the measure proposed by Ebrahimi (1996), $I_R(\alpha, t)$ provides the spectrum of Renyi's information of the remaining life of the system for different values of α. As

Renyi's information is related to the log likelihood, $I_R(\alpha, t)$ is useful in comparing the shapes and the tails of residual lifetime distributions. For more properties and applications of (1.7), we refer to Abraham and Sankaran (2005), Asadi et al. (2005) and Maya and Sunoj (2008).

1.14 Kullback-Leibler divergence

A measure of divergence is generally used as a tool to evaluate the information distance (divergence) between any two populations or functions. It is a measure that quantifies how different the two distributions are. It is not a true distance in the usual sense as it is not a symmetric function of the two distributions. Measures of divergence between two probability distributions have a long history initiated by the pioneering work of Pearson, Mahalanobis, Levy and Kolmogorov. Kullback-Leibler divergence measure and Renyi's divergence measure are among the most popular divergence measures. Accordingly, the present work pays attention on the information divergence measures *viz.* Kullback-Leibler divergence and Renyi's divergence of order α. Let X and Y be two absolutely continuous non-negative rvs that describe the lifetimes of two items. Let f, F, and $\bar{F}$ denote the pdf, the df and the sf of X respectively and g, G and $\bar{G}$, the corresponding functions of Y. As an information distance between F and G, Kullback and Leibler (1951) proposed a divergence measure given by

$$I_{X,Y} = \int_0^\infty f(x) \log \frac{f(x)}{g(x)} dx. \tag{1.8}$$

Equation (1.8) is a ruler to measure the similarity (closeness) between two distributions f and g and it plays an important role in information theory, reliability and other related fields. Further $I_{X,Y} \geq 0$ and equality holds if and only if $f = g$ *a.e.*

For an item under study, the information about the remaining or past lifetime is an important component in many applications. In such cases, the information measures are functions of time and thus are dynamic. Accordingly, Ebrahimi and Kirmani (1996b) defined the Kullback-Leibler divergence measure for the rvs X and Y that have survived time $t > 0$ as

$$I_{X,Y}(t) = \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)} dx. \tag{1.9}$$

Equation (1.9) can be equivalently written as

$$I_{X,Y}(t) = \log \bar{G}(t) - I_X(t) - \frac{1}{\bar{F}(t)} \int_t^\infty f(x) \log g(x) dx.$$

Ebrahimi and Kirmani (1996a) have shown that $I_{X,Y}(t)$ is a constant if and only if X and Y satisfy proportional hazard rate model. On the basis of (1.8) and (1.9) Di Crescenzo and Longobardi (2004) have proposed a measure of discrepancy between past lifetime distributions, given by

$$\bar{I}_{X,Y}(t) = \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)/F(t)}{g(x)/G(t)} dx. \tag{1.10}$$

Analogous to $I_{X,Y}(t)$ the constancy of $\bar{I}_{X,Y}(t)$ is a characterization to reversed proportional hazard rate model. Equation (1.10) can also be written as

$$\bar{I}_{X,Y}(t) = \log G(t) - \bar{I}_X(t) - \frac{1}{F(t)} \int_0^t f(x) \log g(x) dx.$$

Di Crescenzo and Longobardi (2004) have shown that for any strictly increasing bijective transformation ϕ, $\bar{I}_{\phi(X),\phi(Y)}(t) = \bar{I}_{X,Y}(\phi^{-1}(t))$ for $t > 0$. Maya and Sunoj (2008) have proved that $\bar{I}_{X,X^w}(t)$ is a constant if and only if the weight function is of the form $w(t) = (F(t))^{\theta-1}$, $\theta > 0$.

1.15 Kerridge's Inaccuracy

The concept of inaccuracy was introduced by Kerridge (1961). One reasonable measure of uncertainty is the amount of information obtained before certainty is achieved. Inaccuracy can be related to the amount of missing information. Nath (1968) has defined inaccuracy measure in continuous setup (also known as Fraser information) which is given by

$$K_{X,Y} = -\int_0^\infty f(x) \log g(x) dx. \tag{1.11}$$

Taneja et al. (2009) proposed the dynamic measure of inaccuracy for residual lifetime distributions, given by

$$K_{X,Y}(t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx, \tag{1.12}$$

$$= -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx + \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)} dx,$$

$$= I_X(t) + I_{X,Y}(t). \tag{1.13}$$

Note that $K_{X,Y}(t) = K_{X_t,Y_t}$, where $X_t = (X - t|X > t)$ and $Y_t = (Y - t|Y > t)$ are the residual rvs of X and Y respectively. Dynamic measure of inaccuracy for past lifetime rvs $(t - X|X \le t)$ and $(t - Y|Y \le t)$ was proposed by Kumar et al. (2011) and is given by

$$\bar{K}_{X,Y}(t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{g(x)}{G(t)} dx. \tag{1.14}$$

Similar to (1.13) $\bar{K}_{X,Y}(t)$ can be expressed as the sum of $\bar{I}_X(t)$ and $\bar{I}_{X,Y}(t)$. For recent study on inaccuracy measure one can refer to Kundu (2014), Kundu and Nanda (2014) and references therein.

1.16 Present Study

In literature one could find numerous studies in connection with characterization of bivariate dfs using information measures, but a little work could be found in relation with conditional specified or conditional survival approach in identifying the underlying bivariate model. The present study is a collection of such efforts made to characterize bivariate distribution using conditional specified and conditional survival approach with the help of certain well known information measures. The thesis is organized into six chapters. First chapter gives an introduction to the information measures, a literature review of basic reliability measures, basic measures of uncertainty and other concepts that have been used in the study the information measures. The second chapter is devoted to obtain the bounds for Renyi's divergence of order α and Kerridge's inaccuracy of residual and past lifetimes using likelihood ratio ordering. In the third chapter we introduce new entropy measures called cumulative residual entropy for conditionally specified and survival models and study their various properties. We also extend the measure to the past lifetimes and prove results arising out of it. In fourth chapter, a new measure of uncertainty namely cumulative residual Renyi's entropy has been introduced and its properties for the residual rv have been studied. We also examine the properties of the measure in the context of weighted and conditional rvs. A study on residual Kullback Leibler divergence measure, Renyi's divergence measure and Kerridge's inaccuracy measure for conditionally specified and conditional survival rvs are available in chapter five. In the sixth chapter, another two generalized information measures known as residual R norm entropy and divergence measures are studied and proved characterization theorems and bounds based on them in univariate and bivariate setup.

Chapter 2

Bounds for some dynamic information measures

2.1 Introduction

A generalized version of Kullback-Leibler divergence measure, namely Renyi's divergence was proposed by Renyi (see Renyi (1961)) in his studies of information measures. Renyi's divergence is related to Renyi's entropy like Kullback-Leibler divergence measure is related to Shannon entropy. Renyi's divergence possesses similar properties as those of Kullback-Leibler divergence, but has an additional parameter α, called its order. The general properties of Renyi's divergence of order α and its usefulness in characterizing different distributions are discussed in Section 2.2. Di Crescenzo and Longobardi (2004) studied Kullback-Leibler divergence measure for past lives and obtained certain useful bounds for it in terms of reversed hazard rate and past entropy. Motived by this, in this chapter we obtain some bounds for Renyi information divergence of order α and Kerridge's inaccuracy using likelihood ratio ordering. It provides some upper or lower

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bounds to these information measures, where the bounds are functions of hazard (reversed hazard) rates and residual (past) Shannon information measure(s). These bounds are also extended to the weighted models useful in comparing the observed and original distributions.

2.2 Dynamic Renyi's discrimination measure of order α

The wide applicability of KL divergence motivated Alfred Renyi (Renyi (1961)) to introduce a generalized divergence measure known as Renyi's divergence of order α, obtained directly from the Renyi's entropy given in Section 1.13. A formal definition of it for continuous rvs is as follows:

Let X and Y be two absolutely continuous non-negative rvs that describe the lifetimes of two items. Denote by f, F and $\bar{F}$, the pdf, the cdf and the sf of X respectively and g, G and $\bar{G}$, the corresponding functions of Y. Also, let $h_X = f/\bar{F}$ and $h_Y = g/\bar{G}$ be the hazard (failure) rates and $\lambda_X = f/F$ and $\lambda_Y = g/G$ be the reversed hazard rates of X and Y respectively. Then Renyi's information divergence of order α between two distributions f and g is defined by

$$I_{X,Y}(\alpha) = \frac{1}{\alpha-1}\log\int_0^\infty f^\alpha(x)g^{(1-\alpha)}(x)dx = \frac{1}{\alpha-1}\log E_f\left[\frac{f(X)}{g(X)}\right]^{\alpha-1}, \tag{2.1}$$

where $0 < \alpha \neq 1$.

However, in many applied problems *viz.*, reliability, survival analysis, economics, business, actuarial science *etc.* one has information only about the current age of the systems, and thus are dynamic. Then the discrimination information function between two residual lifetime distributions based on Renyi's information divergence of order α is given by

$$I_{X,Y}(\alpha,t)=\frac{1}{\alpha-1}\log\int_t^{\infty}\frac{f^{\alpha}(x)g^{(1-\alpha)}(x)}{\bar{F}^{\alpha}(t)\bar{G}^{(1-\alpha)}(t)}dx, \tag{2.2}$$

with $0<\alpha\neq 1$. Note that $I_{X,Y}(\alpha,t)=I_{X_t,Y_t}(\alpha)$, where $X_t=(X-t|X>t)$ and $Y_t=(Y-t|Y>t)$ are residual lifetimes associated with X and Y respectively. The following example illustrates the role of Renyi's information divergence of order α between two residual rvs.

Example 2.2.1. *Let X and Y_β be the random lifetimes of two items, where X is uniformly distributed on $(0,1)$ and Y_β has the pdf,*

$$g_\beta(t)=\beta\left(t-\frac{1}{2}\right)+1,\quad 0<t<1,\ -2\leq\beta\leq 2.$$

Now using (2.2) we have

$$I_{X,Y_\beta}(\alpha,t)=\frac{1}{\alpha-1}\log\left[\frac{\left(\frac{\beta}{2}+1\right)^{2-\alpha}-\left(\beta\left(t-\frac{1}{2}\right)+1\right)^{2-\alpha}}{\beta(2-\alpha)(1-t)^{\alpha}\left(1-\frac{\beta t^2}{2}+\frac{\beta t}{2}-t\right)^{1-\alpha}}\right]. \tag{2.3}$$

Figure 2.1 represents (2.3) for $\alpha=1.8$, $\beta=1.5$ (lower curve) and $\beta=-1.5$ (upper curve) and figure 2.2 represents (2.3) for $\alpha=1.8$, $\beta=0.15$ (lower curve) and $\beta=-0.15$ (upper curve).

The symmetry of $g_\beta(t)$ with respect to $t=\frac{1}{2}$ and (2.1) imply that the information distance between X and Y_β equals the information distance existing between X and $Y_{-\beta}$ i.e., $I_{X,Y_\beta}(\alpha)=I_{X,Y_{-\beta}}(\alpha)$ for all $\beta\in[-2,2]$. Moreover, for $\alpha\neq 0$ we have (2.3) from which it follows that in general $I_{X,Y_\beta}(\alpha,t)\neq I_{X,X_{-\beta}}(\alpha,t)$ for all $t\in(0,1)$, as is shown in figures 2.1 and 2.2. This illustrates that even if $I_{X,Y_\beta}(\alpha)=I_{X,Y_{-\beta}}(\alpha)$, its dynamic measure $I_{X,Y_\beta}(\alpha,t)$ is generally different from $I_{X,Y_{-\beta}}(\alpha,t)$.

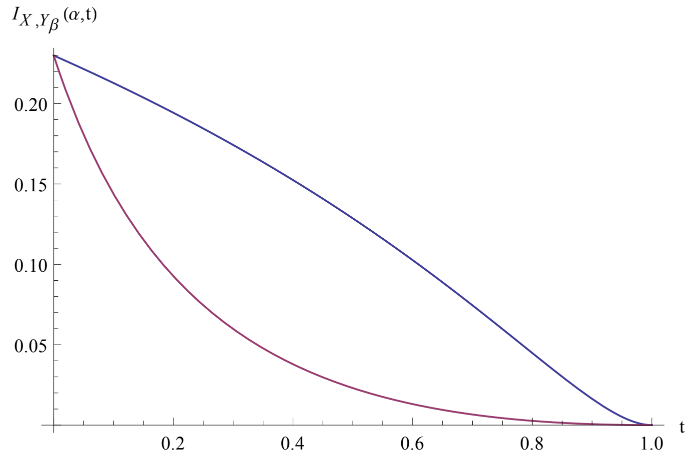

Figure 2.1:

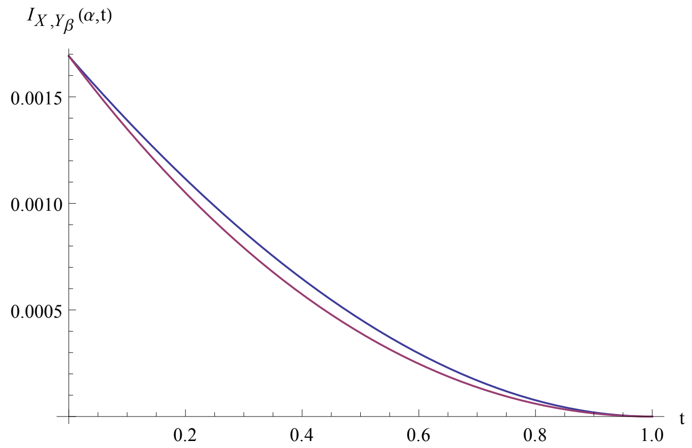

Figure 2.2:

□

Now in the following theorem we show how $I_{X,Y}(\alpha, t)$ is affected by an increasing transformation of X and Y.

Theorem 2.2.1. *If $\phi(\cdot)$ is an increasing function, then $I_{\phi(X),\phi(Y)}(\alpha, t) = I_{X,Y}(\alpha, \phi^{-1}(t))$.*

Proof.

$$\begin{aligned} I_{\phi(X),\phi(Y)}(\alpha, t) &= \frac{1}{\alpha - 1} \log \int_t^\infty \frac{f^\alpha(\phi^{-1}(x)) g^{1-\alpha}(\phi^{-1}(x))}{\bar{F}^\alpha(\phi^{-1}(t)) \bar{G}^{1-\alpha}(\phi^{-1}(t)) \phi'(\phi^{-1}(x))} dx, \\ &= \frac{1}{\alpha - 1} \log \int_{\phi^{-1}(t)}^\infty \frac{f^\alpha(y) g^{1-\alpha}(y)}{\bar{F}^\alpha(\phi^{-1}(t)) \bar{G}^{1-\alpha}(\phi^{-1}(t))} dy, \end{aligned}$$

$$= I_{X,Y}(\alpha, \phi^{-1}(t)). \qquad \square$$

Example 2.2.2. *Let X_1 and X_2 be two Pareto I rvs with pdfs given by $f_1(x) = (\frac{k}{x})^{c_1},\ t > k, k, c_1 > 0$ and $f_2(x) = (\frac{k}{x})^{c_2},\ t > k, k, c_2 > 0$ respectively. Then $I_{X_1,X_2}(\alpha, t)$ is obtained as*

$$I_{X_1,X_2}(\alpha, t) = \frac{1}{\alpha - 1} log \left[\frac{c_1^{\alpha} c_2^{1-\alpha} t^2}{(c_1 + 1)\alpha + (c_2 + 1)(1 - \alpha) - 1} \right].$$

Let $\phi(x) = x - k,\ x > 0, k > 0,$. If $Y_1 = \phi(X_1)$ and $Y_2 = \phi(X_2)$, then Y_1 and Y_2 follow Pareto II distribution with pdfs $g_1(x) = (1 + \frac{x}{k})^{-c_1},\ t > 0, k, c_1 > 0$ and $g_2(x) = (1 + \frac{x}{k})^{-c_2},\ t > 0, k, c_2 > 0$. Then by Theorem 2.2.1 we have

$$I_{Y_1,Y_2}(\alpha, t) = \frac{1}{\alpha - 1} log \left[\frac{c_1^{\alpha} c_2^{1-\alpha} (t + k)^2}{(c_1 + 1)\alpha + (c_2 + 1)(1 - \alpha) - 1} \right]. \qquad \square$$

Another problem of interest that leads to the dynamic information measures is the past lifetime of the individual. In the context of past lifetimes, Asadi et al. (2005) defined Renyi's discrimination implied by F and G between the past lives $(t - X|X \leq t)$ and $(t - Y|Y \leq t)$ as

$$\bar{I}_{X,Y}(\alpha, t) = \frac{1}{\alpha - 1} \log \int_0^t \frac{f^{\alpha}(x) g^{(1-\alpha)}(x)}{F^{\alpha}(t) G^{(1-\alpha)}(t)} dx, \tag{2.4}$$

for α such that $0 < \alpha \neq 1$. Given that at time t, two items have been found to be failing, equation (2.4) measures the disparity between their past lives. In the following example the importance of Renyi's divergence of order α between past lives has been discussed.

Example 2.2.3. *Consider the rvs given in Example 2.2.1. Now using equation* (2.4), *we obtain*

$$\bar{I}_{X,Y_\beta}(\alpha,t) = \frac{1}{\alpha-1}\log\left[\frac{\left(\beta\left(t-\frac{1}{2}\right)+1\right)^{2-\alpha}-\left(1-\frac{\beta}{2}\right)^{2-\alpha}}{\beta(2-\alpha)t^\alpha\left(\frac{\beta t^2}{2}-\frac{\beta t}{2}+t\right)^{1-\alpha}}\right]. \tag{2.5}$$

Figure 2.3 represents (2.5) *for* $\alpha = 1.8$, $\beta = 1.5$ *(upper curve) and* $\beta = -1.5$ *(lower curve) and figure 2.4 represents* (2.5) *for* $\alpha = 1.8$, $\beta = 0.15$ *(upper curve) and* $\beta = -0.15$ *(lower curve). As shown in figures 2.3 and 2.4, even though* $I_{X,Y_\beta}(\alpha) = I_{X,Y_{-\beta}}(\alpha)$, $\bar{I}_{X,Y_\beta}(\alpha,t)$ *is generally different from* $\bar{I}_{X,Y_{-\beta}}(\alpha,t)$.

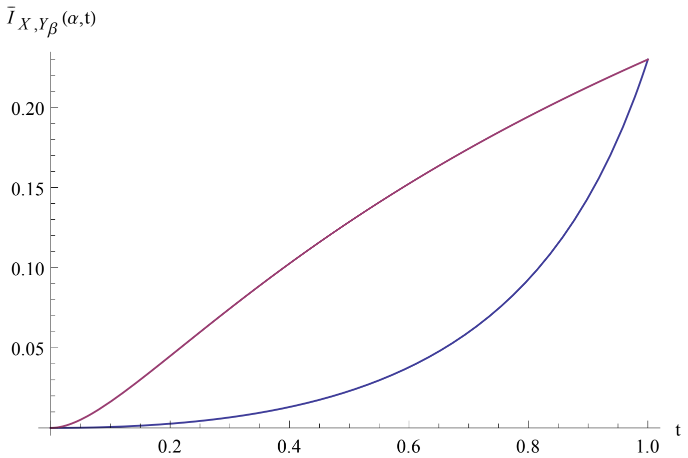

Figure 2.3:

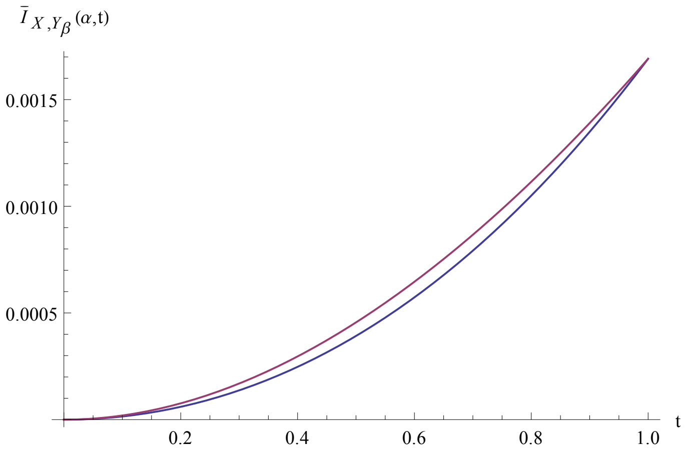

Figure 2.4:

□

In the following theorem we provide a simple relationship between $\bar{I}_{\phi(X),\phi(Y)}(\alpha,t)$ and $\bar{I}_{X,Y}(\alpha,\phi^{-1}(t))$, where ϕ is an increasing function.

Theorem 2.2.2. *If $\phi(\cdot)$ is an increasing function, then $\bar{I}_{\phi(X),\phi(Y)}(\alpha,t) = \bar{I}_{X,Y}(\alpha,\phi^{-1}(t))$.*

Proof.

$$\begin{aligned}
\bar{I}_{\phi(X),\phi(Y)}(\alpha,t) &= \frac{1}{\alpha-1}\log\int_0^t \frac{f^{\alpha}(\phi^{-1}(x))g^{1-\alpha}(\phi^{-1}(x))}{F^{\alpha}(\phi^{-1}(t))G^{1-\alpha}(\phi^{-1}(t))\phi'(\phi^{-1}(x))}dx,\\
&= \frac{1}{\alpha-1}\log\int_0^{\phi^{-1}(t)} \frac{f^{\alpha}(y)g^{1-\alpha}(y)}{F^{\alpha}(\phi^{-1}(t))G^{1-\alpha}(\phi^{-1}(t))}dy,\\
&= \bar{I}_{X,Y}(\alpha,\phi^{-1}(t)).
\end{aligned}$$

□

Example 2.2.4. *Let X_1 and X_2 be two exponential rvs with means $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ respectively. Then $\bar{I}_{X_1,X_2}(\alpha,t)$ can be obtained as*

$$\begin{aligned}
\bar{I}_{X_1,X_2}(\alpha,t) = \log\left[\frac{1-e^{-\lambda_2 t}}{\lambda_2}\right] - \frac{\alpha}{\alpha-1}\log\left[\frac{1-e^{-\lambda_1 t}}{\lambda_1}\right]\\
+ \frac{1}{\alpha-1}\log\left[\frac{1-e^{-(\alpha\lambda_1+(1-\alpha)\lambda_2)t}}{\alpha\lambda_1+(1-\alpha)\lambda_2}\right].
\end{aligned}$$

Let $\phi(x) = x^{1/\gamma}$, $x > 0$, $\gamma > 0$, an increasing function in x. If $Y_1 = \phi(X_1)$ and $Y_2 = \phi(X_2)$ then Y_1 and Y_2 follow Weibull distribution with a common shape parameter γ and scale parameters $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ respectively. Using Theorem 2.2.2. Renyi's divergence for the past lifetimes of Y_1 and Y_2 is given by

$$\begin{aligned}
\bar{I}_{Y_1,Y_2}(\alpha,t) = \log\left[\frac{1-e^{-\lambda_2 t^{\gamma}}}{\lambda_2}\right] - \frac{\alpha}{\alpha-1}\log\left[\frac{1-e^{-\lambda_1 t^{\gamma}}}{\lambda_1}\right]\\
+ \frac{1}{\alpha-1}\log\left[\frac{1-e^{-(\alpha\lambda_1+(1-\alpha)\lambda_2)t^{\gamma}}}{\alpha\lambda_1+(1-\alpha)\lambda_2}\right],
\end{aligned}$$

which using the Weibull pdfs directly is difficult to compute. □

Consider a rv X with pdf f. If $w(\cdot)$ is a non-negative function satisfying $\mu^w =$

$E(w(X)) < \infty$, then the pdf f^w, the df F^w and the sf $\bar{F}^w$ of the corresponding weighted rv X^w are respectively

$$f^w(x) = \frac{w(x)f(x)}{\mu^w}, F^w(x) = \frac{E(w(X)|X \le t)}{\mu^w}F(x)$$

and

$$\bar{F}^w(x) = \frac{E(w(X)|X > t)}{\mu^w}\bar{F}(x).$$

Renyi's discrimination measure for the residual lives of the original and weighted rvs is given by

$$I_{X,X^w}(\alpha, t) = \frac{1}{\alpha - 1}\log\int_t^\infty \frac{(f(x))^\alpha (f^w(x))^{1-\alpha}}{(\bar{F}(t))^\alpha (\bar{F}^w(t))^{1-\alpha}}dx, \tag{2.6}$$

for α such that $0 < \alpha \neq 1$, and that for past lives is given by

$$\bar{I}_{X,X^w}(\alpha, t) = \frac{1}{\alpha - 1}\log\int_0^t \frac{(f(x))^\alpha (f^w(x))^{1-\alpha}}{(F(t))^\alpha (F^w(t))^{1-\alpha}}dx, \tag{2.7}$$

for α such that $0 < \alpha \neq 1$. Equations (2.6) and (2.7) measure the discrepancy between the residual (past) lives of original rv X and weighted rv X^w. More importantly, $I_{X,X^w}(\alpha, t)$ may be a useful tool for measuring how far the true density is distant from a weighted density. On the other hand, when the original and weighted density functions are equal then, $I_{X,X^w}(\alpha, t) = 0$ *a.e.*

Remark 2.2.1. *Equations* (2.6) *and* (2.7) *may be useful in the determination of a weight function and therefore for the selection of a suitable weight function in an observed mechanism, we can choose a weight function for which* (2.6) *or* (2.7) *is small. Note that* (2.6) *and* (2.7) *are asymmetric measures. However, these measures become symmetric when the weight function is unity, i.e., when $f^w = f$ (see Maya and Sunoj(2008)).*

In many instances in applications, stochastic orders and inequalities are very useful for comparing two distributions. In the univariate case, several notions of stochastic orders are available in literature. It is well known that likelihood ratio order is stronger than the other orders such as stochastic order or the hazard rate order (see Shaked and Shanthikumar (2007)), as it implies the latter two. Accordingly, in the following theorems, we use the likelihood ratio ordering to obtain some bounds and inequalities on Renyi's discrimination measure of order α between X and Y and subsequently between X and X^w. We say X is said to be smaller than Y in likelihood ratio ($X \leq_{LR} Y$) if $f(x)/g(x)$ is decreasing in x over the union of the supports of X and Y. For Renyi's information divergence of order α, likelihood ratio ordering provides some upper (lower) bounds which are functions of important reliability measures and/or Shannon information measure. The following theorem provides a simple upper bound for Renyi information of order α with bounds as functions of hazard rates of X and Y.

Theorem 2.2.3. *If $X \leq_{LR} Y$, then*

$$I_{X,Y}(\alpha, t) \leq (\geq) \frac{\alpha}{\alpha - 1} \log \left[\frac{h_X(t)}{h_Y(t)} \right]$$

if $\alpha > 1$ ($0 < \alpha < 1$).

Proof. Since $X \leq_{LR} Y$, $\frac{f(x)}{g(x)}$ is decreasing in x, *i.e.,* $\frac{f(x)}{g(x)} \leq \frac{f(t)}{g(t)}$ for all $x > t$

$$\begin{aligned} I_{X,Y}(\alpha, t) &= \frac{1}{\alpha - 1} \log \int_t^\infty \frac{f^\alpha(x)}{\bar{F}^\alpha(t)} \frac{g^{(1-\alpha)}(x)}{\bar{G}^{(1-\alpha)}(t)} dx, \\ &= \frac{1}{\alpha - 1} \log \int_t^\infty \frac{f^\alpha(x)}{g^\alpha(x)} \frac{g(x)}{\bar{F}^\alpha(t) \bar{G}^{(1-\alpha)}(t)} dx, \end{aligned}$$

Then,

$$I_{X,Y}(\alpha, t) \leq (\geq) \frac{1}{\alpha - 1} \log \int_t^\infty \frac{f^\alpha(t)}{g^\alpha(t)} \frac{g(x)}{\bar{F}^\alpha(t) \bar{G}^{(1-\alpha)}(t)} dx, \text{for } \alpha > 1 (0 < \alpha < 1),$$

$$= \frac{1}{\alpha-1}\log\left[\frac{h_X^\alpha(t)}{h_Y^\alpha(t)}\right] = \frac{\alpha}{\alpha-1}\log\left[\frac{h_X(t)}{h_Y(t)}\right]. \qquad \square$$

Corollory 2.2.1. *If $X \leq_{LR} X^w$, then*

$$I_{X,X^w}(\alpha,t) \leq (\geq) \frac{\alpha}{\alpha-1}\log\left[\frac{E(w(X)|X>t)}{w(t)}\right]$$

if $\alpha > 1$ ($0 < \alpha < 1$).

Proof. For a weighted rv X^w, hazard rate is given by

$$h_{X^w}(t) = \frac{w(x)}{E(w(X)|X>t)}h_X(t),$$

then

$$\frac{h_X(t)}{h_{X^w}(t)} = \frac{E(w(X)|X>t)}{w(x)},$$

now from Theorem 2.2.3 the corollary follows. $\square$

Example 2.2.5. *For Pareto I distribution with pdf $f(x) = ck^c x^{-c-1}$, $x > k$, $k > 0$, $c > 1$, using the weight function $w(x) = x$, we have $X \leq_{LR} X^w$ and also X^w has a Pareto distribution.*

$$\begin{aligned} I_{X,X^w}(\alpha,t) &= \frac{1}{\alpha-1}\log\left[\frac{c^\alpha(c-1)}{(c-1)^\alpha(c+\alpha-1)}\right] \\ &= \frac{\alpha}{\alpha-1}\log\left(\frac{c}{c-1}\right) + \frac{1}{\alpha-1}\log\left(\frac{c-1}{c+\alpha-1}\right). \end{aligned}$$

So

$$I_{X,X^w}(\alpha,t) \leq (\geq) \frac{\alpha}{\alpha-1}\log\left(\frac{c}{c-1}\right) = \frac{\alpha}{\alpha-1}\log\left[\frac{E(w(X)|X>t)}{w(t)}\right],$$

according as $\alpha > 1$ ($0 < \alpha < 1$). $\square$

An important distribution which arises as a special case of weighted distributions is the equilibrium model, obtained when the weight function is $w(\cdot) = \bar{F}/f$. Renyi's discrimination measure for the residual lives of the original and the equilibrium rvs is given by

$$I_{X,X^E}(\alpha, t) = \frac{1}{\alpha - 1} \log \int_t^\infty \frac{(f(x))^\alpha (f^E(x))^{1-\alpha}}{(\bar{F}(t))^\alpha (\bar{F}^E(t))^{1-\alpha}} dx,$$

for α such that $0 < \alpha \neq 1$, and that for past lives is given by

$$\bar{I}_{X,X^E}(\alpha, t) = \frac{1}{\alpha - 1} \log \int_0^t \frac{(f(x))^\alpha (f^E(x))^{1-\alpha}}{(F(t))^\alpha (F^E(t))^{1-\alpha}} dx,$$

for α such that $0 < \alpha \neq 1$.

Corollory 2.2.2. *If $X \leq_{LR} X^E$ then*

$$I_{X,X^E}(\alpha, t) \leq (\geq) \frac{\alpha}{\alpha - 1} \log[1 + r'(t)]$$

if $\alpha > 1$ ($0 < \alpha < 1$) where $r(t)$ is the MRLF of X.

Proof. For the equilibrium rv, hazard rate is given by $h_{X^E}(t) = \frac{1}{r(t)}$ then $\frac{h_X(t)}{h_{X^E}(t)} = h_X(t)r(t) = 1 + r'(t)$, now from Theorem 2.2.3 the corollary follows. □

Even if the past lifetime information measure appears to be a dual of its residual version, however, Di Crescenzo and Longobardi (2004) have shown the importance of past lifetime discrimination measures with residual lifetime and thus a separate study of these discrimination measures for past lifetime is quite worthwhile. Accordingly, in the rest of the chapter, we obtain bounds for these discrimination measures for the past lifetimes. The following theorem provides a lower (upper) bound for $\bar{I}_{X,Y}(\alpha, t)$ using likelihood ordering.

Theorem 2.2.4. *If* $X \leq_{LR} Y$, *then*

$$\bar{I}_{X,Y}(\alpha, t) \geq (\leq) \frac{\alpha}{\alpha - 1} \log \left[\frac{\lambda_X(t)}{\lambda_Y(t)} \right]$$

if $\alpha > 1$ *(*$0 < \alpha < 1$*).*

Proof. The proof is similar to that of Theorem 2.2.3. □

Now we extend the above theorem to weighted models.

Corollory 2.2.3. *If* $X \leq_{LR} X^w$, *then*

$$\bar{I}_{X,X^w}(\alpha, t) \geq (\leq) \frac{\alpha}{\alpha - 1} \log \left[\frac{E(w(X)|X \leq t)}{w(t)} \right]$$

if $\alpha > 1$ *(*$0 < \alpha < 1$*).*

Example 2.2.6. *Suppose* X *is a finite range rv with pdf* $f(x) = cx^{c-1}$, $0 < x < 1$, $c > 0$, *and taking* $w(x) = x^\beta$ *(*$\beta > 0$*) we have* $\frac{f(x)}{f^w(x)} = \frac{c}{\beta+c} x^{-\beta}$ *is decreasing in* x *(i.e.,* $X \leq_{LR} X^w$*). It is easy to show that*

$$\begin{aligned}
\bar{I}_{X,X^w}(\alpha, t) &= \frac{\alpha}{\alpha - 1} \log \left(\frac{c}{\beta + c} \right) + \frac{1}{\alpha - 1} \log \left(\frac{\beta + c}{\beta + c - \beta\alpha} \right), \\
&\geq (\leq) \frac{\alpha}{\alpha - 1} \log \left(\frac{c}{\beta + c} \right), \\
&= \frac{\alpha}{\alpha - 1} \log \left[\frac{E(w(X)|X \leq t)}{w(t)} \right],
\end{aligned}$$

where $E(w(X)|X \leq t) = \frac{1}{F(t)} \int_0^t w(x) f(x) dx = \frac{1}{t^c} \int_0^t x^\beta c x^{c-1} dx = \frac{c}{\beta+c} t^\beta$, *according as* $\alpha > 1$ *(*$0 < \alpha < 1$*), provided* $\beta + c > \beta\alpha$. □

In the study of relative entropies, it is quite useful if we find some close relationships between its different measures and other important reliability/information measures.

Therefore, in the following theorem we derive a lower bound for $I_{X,Y}(\alpha, t)$, which is a function of both hazard rate and residual Renyi's entropy function.

Theorem 2.2.5. *If $g(x)$ is decreasing in x, then*

$$I_{X,Y}(\alpha, t) \geq -\log h_Y(t) - I_X(\alpha, t), \alpha \neq 1,$$

where $I_X(\alpha, t) = \frac{1}{1-\alpha} \log \int_t^\infty \frac{f^\alpha(x)}{\bar{F}^\alpha(t)} dx$, the residual Renyi's entropy function.

Proof. Since $g(x)$ is decreasing in x, $g(x) \leq g(t)$ $\forall x > t$. Therefore,

$$\begin{aligned} I_{X,Y}(\alpha, t) &= \frac{1}{\alpha - 1} \log \int_t^\infty \frac{f^\alpha(x) g^{1-\alpha}(x)}{\bar{F}^\alpha(t) \bar{G}^{1-\alpha}(t)} dx, \\ &\geq \frac{1}{\alpha - 1} \log \int_t^\infty \frac{f^\alpha(x)}{\bar{F}^\alpha(t)} dx + \frac{1}{\alpha - 1} \log \frac{g^{(1-\alpha)}(t)}{\bar{G}^{(1-\alpha)}(t)}, \quad \text{for } \alpha \neq 1, \\ &= -I_X(\alpha, t) - \log h_Y(t). \end{aligned}$$

□

Corollory 2.2.4. *If $f^w(x)$ is decreasing in x, then*

$$I_{X,X^w}(\alpha, t) \geq -\log \left(\frac{w(t) h_X(t)}{E(w(X)|X > t)} \right) - I_X(\alpha, t), \quad \alpha \neq 1.$$

Example 2.2.7. *Applying the same pdf and weight function used in Example 2.2.6, we can easily illustrate Corollary 2.2.4.* □

The analogous results are straightforward for the past lifetimes, the statements are as follows:

Theorem 2.2.6. *If $g(x)$ is increasing in x, then*

$$\bar{I}_{X,Y}(\alpha, t) \geq -\log \lambda_Y(t) - \bar{I}_X(\alpha, t), \alpha \neq 1, \alpha > 0,$$

where $\bar{I}_X(\alpha, t) = \frac{1}{1-\alpha} \log \int_0^t \frac{f^\alpha(x)}{F^\alpha(t)} dx$ is the Renyi's past entropy function.

Corollory 2.2.5. *If $f^w(x)$ is increasing in x, then*

$$\bar{I}_{X,X^w}(\alpha,t) \geq -\log\left(\frac{w(t)\lambda_X(t)}{E(w(X)|X\leq t)}\right) - \bar{I}_X(\alpha,t), \quad \alpha\neq 1,\ \alpha>0.$$

Example 2.2.8. *It is easy to show that for the Power function rv with pdf $f(x)=cx^{c-1}$, $0<x<1$, $c>1$ and taking $w(x)=x$, we have $f^w(x)=(c+1)x^c$ increasing in x and hence Corollary 2.2.5 follows.* □

In the following theorems, we establish an upper (lower) bound for $I_{X,Y}(\alpha,t)$, when there are more than two rvs, taken two at a time.

Theorem 2.2.7. *Let X_1, X_2 and Y be 3 non-negative absolutely continuous rvs with densities f_1, f_2 and g, sfs $\bar{F}_1$, $\bar{F}_2$ and $\bar{G}$ and hazard rates h_{X_1}, h_{X_2} and h_Y respectively. If $X_1 \leq_{LR} X_2$, then*

$$I_{X_1,Y}(\alpha,t) \leq (\geq) \frac{\alpha}{\alpha-1}\log\left[\frac{h_{X_1}(t)}{h_{X_2}(t)}\right] + I_{X_2,Y}(\alpha,t)$$

if $\alpha>1$ ($0<\alpha<1$).

Proof. Since $X_1 \leq_{LR} X_2$, $\frac{f_1(x)}{f_2(x)}$ is decreasing in x, so $\frac{f_1^\alpha(x)}{f_2^\alpha(x)}$ is decreasing in x for $\alpha>0$

$$\begin{aligned}
I_{X_1,Y}(\alpha,t) &= \frac{1}{\alpha-1}\log\int_t^\infty \frac{f_1^\alpha(x)g^{(1-\alpha)}(x)}{\bar{F}_1^\alpha(t)\bar{G}^{(1-\alpha)}(t)}dx,\\
&= \frac{1}{\alpha-1}\log\int_t^\infty \frac{f_1^\alpha(x)}{f_2^\alpha(x)}\frac{f_2^\alpha(x)}{\bar{F}_2^\alpha(t)}\frac{\bar{F}_2^\alpha(t)}{\bar{F}_1^\alpha(t)}\frac{g^{(1-\alpha)}(x)}{\bar{G}^{(1-\alpha)}(t)}dx,
\end{aligned}$$

provided $\alpha\neq 1$.

For $\alpha>1$,

$$I_{X_1,Y}(\alpha,t) \leq \frac{1}{\alpha-1}\left[\log\left[\frac{h_{X_1}(t)}{h_{X_2}(t)}\right]^\alpha + \log\int_t^\infty \frac{f_2^\alpha(x)g^{(1-\alpha)}(x)}{\bar{F}_2^\alpha(t)\bar{G}^{(1-\alpha)}(t)}dx\right],$$

$$= \frac{\alpha}{\alpha-1}\log\left[\frac{h_{X_1}(t)}{h_{X_2}(t)}\right] + I_{X_2,Y}(\alpha,t).$$

For $0 < \alpha < 1$,

$$\begin{aligned} I_{X_1,Y}(\alpha,t) &\geq \frac{1}{\alpha-1}\left[\log\left[\frac{h_{X_1}(t)}{h_{X_2}(t)}\right]^{\alpha} + \log\int_t^{\infty}\frac{f_2^{\alpha}(x)g^{(1-\alpha)}(x)}{\bar{F}_2^{\alpha}(t)\bar{G}^{(1-\alpha)}(t)}dx\right], \\ &= \frac{\alpha}{\alpha-1}\log\left[\frac{h_{X_1}(t)}{h_{X_2}(t)}\right] + I_{X_2,Y}(\alpha,t). \end{aligned}$$

□

Example 2.2.9. *Let X_1 and X_2 be two independent exponential rvs with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$ respectively such that $\lambda_1 > \lambda_2$. Then*

$$\frac{f_1(x)}{f_2(x)} = \frac{\lambda_1}{\lambda_2}\exp[-(\lambda_1-\lambda_2)x]$$

is decreasing in x. Let $Y = \min(X_1, X_2)$, then

$$\begin{aligned} I_{X_1,Y}(\alpha,t) &= \frac{1}{\alpha-1}\log\left[\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{\alpha}\left(\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-\lambda_2\alpha}\right)\right], \\ &= \frac{\alpha}{\alpha-1}\log\left(\frac{\lambda_1}{\lambda_2}\right) + \frac{1}{\alpha-1}\log\left[\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{\alpha}\left(\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-\lambda_1\alpha}\right)\right] \\ &\quad + \frac{1}{\alpha-1}\log\left(\frac{\lambda_1+\lambda_2-\lambda_1\alpha}{\lambda_1+\lambda_2-\lambda_2\alpha}\right), \\ &\leq (\geq)\frac{\alpha}{\alpha-1}\log\left(\frac{\lambda_1}{\lambda_2}\right) + \frac{1}{\alpha-1}\log\left[\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{\alpha}\left(\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-\lambda_1\alpha}\right)\right], \\ &= \frac{\alpha}{\alpha-1}\log\left[\frac{h_{X_1}(t)}{h_{X_2}(t)}\right] + I_{X_2,Y}(\alpha,t), \end{aligned}$$

according as $\alpha > 1$ $(0 < \alpha < 1)$, provided $\lambda_i + \lambda_j - \lambda_i\alpha > 0$, $i \neq j$, $i, j = 1, 2$. □

Theorem 2.2.8. *Let X_1, X_2 and Y be 3 non-negative absolutely continuous rvs with densities f_1, f_2 and g, dfs F_1, F_2 and G and reversed hazard rates λ_{X_1}, λ_{X_2} and λ_Y respectively. If $X_1 \leq_{LR} X_2$, then*

$$\bar{I}_{X_1,Y}(\alpha,t) \geq (\leq) \frac{\alpha}{\alpha-1}\log\left[\frac{\lambda_{X_1}(t)}{\lambda_{X_2}(t)}\right] + \bar{I}_{X_2,Y}(\alpha,t)$$

if $\alpha > 1$ *(*$0 < \alpha < 1$*).*

Example 2.2.10. *Let* X_1 *and* X_2 *be two independent Power function rvs with densities given by* $f_1(x) = c_1 x^{c_1-1}$*;* $0 < x < 1$*,* $c_1 > 0$ *and* $f_2(x) = c_2 x^{c_2-1}$*;* $0 < x < 1$*,* $c_2 > 0$ *respectively such that* $c_1 < c_2$*, so*

$$\frac{f_1(x)}{f_2(x)} = \frac{c_1}{c_2} x^{c_1-c_2}$$

is decreasing in x*. Letting* $Y = \max(X_1, X_2)$*, it is easy to show that Theorem 2.2.8 follows.* □

Theorem 2.2.9. *Let* X, Y_1 *and* Y_2 *be 3 non-negative absolutely continuous rvs with pdfs* f*,* g_1 *and* g_2*, sfs* $\bar{F}$*,* $\bar{G}_1$ *and* $\bar{G}_2$ *and hazard rates* h_X*,* h_{Y_1} *and* h_{Y_2} *respectively. If* $Y_1 \leq_{LR} Y_2$*, then*

$$I_{X,Y_1}(\alpha,t) \geq \log\left[\frac{h_{Y_1}(t)}{h_{Y_2}(t)}\right] + I_{X,Y_2}(\alpha,t) \text{ for } \alpha \neq 1, \alpha > 0.$$

Example 2.2.11. *Let* Y_1 *and* Y_2 *be two independent Pareto I rvs with densities given by* $g_1(x) = c_1 k_1^{c_1} x^{-c_1-1}$*;* $x > k_1 > 0$*,* $c_1 > 0$ *and* $g_2(x) = c_2 k_2^{c_2} x^{-c_2-1}$*;* $x > k_2 > 0$*,* $c_2 > 0$ *respectively such that* $c_1 > c_2$*. Then*

$$\frac{g_1(x)}{g_2(x)} = \frac{c_1 k_1^{c_1}}{c_2 k_2^{c_2}} x^{-(c_1-c_2)}$$

is decreasing in x*. Consider* $X = \min(Y_1, Y_2)$*, then Theorem 2.2.9 follows.* □

Theorem 2.2.10. *Let* X*,* Y_1 *and* Y_2 *be 3 non-negative absolutely continuous rvs with densities* f, g_1 *and* g_2*, dfs* F, G_1 *and* G_2 *and reversed hazard rates* λ_X*,* λ_{Y_1} *and* λ_{Y_2}

respectively. If $Y_1 \leq_{LR} Y_2$*, then*

$$\bar{I}_{X,Y_1}(\alpha, t) \leq \log\left[\frac{\lambda_{Y_2}(t)}{\lambda_{Y_1}(t)}\right] + \bar{I}_{X,Y_2}(\alpha, t)$$

for $\alpha \neq 1$, $\alpha > 0$.

Example 2.2.12. *Let* Y_1 *and* Y_2 *be two independent Power function rvs with densities given by* $g_1(x) = c_1 x^{c_1-1}$; $0 < x < 1$, $c_1 > 0$ *and* $g_2(x) = c_2 x^{c_2-1}$; $0 < x < 1$, $c_2 > 0$ *respectively such that* $c_1 < c_2$*, so*

$$\frac{g_1(x)}{g_2(x)} = \frac{c_1}{c_2} x^{c_1-c_2}$$

is decreasing in x*. Using* $X = \max(Y_1, Y_2)$*, we can illustrate the theorem.* □

2.3 Dynamic Kerridge's inaccuracy measure

As explained in Chapter 1, the inaccuracy measure due to Kerridge (1961) is a useful tool to measure the inaccuracy between two distributions f and g. It can also be expressed as

$$K_{X,Y} = I_X + I_{X,Y}, \text{ where } I_{X,Y} = \int_0^\infty f(x)\log(f(x)/g(x))dx$$

is the Kullback-Leibler (KL) divergence between X and Y and $I_X = -\int\limits_0^\infty f(x)\log f(x)dx$, Shannon measure of information of X. Taneja et al. (2009) introduced a dynamic version of Kerridge's measure, given in (1.12) where $K_{X,Y}(t) = K_{X_t,Y_t}$. Clearly, when $X = Y$, equation (1.12) becomes the dynamic measure of uncertainty (residual entropy) due to Ebrahimi (1996). A similar expression for the inactivity times is available in Kumar et al.(2011) and given by (1.14). If we consider the rvs given in Example 2.2.1, we could see that $K_{X,Y_\beta}(t) \neq K_{X,Y_{-\beta}}(t)$ and $\bar{K}_{X,Y_\beta}(t) \neq \bar{K}_{X,Y_{-\beta}}(t)$ for some β. The figures

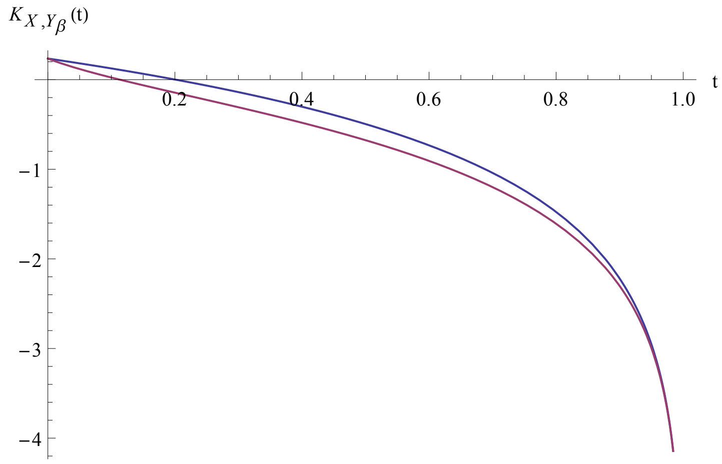

Figure 2.5:

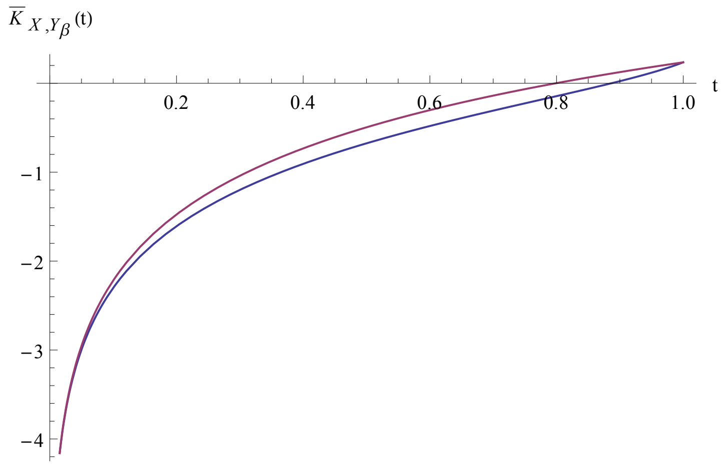

Figure 2.6:

2.5 and 2.6 illustrate it. Figure 2.5 represents $K_{X,Y_\beta}(t)$ for $\beta = 1.9$ (lower curve) and $\beta = -1.9$ (upper curve), and figure 2.6 represents $\bar{K}_{X,Y_\beta}(t)$ for $\beta = 1.9$ (upper curve) and $\beta = -1.9$ (lower curve).

In the rest of the section we obtain bounds similar to $I_{X,Y}(\alpha, t)$ and $\bar{I}_{X,Y}(\alpha, t)$ that is given in Section 2.2 for the Kerridge's inaccuracy measures. Let X and Y be the rvs defined in Section 2.2. Then the dynamic inaccuracy measure for residual and past lives of the original and weighted distributions are given by

$$K_{X,X^w}(t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log\left[\frac{w(x)f(x)}{E(w(X)|X>t)\bar{F}(t)}\right] dx, \tag{2.8}$$

and

$$\bar{K}_{X,X^w}(t) = -\int_0^t \frac{f(x)}{F(t)} \log\left[\frac{w(x)f(x)}{E(w(X)|X\leq t)F(t)}\right] dx. \tag{2.9}$$

Remark 2.3.1. *From the above definition, it is easy to obtain,* $K_{X,X^E}(t) = 1 + \log r(t)$.

The following theorem gives a simple lower bound for Kerridge's inaccuracy measures using likelihood ordering.

Theorem 2.3.1. *If* $g(x)$ *is decreasing in* x*, then* $K_{X,Y}(t) \geq -\log h_Y(t)$.

Proof. Since $g(x)$ is decreasing in x, we have $g(x) \leq g(t)$ for all $x > t$. Then,

$$\begin{aligned} K_{X,Y}(t) = -\frac{1}{\bar{F}(t)}\int_t^\infty f(x)\log\frac{g(x)}{\bar{G}(t)}dx &\geq -\frac{1}{\bar{F}(t)}\int_t^\infty f(x)\log\frac{g(t)}{\bar{G}(t)}dx, \\ &= -\log h_Y(t). \end{aligned}$$

□

Corollary 2.3.1. *If* $f^w(x)$ *is decreasing in* x*, then*

$$K_{X,X^w}(t) \geq \log\left(\frac{E(w(X)|X>t)}{w(t)h_X(t)}\right).$$

Analogous results are obtained for past lifetimes in the following theorems.

Theorem 2.3.2. *If* $g(x)$ *is increasing in* x*, then* $\bar{K}_{X,Y}(t) \geq -\log \lambda_Y(t)$.

Corollary 2.3.2. *If* $f^w(x)$ *is increasing in* x*, then*

$$\bar{K}_{X,X^w}(t) \geq \log\left(\frac{E(w(X)|X\leq t)}{w(t)h_X(t)}\right).$$

Example 2.3.1. *Suppose* X *is a Uniform rv with pdf* $f(x) = \frac{1}{a}$; $0 < x < a$, $a > 0$.

Taking the weight function $w(x) = x$, $f^w(x) = \frac{2x}{a^2}$ *is increasing in* x. *Then,*

$$\bar{K}_{X,Y}(t) = 1 + \log(t/2) \geq \log(t/2) = \log\left(\frac{E(w(X)|X \leq t)}{w(t)h_X(t)}\right). \qquad \square$$

In the following theorem we have a simple bound for Kerridge's inaccuracy measure between X and X^w which are functions of hazard rates of the same rvs and residual entropy of X.

Theorem 2.3.3. *If the weight function* $w(x)$ *is increasing in* x, *then*

$$K_{X,X^w}(t) \leq \log\left(\frac{E(w(X)|X > t)}{w(t)}\right) + I_X(t),$$

where $I_X(t) = -\frac{1}{\bar{F}(t)}\int_t^\infty f(x)\log\frac{f(x)}{\bar{F}(t)}dx$ *is the residual entropy function.*

Proof. Since $w(x)$ is increasing in x, we have $w(x) \geq w(t)$ for all $x > t$. Now using equation (2.8) we have

$$\begin{aligned} K_{X,X^w}(t) &\leq -\frac{1}{\bar{F}(t)}\int_t^\infty f(x)\log\left[\frac{w(t)f(x)}{E(w(X)|X > t)\bar{F}(t)}\right]dx, \\ &= \log\left(\frac{E(w(X)|X > t)}{w(t)}\right) + I_X(t). \end{aligned} \qquad \square$$

Example 2.3.2. *Let* X *be a Pareto I rv with pdf* $f(x) = ck^c x^{-c-1}$; $c > 1$, $x > k > 0$. *Take the weight function as* $w(x) = x$, *which is an increasing function in* x. *Then*

$$\begin{aligned} K_{X,X^w}(t) = \log\left(\frac{t}{c-1}\right) + 1 &= \log\left(\frac{c}{c-1}\right) + \log\left(\frac{t}{c}\right) + \left(\frac{c+1}{c}\right) - \frac{1}{c}, \\ &\leq \log\left(\frac{c}{c-1}\right) + \log\left(\frac{t}{c}\right) + \left(\frac{c+1}{c}\right), \\ &= \log\left(\frac{E(w(X)|X > t)}{w(t)}\right) + I_X(t). \end{aligned} \qquad \square$$

The following theorem is an analogous result of Theorem 2.3.3 for past lifetime.

Theorem 2.3.4. *If the weight function $w(x)$ is decreasing in x, then*

$$\bar{K}_{X,X^w}(t) \leq \log\left(\frac{E(w(X)|X \leq t)}{w(t)}\right) + \bar{I}_X(t),$$

where $\bar{I}_X(t) = -\frac{1}{F(t)}\int_0^t f(x)\log\frac{f(x)}{F(t)}dx$ is the past entropy function.

Example 2.3.3. *Consider a finite range rv X with density function given by $f(x) = cx^{c-1}$; $0 < x < 1$, $c > 1$. Let $w(x) = \frac{1}{x}$, a decreasing function in x, then using equation* (2.9) *we have*

$$\begin{aligned}\bar{K}_{X,X^w}(t) = \log\left(\frac{t}{c-1}\right) + \left(\frac{c-2}{c}\right) &= \log\left(\frac{c}{c-1}\right) + \log\left(\frac{t}{c}\right) + \left(\frac{c-1}{c}\right) - \frac{1}{c},\\ \leq \log\left(\frac{c}{c-1}\right) + \log\left(\frac{t}{c}\right) + \left(\frac{c-1}{c}\right) &= \log\left(\frac{E(w(X)|X \leq t)}{w(t)}\right) + \bar{I}_X(t). \quad \square\end{aligned}$$

Theorem 2.3.5. *If $X \leq_{LR} Y$, then*

$$K_{X,Y}(t) \leq I_X(t) + \log\left(\frac{h_X(t)}{h_Y(t)}\right).$$

Proof. From the definition (1.12), we have

$$\begin{aligned}K_{X,Y}(t) &= -\int_t^\infty \frac{f(x)}{\bar{F}(t)}\log\left(\frac{f(x)}{\bar{F}(t)}\frac{g(x)}{f(x)}\frac{\bar{F}(t)}{\bar{G}(t)}\right)dx,\\ &= I_X(t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)}\log\left(\frac{g(x)}{f(x)}\frac{\bar{F}(t)}{\bar{G}(t)}\right)dx,\\ &\leq I_X(t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)}\log\left(\frac{g(t)}{f(t)}\frac{\bar{F}(t)}{\bar{G}(t)}\right)dx,\\ &= I_X(t) - \log\left(\frac{g(t)}{f(t)}\frac{\bar{F}(t)}{\bar{G}(t)}\right) = I_X(t) + \log\left(\frac{h_X(t)}{h_Y(t)}\right),\end{aligned}$$

where the inequality is obtained by using the fact that $g(x)/f(x)$ is increasing. $\square$

A similar statement exists for the past lifetime.

Theorem 2.3.6. *If $X \leq_{LR} Y$, then*

$$\bar{K}_{X,Y}(t) \geq \bar{I}_X(t) + \log\left(\frac{\lambda_X(t)}{\lambda_Y(t)}\right).$$

Similar to Theorems 2.2.7 to 2.2.10, in the following theorems we obtain some bounds for Kerridge's inaccuracy, when there are more than two rvs, taken two at a time.

Theorem 2.3.7. *Let X, Y_1 and Y_2 be 3 non-negative absolutely continuous rvs with pdfs f, g_1 and g_2, sfs $\bar{F}$, $\bar{G}_1$ and $\bar{G}_2$ and hazard rates h_X, h_{Y_1} and h_{Y_2} respectively. If $Y_1 \leq_{LR} Y_2$, then*

$$K_{X,Y_1}(t) \geq K_{X,Y_2}(t) + \log\left[\frac{h_{Y_2}(t)}{h_{Y_1}(t)}\right].$$

Proof. From the definition (1.12), we have

$$\begin{aligned}
K_{X,Y_1}(t) &= -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log\left(\frac{g_2(x)}{\bar{G}_2(t)}\frac{g_1(x)}{g_2(x)}\frac{\bar{G}_2(t)}{\bar{G}_1(t)}\right) dx,\\
&= K_{X,Y_2}(t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log\left(\frac{g_1(x)}{g_2(x)}\frac{\bar{G}_2(t)}{\bar{G}_1(t)}\right) dx,\\
&\geq K_{X,Y_2}(t) - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log\left(\frac{g_1(t)}{g_2(t)}\frac{\bar{G}_2(t)}{\bar{G}_1(t)}\right) dx,\\
&= K_{X,Y_2}(t) + \log\left[\frac{h_{Y_2}(t)}{h_{Y_1}(t)}\right].
\end{aligned}$$

□

Example 2.3.4. *Let Y_1 and Y_2 be two independent Pareto II rvs with pdfs $g_1(x) = ac_1(1+ax)^{-c_1-1}$; $x > 0$, $a, c_1 > 0$ and $g_2(x) = ac_2(1+ax)^{-c_2-1}$; $x > 0$, $a, c_2 > 0$ such that $c_1 > c_2$, then*

$$\frac{g_1(x)}{g_2(x)} = \frac{c_1}{c_2}(1+ax)^{-(c_1-c_2)}$$

is decreasing in x. Let $X = \min(Y_1, Y_2)$, then

$$K_{X,Y_1}(t) = \log\left(\frac{1+at}{ac_1}\right) + \left(\frac{c_1+1}{c_1+c_2}\right),$$

$$= \log\left(\frac{c_2}{c_1}\right) + \log\left(\frac{1+at}{ac_2}\right) + \left(\frac{c_2+1}{c_1+c_2}\right) + \left(\frac{c_1-c_2}{c_1+c_2}\right),$$
$$\geq \log\left(\frac{c_2}{c_1}\right) + \log\left(\frac{1+at}{ac_2}\right) + \left(\frac{c_2+1}{c_1+c_2}\right) = K_{X,Y_2}(t) + \log\left[\frac{h_{Y_2}(t)}{h_{Y_1}(t)}\right]. \qquad \square$$

Next we obtain an analogous result for the past lifetime.

Theorem 2.3.8. *Let X, Y_1 and Y_2 be 3 non-negative rvs with pdfs f, g_1 and g_2, dfs F, G_1 and G_2 and reversed hazard rates λ_X, λ_{Y_1} and λ_{Y_2} respectively. If $Y_1 \leq_{LR} Y_2$ then*

$$\bar{K}_{X,Y_1}(t) \leq \bar{K}_{X,Y_2}(t) + \log\left[\frac{\lambda_{Y_2}(t)}{\lambda_{Y_1}(t)}\right].$$

Example 2.3.5. *Let Y_1 and Y_2 be 2 independent finite range rvs with pdfs given by $g_1(x) = c_1 x^{c_1-1}$; $0 < x < 1$, $c_1 > 0$ and $g_2(x) = c_2 x^{c_2-1}$; $0 < x < 1$, $c_2 > 0$ such that $c_1 < c_2$, then*

$$\frac{g_1(x)}{g_2(x)} = \frac{c_1}{c_2} x^{c_1-c_2}$$

is decreasing in x. Let $X = \max(Y_1, Y_2)$, we get

$$\bar{K}_{X,Y_1}(t) = \log\left(\frac{t}{c_1}\right) + \left(\frac{c_1-1}{c_1+c_2}\right) = \log\left(\frac{c_2}{c_1}\right) + \log\left(\frac{t}{c_2}\right) + \left(\frac{c_2-1}{c_1+c_2}\right) + \left(\frac{c_1-c_2}{c_1+c_2}\right),$$
$$\leq \log\left(\frac{c_2}{c_1}\right) + \log\left(\frac{t}{c_2}\right) + \left(\frac{c_2-1}{c_1+c_2}\right) = \bar{K}_{X,Y_2}(t) + \log\left[\frac{\lambda_{Y_2}(t)}{\lambda_{Y_1}(t)}\right]. \qquad \square$$

Chapter 3

Cumulative measure of entropy for conditionally specified and conditional survival models

3.1 Introduction

Although Shannon's entropy has been widely applied in many areas of research, Rao et al. (2004) identified some limitations of it in measuring the randomness of certain systems (Rao (2005)) and introduced an alternate measure of uncertainty called cumulative residual entropy (CRE). CRE is obtained by replacing the pdf with the cdf in the Shannon entropy (1.4). CRE is more useful than Shannon entropy in certain systems as it uses the cdf, which always exists and hence more regular than the pdf. The definition of CRE in the univariate case and for the non-negative rvs is as follows:

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$$\xi = -\int_0^\infty \bar{F}(x)\log \bar{F}(x)dx, \tag{3.1}$$

Clearly, ξ measures the uncertainty contained in the sf of X. For used items, (3.1) is not adequate for measuring uncertainty, hence dynamic versions of it are important. Based on this idea, Asadi and Zohrevand (2007) extended (3.1) to the residual time, called dynamic cumulative residual entropy (DCRE), given by

$$\xi(t) = -\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)}\log \frac{\bar{F}(x)}{\bar{F}(t)}dx. \tag{3.2}$$

After the unit has elapsed time t, $\xi(t)$ measures the uncertainty or randomness contained in the conditional sf of $X-t$ given $X>t$ about the predictability of remaining lifetime of the unit.

Analogous to CRE, Di Crescenzo and Longobardi (2009) introduced cumulative entropy useful in measuring the inactivity of a system. It is a dual concept of the CRE and is suitable to measure the uncertainty on past lifetimes of the system. It is defined as

$$C\xi = -\int_0^\infty F(x)\log F(x)dx.$$

In analogy with (3.2), Di Crescenzo and Longobardi (2009) proposed a dynamic cumulative entropy for the past lifetime, defined by

$$C\xi(t) = -\int_0^t \frac{F(x)}{F(t)}\log \frac{F(x)}{F(t)}dx.$$

It measures the uncertainty related to past lifetime, *i.e.*, it explains the uncertainty related to the rv $(t-X|X\leq t)$. For more recent works and details on cumulative residual entropy and cumulative entropy we refer to Di Crescenzo and Longobardi (2009), Abbasnejad et al. (2010), Navarro et al. (2010), Baratpour and Khodadadi (2012), Thapliyal et al. (2013) and references therein.

The study of reliability properties of conditionally specified models is quite recent. Arnold (1995, 1996) and Arnold and Kim (1996) have studied several classes of conditional survival models. The identification of the joint distribution of (X_1, X_2), when conditional distributions of $(X_1|X_2 = t_2)$ and $(X_2|X_1 = t_1)$ are known, has been an important problem studied by many researchers in the past. This approach of identifying a bivariate density using the conditionals is called the conditional specification of the joint distribution (see Arnold et al. (1999)). These conditional models are often useful in many two-component reliability systems where the operational status of one of the components is known in advance. Another important problem closely associated to this is the identification of the joint distribution of (X_1, X_2) when the conditional distribution or corresponding reliability measures of the rvs $(X_1|X_2 > t_2)$ and $(X_2|X_1 > t_1)$ are known. That is, instead of conditioning on a component failing (down) at a specified time, we study the system when the survival time of one of the components is known (see Arnold (1987)). For a recent study of these models, we refer to Sunoj and Sankaran (2005), Navarro and Sarabia (2013) and Navarro et al. (2011) and the references therein.

Although variety of research is available for cumulative entropies, a study of the same for conditionally specified models does not appear to have been taken up. In this chapter, we study two classes of cumulative measure of uncertainty based on conditioning two types of events *viz.* $\{X_1 = t_1\}$ and $\{X_2 = t_2\}$ and $\{X_1 > t_1\}$ and $\{X_2 > t_2\}$ respectively. The important characterization properties of these measures are studied. We also define cumulative entropies of rvs $(X_1|X_2 \leq t_2\}$ and $(X_2|X_1 \leq t_1\}$ and discuss some characterizations using these measures. Finally, an application of the univariate dynamic cumulative residual entropy is studied using the maximum entropy principle.

3.2 Cumulative measure of uncertainty for conditionally specified models

In the first case we consider events of the form $\{X_1 = t_1\}$ and $\{X_2 = t_2\}$. Let (X_1, X_2) be a bivariate rv with support $S = (0, \infty) \times (0, \infty)$. Suppose $f_i(t_i|t_j)$ be the conditional pdf of $(X_i|X_j = t_j)$. Then a direct extension of (3.1) to this conditional rv is given by

$$\xi_i(t_j) = -\int_0^{\infty} \bar{F}_i(x_i|t_j) \log \bar{F}_i(x_i|t_j) dx_i,$$

where $\bar{F}_i(t_i|t_j) = \int_{t_i}^{\infty} f_i(x_i|t_j) dx_i$ is the conditional sf of $(X_i|X_j = t_j)$ for $i, j = 1, 2$, $i \neq j$, with $\bar{F}_i(x_i|t_j) > 0$. An equivalent dynamic version of (3.2) for the conditional rv $(X_i|X_j = t_j)$ called as cumulative measure of uncertainty of type 1 (CMU_1) is defined as

$$\xi_i(t_1, t_2) = -\int_{t_i}^{\infty} \frac{\bar{F}_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log \frac{\bar{F}_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} dx_i. \tag{3.3}$$

The identity (3.3) measures the uncertainty contained in the conditional distributions of $(X_i - t_i|X_i > t_i, X_j = t_j)$ for $i, j = 1, 2$, $i \neq j$. Equation (3.3) is equivalent to

$$\xi_i(t_1, t_2) = -\frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} \bar{F}_i(x_i|t_j) \log \bar{F}_i(x_i|t_j) dx_i + \frac{\log \bar{F}_i(t_i|t_j)}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} \bar{F}_i(x_i|t_j) dx_i. \tag{3.4}$$

Let $r_i(t_i|t_j)$ denote conditional mean residual life function of $(X_i|X_j = t_j)$ and it be defined as

$$r_i(t_i|t_j) = E(X_i - t_i|X_i > t_i, X_j = t_j) = \frac{-1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} (x_i - t_i) d\bar{F}_i(x_i|t_j) dx_i,$$

$$= \frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} \bar{F}_i(x_i|t_j)dx_i,$$

for $i, j = 1, 2,\ i \neq j$. Equation (3.4) thus becomes

$$\xi_i(t_1, t_2) = -\frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} \bar{F}_i(x_i|t_j) \log \bar{F}_i(x_i|t_j)dx_i + r_i(t_i|t_j) \log \bar{F}_i(t_i|t_j). \tag{3.5}$$

Recently, Navarro and Sarabia (2013) studied the reliability properties in two classes of bivariate continuous distributions based on the specification of conditional hazard functions. These classes were constructed by conditioning on two types of events *viz.* $\{X_1 = t_1\}$ and $\{X_2 = t_2\}$ and $\{X_1 > t_1\}$ and $\{X_2 > t_2\}$ respectively (see also Arnold and Kim (1996)). In survival studies the most widely used semi-parametric regression model is the proportional hazard rate (PHR) model. The univariate Cox PHR model is a class of modelling distributions with pdf and sf given by

$$f(t; \alpha) = \alpha\lambda^o(t) \exp\{-\alpha\Lambda^o(t)\},\ t \geq 0, \tag{3.6}$$

and $\bar{F}(t; \alpha) = \exp\{-\alpha\Lambda^o(t)\}$, $t \geq 0$ where $\alpha > 0$, $\lambda^o(t)$ is the baseline hazard rate function and $\Lambda^o(t) = \int_0^t \lambda^o(x)dx$ is the baseline cumulative hazard function, where both $\lambda^o(t)$ and $\Lambda^o(t)$ might involve parameter θ, besides the parameter α. The hazard (or failure) rate function of $f(t; \alpha)$ is $h(t; \alpha) = f(t; \alpha)/\bar{F}(t; \alpha) = \alpha\lambda^o(t)$. A rv with the pdf (3.6) can be denoted by $X \sim PHR\,(\alpha; \Lambda^o(t))$. Special cases of $f(t; \alpha)$ and $\bar{F}(t; \alpha)$ include Exponential ($\Lambda^o(t) = t$), Burr ($\Lambda^o(t) = \log \frac{\beta + t^\gamma}{\beta}$), Pareto ($\Lambda^o(t) = \log \frac{\beta + t}{\beta}$) and Weibull ($\Lambda^o(t) = t^\gamma$). Navarro and Sarabia (2013) obtained a general form of a bivariate pdf with conditional distributions satisfying $(X_i|X_j = t_j) \sim PHR\,(\alpha_i(t_j); \Lambda_i^o(t_i))$ for

$i, j = 1, 2,\ i \neq j$ and is given by

$$f(t_1, t_2) = c(\phi)a_1a_2\lambda_1^o(t_1)\lambda_2^o(t_2)\exp[-a_1\Lambda_1^o(t_1) - a_2\Lambda_2^o(t_2) - \phi a_1a_2\Lambda_1^o(t_1)\Lambda_2^o(t_2)], \quad (3.7)$$

for $t_1, t_2 \geq 0$, $a_1, a_2 > 0$ and $\phi \geq 0$. Letting $(X_i|X_j = t_j) \sim PHR(a_i(1 + \phi a_j t_j); t_i)$ and using the conditional densities of (3.7) with sfs $\bar{F}_i(t_i|t_j) = \exp\{-a_i[1 + \phi a_j t_j]t_i\}$, $t_i > 0$ then CMU$_1$ for $(X_i|X_j = t_j)$, using (3.4) is obtained as

$$\xi_i(t_i|t_j) = \frac{1}{a_i(1 + \phi a_j t_j)}, \quad (3.8)$$

Suppose $h_i(t_i|t_j)$ denotes conditional hazard (failure) rate function of $(X_i|X_j = t_j)$ and is defined by

$$h_i(t_i|t_j) = -\frac{\partial}{\partial t_i}\log \bar{F}_i(t_i|t_j) = \frac{f_i(t_i|t_j)}{\bar{F}_i(t_i|t_j)}.$$

Now we have the following characterization theorem.

Theorem 3.2.1. *For bivariate rvs* (X_1, X_2)*,* $\xi_i(t_1, t_2)$ *is independent of* t_i *if and only if* (X_1, X_2) *follows the joint pdf* (3.7) *with* $\Lambda_i^o(t_i) = t_i$*,* $\alpha_i(t_j) = a_i(1 + \phi a_j t_j)$*,* $i, j = 1, 2$*,* $i \neq j$*.*

Proof. If (X_1, X_2) follows the joint pdf (3.7) with $\Lambda_i(t_i) = t_i$, $\alpha_i(t_j) = a_i(1 + \phi a_j t_j)$, $i, j = 1, 2$, $i \neq j$, a direct computation yields (3.8). This proves the 'only if' part.

To prove the 'if' part, assume that $\xi_i(t_1, t_2)$ is independent of t_i. Then we can write

$$\frac{-1}{\bar{F}_i(t_i|t_j)}\int_{t_i}^{\infty}\bar{F}_i(x_i|t_j)\log \bar{F}_i(x_i|t_j)dx_i + \frac{\log \bar{F}_i(t_i|t_j)}{\bar{F}_i(t_i|t_j)}\int_{t_i}^{\infty}\bar{F}_i(x_i|t_j)dx_i = c_i(t_j).$$

That is,

$$-\int_{t_i}^{\infty}\bar{F}_i(x_i|t_j)\log\bar{F}_i(x_i|t_j)dx_i+\log\bar{F}_i(t_i|t_j)\int_{t_i}^{\infty}\bar{F}_i(x_i|t_j)dx_i=c_i(t_j)\bar{F}_i(t_i|t_j). \quad (3.9)$$

Differentiating (3.9) with respect to t_i and simplifying, we get $r_i(t_i|t_j) = c_i(t_j)$, or equivalently $\int_{t_i}^{\infty}\bar{F}_i(x_i|t_j)dx_i = c_i(t_j)\bar{F}_i(t_i|t_j)$ which on differentiation yields $h_i(t_i|t_j) = \frac{1}{c_i(t_j)}$. The rest of the proof follows from Theorem 2.1 of Navarro and Sarabia (2013). □

Theorem 3.2.2. *The relationship*

$$\xi_i(t_1,t_2) = Cr_i(t_i|t_j) \quad (3.10)$$

where C is a constant independent of t_1 and t_2, holds if and only if (X_1,X_2) is distributed as either bivariate distribution with Pareto conditionals (Arnold (1987)) specified by the pdf

$$f(t_1,t_2) = K_1(1+a_1t_1+a_2t_2+bt_1t_2)^{-c};\ a_1,a_2>0, b\geq 0,\ c>2,$$
$$K_1>0,\ \textit{the normalizing constant};\ t_1,t_2>0, \quad (3.11)$$

or bivariate distribution with exponential conditionals (Arnold and Strauss (1988)) with pdf

$$f(t_1,t_2) = K_2\exp(-\lambda_1t_1-\lambda_2t_2-\theta t_1t_2);\ \lambda_1,\lambda_2>0, \theta\geq 0,$$
$$K_2>0,\ \textit{the normalizing constant};\ t_1,t_2>0, \quad (3.12)$$

or bivariate distribution with beta conditionals with pdf

$$f(t_1,t_2) = K_3(1-p_1t_1-p_2t_2+qt_1t_2)^d;\ p_1,p_2,d>0, q\geq 0, K_3>0,\ \textit{the normalizing constant};\ 0<t_1<\frac{1}{p_1},\ 0<t_2<\frac{1-p_1t_1}{p_2-qt_1}, \quad (3.13)$$

according as $C>1, C=1$ *or* $0<C<1$.

Remark 3.2.1. *The models* (3.11), (3.12) *and* (3.13) *are particular cases of bivariate model* (3.7).

Proof. Assume that (3.10) holds. Now using (3.5), we can write

$$\frac{-1}{\bar{F}_i(t_i|t_j)}\int_{t_i}^{\infty}\bar{F}_i(x_i|t_j)\log\bar{F}_i(x_i|t_j)dx_i + r_i(t_i|t_j)log\bar{F}_i(t_i|t_j) = Cr_i(t_i|t_j). \qquad (3.14)$$

Differentiating (3.14) with respect to t_i, we get,

$$\frac{-f_i(t_i|t_j)}{(\bar{F}_i(t_i|t_j))^2}\int_{t_i}^{\infty}\bar{F}_i(x_i|t_j)\log\bar{F}_i(x_i|t_j)dx_i + \log\bar{F}_i(t_i|t_j) - \frac{f_i(t_i|t_j)}{\bar{F}_i(t_i|t_j)}r_i(t_i|t_j) + \log\bar{F}_i(t_i|t_j)\frac{\partial}{\partial t_i}r_i(t_i|t_j) = C\frac{\partial}{\partial t_i}r_i(t_i|t_j). \quad (3.15)$$

Now using the relationship $1+\frac{\partial}{\partial t_i}r_i(t_i|t_j) = r_i(t_i|t_j)h_i(t_i|t_j)$, (3.15) reduces to $r_i(t_i|t_j)$ $h_i(t_i|t_j) = C$, or $\frac{\partial}{\partial t_i}r_i(t_i|t_j) = C-1$. Thus we obtain $r_i(t_i|t_j) = (C-1)t_i + D_i(t_j)$. Now by a characterization theorem due to Sankaran and Nair (1993a), (X_1,X_2) follows the models (3.11), (3.12) and (3.13) according as $C>1, C=1$ or $0<C<1$.

To prove the converse part, suppose that (X_1,X_2) follows the bivariate model with pdf (3.11). Now using (3.4), we have

$$\xi_i(t_1,t_2) = \frac{(c-1)(1+a_1t_1+a_2t_2+bt_1t_2)}{(c-2)^2(a_i+bt_j)} = \frac{(c-1)}{(c-2)}r_i(t_i|t_j),$$
$$= Cr_i(t_i|t_j),$$

where $C = \frac{(c-1)}{(c-2)} > 1$. When (X_1, X_2) has a joint pdf (3.13), we obtain

$$\xi_i(t_1, t_2) = \frac{(d+1)(1 - p_1 t_1 - p_2 t_2 + q t_1 t_2)}{(d+2)^2 (p_i - q t_j)} = \frac{(d+1)}{(d+2)} r_i(t_i|t_j),$$
$$= C r_i(t_i|t_j),$$

such that $0 < C = \frac{(d+1)}{(d+2)} < 1$. The model for (3.12) is similar. □

Next we prove a theorem for CMU_1 following a similar result in Asadi and Zohrevand (2007).

Theorem 3.2.3. *For any random vector (X_1, X_2), the following relationship holds*

$$\xi_i(t_1, t_2) = E[r_i(X_i|t_j)|X_i > t_i, X_j = t_j], \; i, j = 1, 2, \; i \neq j.$$

Proof. By definition,

$$E[r_i(X_i|t_j)|X_i > t_i, X_j = t_j] = \frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} r_i(x_i|t_j) f_i(x_i|t_j) dx_i,$$
$$= \frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} \frac{1}{\bar{F}_i(x_i|t_j)} \left(\int_{x_i}^{\infty} \bar{F}_i(u|t_j) du \right) f_i(x_i|t_j) dx_i,$$
$$= \frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} \left(\int_{t_i}^{u} h_i(x_i|t_j) dx_i \right) \bar{F}_i(u|t_j) du,$$
$$= \frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} \left(\int_{0}^{u} h_i(x_i|t_j) dx_i - \int_{0}^{t_i} h_i(x_i|t_j) dx_i \right) \bar{F}_i(u|t_j) du,$$
$$= \frac{-1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} \bar{F}_i(x_i|t_j) \log \bar{F}_i(x_i|t_j) dx_i + \frac{\log \bar{F}_i(t_i|t_j)}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} \bar{F}_i(x_i|t_j) dx_i,$$
$$= \xi_i(t_1, t_2).$$

□

Theorem 3.2.4. *If $\xi_i(t_1,t_2)$ is increasing (decreasing) in t_i, then $\xi_i(t_1,t_2) \geq (\leq) r_i(t_i|t_j), \forall t_j$.*

Proof. Assume that $\xi_i(t_1,t_2)$ is increasing (decreasing) in t_i. Then $\frac{\partial}{\partial t_i}\xi_i(t_1,t_2) \geq (\leq) 0$. Differentiating (3.4) with respect to t_i, we get

$$\frac{-h_i(t_i|t_j)}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} \bar{F}_i(x_i|t_j) \log \bar{F}_i(x_i|t_j) dx_i + \log \bar{F}_i(t_i|t_j) \\ - h_i(t_i|t_j) r_i(t_i|t_j) + \log \bar{F}_i(t_i|t_j) \frac{\partial}{\partial t_i} r_i(t_i|t_j) \geq (\leq) 0.$$

Equivalently,

$$h_i(t_i|t_j)[\xi_i(t_1,t_2) - \log \bar{F}_i(t_i|t_j) r_i(t_i|t_j) - r_i(t_i|t_j)] \\ + \log \bar{F}_i(t_i|t_j) h_i(t_i|t_j) r_i(t_i|t_j) \geq (\leq) 0,$$

$$h_i(t_i|t_j)[\xi_i(t_1,t_2) - r_i(t_i|t_j)] \geq (\leq) 0.$$

This proves the theorem. □

Example 3.2.1. *Suppose (X_1, X_2) follows bivariate distribution with Pareto conditionals (Arnold (1987)) specified by the pdf (3.11), then it is easy to show from the converse part of Theorem 3.2.2 that $\xi_i(t_1,t_2)$ is increasing in t_i, and $\xi_i(t_1,t_2) \geq r_i(t_i|t_j)$ for $i,j = 1,2$, $i \neq j$.* □

3.3 Cumulative measure of uncertainty for conditional survival models

In the case of bivariate survival models, instead of conditioning on a component failing at a specified time, it is sometimes more natural to condition on the components having survived beyond a specified time (see Arnold (1996), Navarro and Sarabia (2013)), *i.e.*, conditioning on events of the form $\{X_1 > t_1\}$ and $\{X_2 > t_2\}$. Let $f_i^*(t_i|t_j)$ be the pdf of the rv $(X_i|X_j > t_j)$. Then as a direct extension of (3.1) we get cumulative measure of uncertainty for $(X_i|X_j > t_j)$ and is given by

$$\xi_i^*(t_j) = -\int_0^\infty \bar{F}_i^*(x_i|t_j)\log \bar{F}_i^*(x_i|t_j)dx_i, \tag{3.16}$$

where $\bar{F}_i^*(t_i|t_j) = \int_{t_i}^\infty f_i^*(x_i|t_j)dx_i$ is the conditional sf of $(X_i|X_j > t_j)$ with $\bar{F}_i^*(x_i|t_j) > 0$. Assume that $\frac{\bar{F}_1^*(t_1|t_2)}{\bar{F}_2^*(t_2|t_1)} = u(t_1)v(t_2)$ where $u(t_1)$ and $1/v(t_2)$ are two reliability functions, the necessary conditions for the existence of the conditional sfs $\bar{F}_1^*$ and $\bar{F}_2^*$ (see Navarro and Sarabia (2010)). The equivalent dynamic version of (3.2) for the conditional rvs $(X_i|X_j > t_j)$ called as cumulative measure of uncertainty of type 2 (CMU$_2$) is defined as

$$\xi_i^*(t_1, t_2) = -\int_{t_i}^\infty \frac{\bar{F}_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)}\log \frac{\bar{F}_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)}dx_i. \tag{3.17}$$

Like CMU$_1$, (3.17) measures the uncertainty contained in the conditional distribution of $(X_i - t_i|X_i > t_i, X_j > t_j)$. If $r_i^*(t_i|t_j)$ denotes the MRLF of $X_i|X_j > t_j$, defined as

$$\begin{aligned} r_i^*(t_i|t_j) &= E(X_i - t_i|X_1 > t_1, X_2 > t_2) = \frac{1}{\bar{F}_i^*(t_i|t_j)}\int_{t_i}^\infty \bar{F}_i^*(x_i|t_j)dx_i, \\ &= \frac{1}{\bar{F}(t_1, t_2)}\int_{t_i}^\infty \bar{F}(x_i, t_j)dx_i = r_i(t_1, t_2), \end{aligned}$$

which is the i^{th} component of vector-valued MRLF in bivariate case, where $\bar{F}(t_1, t_2) = P(X_1 > t_1, X_2 > t_2)$ is the bivariate sf, equation (3.17) becomes

$$\xi_i^*(t_1, t_2) = \frac{-1}{\bar{F}_i^*(t_i|t_j)}\int_{t_i}^\infty \bar{F}_i^*(x_i|t_j)\log \bar{F}_i^*(x_i|t_j)dx_i + \log \bar{F}_i^*(t_i|t_j)r_i(t_1, t_2). \tag{3.18}$$

Now we prove a theorem which obtains bounds for (3.16) using stochastic ordering for the conditional survival models $(X_1|X_2 > t_2)$ and $(X_2|X_1 > t_1)$. We say the conditional rv

$(X_i|X_j > t_j)$ is said to be smaller than $(Y_i|Y_j > t_j)$ in the usual stochastic order, denoted by $(X_i|X_j > t_j) \leq_{ST} (Y_i|Y_j > t_j)$, if $\bar{F}^*_{X_i}(t_i|t_j) \leq \bar{F}^*_{Y_i}(t_i|t_j)$ for all t_i, t_j, $i, j = 1, 2$, $i \neq j$, where $\bar{F}^*_{X_i}(t_i|t_j)$ and $\bar{F}^*_{Y_i}(t_i|t_j)$ are the sfs of $(X_i|X_j > t_j)$ and $(Y_i|Y_j > t_j)$ respectively, for $i, j = 1, 2$, $i \neq j$.

Theorem 3.3.1. *If* $(X_i|X_j > t_j) \geq_{ST} (Y_i|Y_j > t_j), \forall t_j$, *then*

$$\xi^*_{X_i}(t_j) \leq \xi^*_{Y_i}(t_j) - E(X_i|X_j > t_j) \log \left[\frac{E(X_i|X_j > t_j)}{E(Y_i|Y_j > t_j)} \right], \ i, j = 1, 2, \ i \neq j.$$

Proof. Using (3.16), we get

$$\begin{aligned}
\xi^*_{X_i}(t_j) &= -\int_0^\infty \bar{F}^*_{X_i}(x_i|t_j) \log \bar{F}^*_{X_i}(x_i|t_j) dx_i, \\
&= -\int_0^\infty \bar{F}^*_{X_i}(x_i|t_j) \log \left(\frac{\bar{F}^*_{X_i}(x_i|t_j)}{\bar{F}^*_{Y_i}(x_i|t_j)} \bar{F}^*_{Y_i}(x_i|t_j) \right) dx_i, \\
&= -\int_0^\infty \bar{F}^*_{X_i}(x_i|t_j) \left(\log \frac{\bar{F}^*_{X_i}(x_i|t_j)}{\bar{F}^*_{Y_i}(x_i|t_j)} \right) dx_i - \int_0^\infty \bar{F}^*_{X_i}(x_i|t_j)(\log \bar{F}^*_{Y_i}(x_i|t_j)) dx_i. \qquad (3.19)
\end{aligned}$$

Using the log sum inequality, we have

$$\int_0^\infty \bar{F}^*_{X_i}(x_i|t_j) \left(\log \frac{\bar{F}^*_{X_i}(x_i|t_j)}{\bar{F}^*_{Y_i}(x_i|t_j)} \right) dx_i \geq E(X_i|X_j > t_j) \log \left[\frac{E(X_i|X_j > t_j)}{E(Y_i|Y_j > t_j)} \right],$$

or equivalently,

$$-\int_0^\infty \bar{F}^*_{X_i}(x_i|t_j) \left(\log \frac{\bar{F}^*_{X_i}(x_i|t_j)}{\bar{F}^*_{Y_i}(x_i|t_j)} \right) dx_i \leq -E(X_i|X_j > t_j) \log \left[\frac{E(X_i|X_j > t_j)}{E(Y_i|Y_j > t_j)} \right].$$

Then (3.19) becomes

$$\begin{aligned}
\xi^*_{X_i}(t_j) &\leq -E(X_i|X_j > t_j) \log \left[\frac{E(X_i|X_j > t_j)}{E(Y_i|Y_j > t_j)} \right] - \int_0^\infty \bar{F}^*_{X_i}(x_i|t_j) \log \bar{F}^*_{Y_i}(x_i|t_j) dx_i, \\
&\leq -E(X_i|X_j > t_j) \log \left[\frac{E(X_i|X_j > t_j)}{E(Y_i|Y_j > t_j)} \right] - \int_0^\infty \bar{F}^*_{Y_i}(x_i|t_j) \log \bar{F}^*_{Y_i}(x_i|t_j) dx_i,
\end{aligned}$$

where the second term on the right hand side of the inequality is obtained from the condition $(X_i|X_j > t_j) \geq_{ST} (Y_i|Y_j > t_j)$, which proves the result. □

Theorem 3.3.2. *For any random vector (X_1, X_2), the following relationship holds*

$$\xi_i^*(t_1, t_2) = E[r_i(X_i, t_j)|X_1 > t_1, X_2 > t_2], \quad i, j = 1, 2, \ i \neq j.$$

Proof. Applying the similar steps as in Theorem 3.2.3, the result follows. □

Theorem 3.3.3. *The relationship*

$$\xi_i^*(t_1, t_2) = Cr_i(t_1, t_2), \tag{3.20}$$

for all t_1 and t_2, where C is a constant independent of t_1 and t_2, holds if and only if (X_1, X_2) is distributed as bivariate Pareto distribution with joint sf

$$\bar{F}(t_1, t_2) = (1 + a_1t_1 + a_2t_2 + bt_1t_2)^{-c}; \ a_1, a_2, c > 0, 0 \leq b \leq (c+1)a_1a_2, t_1, t_2 > 0, \tag{3.21}$$

or Gumbel's bivariate exponential distribution with joint sf

$$\bar{F}(t_1, t_2) = \exp(-\lambda_1t_1 - \lambda_2t_2 - \theta t_1t_2); \ \lambda_1, \lambda_2 > 0, 0 \leq \theta < \lambda_1\lambda_2, t_1, t_2 > 0, \tag{3.22}$$

or bivariate beta distribution with joint sf

$$\begin{aligned}\bar{F}(t_1, t_2) = (1 - p_1t_1 - p_2t_2 + qt_1t_2)^d; \ p_1, p_2, d > 0,\\ 1 - d \leq \frac{q}{p_1p_2} \leq 1, 0 < t_1 < \frac{1}{p_1}, \ 0 < t_2 < \frac{1 - p_1t_1}{p_2 - qt_1},\end{aligned} \tag{3.23}$$

according as $C > 1, C = 1$ or $0 < C < 1$.

Proof. Assume that (3.20) holds. Now using (3.17), we have

$$\frac{-1}{\bar{F}_i^*(t_i|t_j)}\int_{t_i}^{\infty}\bar{F}_i^*(x_i|t_j)\log\bar{F}_i^*(x_i|t_j)dx_i+\log\bar{F}_i^*(t_i|t_j)r_i(t_1,t_2)=Cr_i(t_1,t_2).$$

Applying similar steps as in the proof of Theorem 3.2.2, we get

$$r_i(t_1,t_2)=(C-1)t_i+D_i(t_j),\ i,j=1,2,\ i\neq j.$$

Now using a characterization Theorem due to Roy (1989), (X_1,X_2) follows the models (3.21), (3.22) and (3.23) according as $C>1, C=1$ or $0<C<1$. The converse part is quite straightforward. □

Remark 3.3.1. *The models* (3.21), (3.22) *and* (3.23) *are special cases of the general bivariate sf*

$$\bar{F}(t_1,t_2)=\exp[-a_1\Lambda_1^o(t_1)-a_2\Lambda_2^o(t_2)-\phi a_1a_2\Lambda_1^o(t_1)\Lambda_2^o(t_2)],$$

for $t_1,t_2\geq 0$, where $a_1,a_2>0$ and $0\leq\phi\leq 1$ due to Navarro and Sarabia (2013), which is constructed by taking conditional PHR models of the form $\bar{F}_1^*(t_1|t_2)=\exp[-\alpha_1t_2\Lambda_1^o(t_1)]$ and $\bar{F}_2^*(t_2|t_1)=\exp[-\alpha_2(t_1)\Lambda_2^o(t_2)]$.

Theorem 3.3.4. *If $\xi_i^*(t_1,t_2)$ is an increasing (decreasing) function in t_i, then $\xi_i^*(t_1,t_2)\geq(\leq)r_i(t_1,t_2)$, $\forall t_j$.*

The proof is similar to that of Theorem 3.2.4.

Theorem 3.3.5. *$\xi_i^*(t_1,t_2)$, $i=1,2$, is independent of t_i if and only if (X_1,X_2) follows Gumbel's bivariate exponential distribution* (3.22).

3.4 Cumulative entropy of conditional rvs of the form X_i given $X_j \leq t_j$

In this section we define a new measure of cumulative entropy in bivariate setup by conditioning on events of the form $\{X_1 \leq t_1\}$ and $\{X_2 \leq t_2\}$ namely cumulative measure of uncertainty of type 3 (CMU$_3$). It measures the uncertainty of past life of a component X_i when the other component X_j was found failed at time t_j. It is defined as follows: Let $f_i^{\#}(t_i|t_j)$, $F_i^{\#}(t_i|t_j)$, $\lambda_i^{\#}(t_i|t_j)$ and $v_i^{\#}(t_i|t_j)$ be the pdf, the df, the reversed hazard rate and the reversed mean residual life (mean inactivity time) of the conditional rv $(X_i|X_j \leq t_j)$. Then CMU$_3$ denoted by $C\xi_i^{\#}(t_1, t_2)$ is defined as

$$C\xi_i^{\#}(t_1, t_2) = \frac{-1}{F_i^{\#}(t_i|t_j)} \int_0^{t_i} F_i^{\#}(x_i|t_j) \log\left(\frac{F_i^{\#}(x_i|t_j)}{F_i^{\#}(t_i|t_j)}\right) dx_i,$$

with $F_i^{\#}(x_i|t_j) > 0$.

Theorem 3.4.1. *For any rv (X_1, X_2), the relationship*

$$C\xi_i^{\#}(t_1, t_2) = E(v_i^{\#}(X_i|t_j)|X_i \leq t_i, X_j \leq t_j)$$

holds for all $i, j = 1, 2$, $i \neq j$.

Proof. For $i = 1$,

$$\begin{aligned}
E(v_1^{\#}(X_1|t_2)|X_1 \leq t_1, X_2 \leq t_2) &= \int_0^{t_1} \frac{v_1^{\#}(x_1|t_2)}{F_1^{\#}(t_1|t_2)} f_1^{\#}(x_1|t_2) dx_1, \\
&= \frac{1}{F_1^{\#}(t_1|t_2)} \int_0^{t_1} \left(\int_0^{x_1} \frac{F_1^{\#}(u|t_2)}{F_1^{\#}(x_1|t_2)} du\right) f_1^{\#}(x_1|t_2) dx_1, \\
&= \frac{1}{F_1^{\#}(t_1|t_2)} \int_0^{t_1} \left(\int_u^{t_1} \frac{f_1^{\#}(x_1|t_2)}{F_1^{\#}(x_1|t_2)} dx_1\right) F_1^{\#}(u|t_2) du,
\end{aligned}$$

$$= \frac{1}{F_1^{\#}(t_1|t_2)} \int_0^{t_1} \left(\int_u^{\infty} \lambda_1^{\#}(x_1|t_2)dx_1 - \int_{t_1}^{\infty} \lambda_1^{\#}(x_1|t_2)dx_1 \right) F_1^{\#}(u|t_2)du,$$

$$= \frac{1}{F_1^{\#}(t_1|t_2)} \int_0^{t_1} (-\log F_1^{\#}(u|t_2) + \log F_1^{\#}(t_1|t_2))F_1^{\#}(u|t_2)du,$$

$$= -\frac{1}{F_1^{\#}(t_1|t_2)} \int_0^{t_1} F_1^{\#}(u|t_2) \log \left(\frac{F_1^{\#}(u|t_2)}{F_1^{\#}(t_1|t_2)} \right) du = C\xi_1^{\#}(t_1, t_2).$$

Similarly for $i = 2$, we can prove

$$C\xi_2^{\#}(t_1, t_2) = E(v_2^{\#}(X_2|t_1)|X_2 \leq t_2, X_1 \leq t_1). \qquad \square$$

Theorem 3.4.2. *If (X_1, X_2) is a random vector in the support $(0, b_1) \times (0, b_2)$ admitting an absolutely continuous df F, then the relationship*

$$C\xi_i^{\#}(t_1, t_2) = a_i v_i^{\#}(t_i|t_j) \tag{3.24}$$

holds if and only if (X_1, X_2) is distributed as

$$F(x_1, x_2) = \left(\frac{x_1}{b_1}\right)^{k_1} \left(\frac{x_2}{b_2}\right)^{k_2}, \quad k_1, k_2, b_1, b_2 > 0, \tag{3.25}$$

where $k_i = (1 - a_i)^{-1} - 1$, $i = 1, 2$.

Proof. Suppose (X_1, X_2) has joint df (3.25) then, for $i = 1$

$$C\xi_1^{\#}(t_1, t_2) = -\frac{1}{F_1^{\#}(t_1|t_2)} \int_0^{t_1} F_1^{\#}(x_1|t_2) \log \frac{F_1^{\#}(x_1|t_2)}{F_1^{\#}(t_1|t_2)} dx_1,$$

$$= -\frac{1}{t_1^{k_1}} \int_0^{t_1} x_1^{k_1} \log \left(\frac{x_1^{k_1}}{t_1^{k_1}} \right) dx_1,$$

$$= -\frac{1}{t_1^{k_1}} \int_0^{t_1} x_1^{k_1} [k_1 \log x_1 - k_1 \log t_1] dx_1,$$

$$= \frac{k_1 \log t_1}{t_1^{k_1}} \left(\frac{t_1^{k_1+1}}{k_1 + 1} \right) - \frac{k_1}{t_1^{k_1}} \left[\log t_1 \left(\frac{t_1^{k_1+1}}{k_1 + 1} \right) - \frac{t_1^{k_1+1}}{(k_1 + 1)^2} \right],$$

$$
\begin{aligned}
&= \frac{k_1 t_1 \log t_1}{(k_1+1)} - \frac{k_1 t_1 \log t_1}{(k_1+1)} + \frac{k_1 t_1}{(k_1+1)^2}, \\
&= \frac{k_1 t_1}{(k_1+1)^2} = \left(\frac{k_1}{k_1+1}\right)\left(\frac{t_1}{k_1+1}\right), \\
&= a_1 v_1^{\#}(t_1|t_2).
\end{aligned}
$$

The case for $i = 2$ is similar.

Conversely, suppose (3.24) holds, then for $i = 1$, we have

$$
\begin{aligned}
&-\frac{1}{F_1^{\#}(t_1|t_2)} \int_0^{t_1} F_1^{\#}(x_1|t_2) \log F_1^{\#}(x_1|t_2) dx_1 \\
&\qquad\qquad + v_1^{\#}(t_1|t_2) \log F_1^{\#}(t_1|t_2) = a_1 v_1^{\#}(t_1|t_2). \quad (3.26)
\end{aligned}
$$

Differentiating equation (3.26) with respect to t_1, we obtain

$$
\begin{aligned}
&\frac{f_1^{\#}(t_1|t_2)}{(F_1^{\#}(t_1|t_2))^2} \int_0^{t_1} F_1^{\#}(x_1|t_2) \log F_1^{\#}(x_1|t_2) dx_1 - \log F_1^{\#}(t_1|t_2) + v_1^{\#}(t_1|t_2) \frac{f_1^{\#}(t_1|t_2)}{F_1^{\#}(t_1|t_2)} \\
&\qquad\qquad + \log F_1^{\#}(t_1|t_2) \frac{\partial}{\partial t_1} v_1^{\#}(t_1|t_2) = a_1 \frac{\partial}{\partial t_1} v_1^{\#}(t_1|t_2), \\
&\frac{\lambda_1^{\#}(t_1|t_2)}{(F_1^{\#}(t_1|t_2))} \int_0^{t_1} F_1^{\#}(x_1|t_2) \log F_1^{\#}(x_1|t_2) dx_1 - \log F_1^{\#}(t_1|t_2) + v_1^{\#}(t_1|t_2) \lambda_1^{\#}(t_1|t_2) \\
&\qquad\qquad + (\log F_1^{\#}(t_1|t_2) - a_1) \frac{\partial}{\partial t_1} v_1^{\#}(t_1|t_2) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\lambda_1^{\#}(t_1|t_2) v_1^{\#}(t_1|t_2)[\log F_1^{\#}(t_1|t_2) - a_1] - \log F_1^{\#}(t_1|t_2) \\
&\qquad\qquad + v_1^{\#}(t_1|t_2) \lambda_1^{\#}(t_1|t_2) + [\log F_1^{\#}(t_1|t_2) - a_1] \frac{\partial}{\partial t_1} v_1^{\#}(t_1|t_2) = 0,
\end{aligned}
$$

$$\left(1-\frac{\partial}{\partial t_1}v_1^{\#}(t_1|t_2)\right)(\log F_1^{\#}(t_1|t_2)-a_1)-\log F_1^{\#}(t_1|t_2)$$
$$+1-\frac{\partial}{\partial t_1}v_1^{\#}(t_1|t_2)+[\log F_1^{\#}(t_1|t_2)-a_1]\frac{\partial}{\partial t_1}v_1^{\#}(t_1|t_2)=0,$$

$$-a_1+1-\frac{\partial}{\partial t_1}v_1^{\#}(t_1|t_2)=0,$$
$$\frac{\partial}{\partial t_1}v_1^{\#}(t_1|t_2)=1-a_1,$$

which implies

$$v_1^{\#}(t_1|t_2)=(1-a_1)t_1+A_1(t_2). \tag{3.27}$$

Similarly for $i=2$ we get

$$v_2^{\#}(t_2|t_1)=(1-a_2)t_2+A_2(t_1). \tag{3.28}$$

Equivalently we can write (3.27) and (3.28) as

$$v_i(t_1,t_2)=(1-a_i)t_i+A_i(t_j).$$

Using the condition $v_i(t_1,t_2)\to 0$ as $t_i\to 0$, we have $A_i(t_j)=0$, which implies that

$$v_i(t_1,t_2)=(1-a_i)t_i.$$

Now using the expressions (see Nair and Asha (2008))

$$F(x_1,x_2)=\frac{v_1(b_1,b_2)v_2(x_1,b_2)}{v_1(x_1,b_2)v_2(x_1,x_2)}e^{-\int_{x_1}^{b_1}\frac{du}{v_1(u,b_2)}-\int_{x_2}^{b_2}\frac{du}{v_2(x_1,u)}}$$

and

$$F(x_1, x_2) = \frac{v_1(b_1, x_2)v_2(b_1, b_2)}{v_1(x_1, x_2)v_2(b_1, x_2)} e^{-\int_{x_1}^{b_1} \frac{du}{v_1(u, x_2)} - \int_{x_2}^{b_2} \frac{du}{v_2(b_1, u)}},$$

which uniquely determines the joint df, we deduce

$$F(x_1, x_2) = \left(\frac{x_1}{b_1}\right)^{k_1} \left(\frac{x_2}{b_2}\right)^{k_2}, \quad k_i, b_i > 0, 0 < x_i < b_i,\ i = 1, 2$$

where $k_i = \frac{a_i}{1-a_i}$. □

Theorem 3.4.3. *If (X_1, X_2) is any random vector in the support $(0, b_1) \times (0, b_2)$ admitting an absolutely continuous df F, then the relationship*

$$C\xi_i^{\#}(t_1, t_2) = a_i(t_j)v_i^{\#}(t_i|t_j), \quad i, j = 1, 2,\ i \neq j, \tag{3.29}$$

where $a_i(\cdot)$ is a non-negative function, holds if and only if (X_1, X_2) has Power function distribution (see Nair and Asha (2008))

$$F(x_1, x_2) = \left(\frac{x_1}{b_1}\right)^{k_1} \left(\frac{x_2}{b_2}\right)^{k_2 + \theta \log(\frac{x_1}{b_1})}, \ k_i, b_i > 0, 0 < x_i < b_i, i = 1, 2, \theta \leq 0, \tag{3.30}$$

where $k_i = [1 - a_i(b_j)]^{-1} - 1$.

Proof. Suppose (X_1, X_2) follows (3.30). Using Theorem 3.4.1 we have,

$$C\xi_i^{\#}(t_1, t_2) = \frac{1}{F_i^{\#}(t_i|t_j)} \int_0^{t_i} v_i^{\#}(x_i|t_j) f_i^{\#}(x_i|t_j) dx_i,$$
$$C\xi_i^{\#}(t_1, t_2) = \frac{1}{F_i^{\#}(t_i|t_j)} \int_0^{t_i} v_i(x_i, t_j) f_i^{\#}(x_i|t_j) dx_i,$$

For $i = 1$,

$$C\xi_1^{\#}(t_1, t_2) = \frac{1}{F_1^{\#}(t_1|t_2)} \int_0^{t_1} v_1(x_1, t_2) f_1^{\#}(x_1|t_2) dx_1,$$

$$
\begin{aligned}
&= \frac{1}{F_1^{\#}(t_1|t_2)} \int_0^{t_1} v_1(x_1,t_2) f_1^{\#}(x_1|t_2) dx_1, \\
&= \frac{1}{\left(\frac{t_1}{b_1}\right)^{k_1} \left(\frac{t_2}{b_2}\right)^{k_2+\theta\log(\frac{t_1}{b_1})}} \int_0^{t_1} \frac{x_1}{\left(1+k_1+\theta\log\left(\frac{t_2}{b_2}\right)\right)} \\
&\left[\frac{k_1 x_1^{k_1-1}}{b_1^{k_1}} \left(\frac{t_2}{b_2}\right)^{k_2+\theta\log(\frac{x_1}{b_1})} + \left(\frac{x_1}{b_1}\right)^{k_1} \left(\frac{t_2}{b_2}\right)^{k_2+\theta\log(\frac{x_1}{b_1})} \log\left(\frac{t_2}{b_2}\right) \frac{\theta}{x_1}\right] dx_1, \\
&= \frac{1}{\left(\frac{t_1}{b_1}\right)^{k_1} \left(\frac{t_2}{b_2}\right)^{k_2+\theta\log(\frac{t_1}{b_1})}} \\
&\left[\int_0^{t_1} \frac{k_1}{\left(1+k_1+\theta\log\left(\frac{t_2}{b_2}\right)\right)} \left(\frac{x_1}{b_1}\right)^{k_1} \left(\frac{t_2}{b_2}\right)^{k_2+\theta\log(\frac{x_1}{b_1})} dx_1\right] \\
&\quad + \frac{1}{\left(\frac{t_1}{b_1}\right)^{k_1} \left(\frac{t_2}{b_2}\right)^{k_2+\theta\log(\frac{t_1}{b_1})}} \\
&\left[\int_0^{t_1} \frac{\theta\log\left(\frac{t_2}{b_2}\right)}{\left(1+k_1+\theta\log\left(\frac{t_2}{b_2}\right)\right)} \left(\frac{x_1}{b_1}\right)^{k_1} \left(\frac{t_2}{b_2}\right)^{k_2+\theta\log(\frac{x_1}{b_1})} dx_1\right], \\
&= \frac{\left(k_1+\theta\log\left(\frac{t_2}{b_2}\right)\right)}{\left(\frac{t_1}{b_1}\right)^{k_1} \left(\frac{t_2}{b_2}\right)^{k_2+\theta\log(\frac{t_1}{b_1})} \left(1+k_1+\theta\log\left(\frac{t_2}{b_2}\right)\right)} \left[\int_0^{t_1} \left(\frac{x_1}{b_1}\right)^{k_1} \left(\frac{t_2}{b_2}\right)^{k_2+\theta\log(\frac{x_1}{b_1})} dx_1\right], \\
&= \frac{\left(k_1+\theta\log\left(\frac{t_2}{b_2}\right)\right)}{\left(\frac{t_1}{b_1}\right)^{k_1} \left(\frac{t_2}{b_2}\right)^{k_2+\theta\log(\frac{t_1}{b_1})} \left(1+k_1+\theta\log\left(\frac{t_2}{b_2}\right)\right)} \left[\frac{t_1 \left(\frac{t_1}{b_1}\right)^{k_1} \left(\frac{t_2}{b_2}\right)^{k_2+\theta\log(\frac{t_1}{b_1})}}{\left(1+k_1+\theta\log\left(\frac{t_2}{b_2}\right)\right)}\right], \\
&= \frac{t_1\left(k_1+\theta\log\left(\frac{t_2}{b_2}\right)\right)}{\left(1+k_1+\theta\log\left(\frac{t_2}{b_2}\right)\right)^2}, \\
&= a_1(t_2) v_1^{\#}(t_1|t_2).
\end{aligned}
$$

The case for $i = 2$ is similar.

Conversely, suppose that (3.29) holds. Then

$$
\frac{1}{F_i^{\#}(t_i|t_j)} \int_0^{t_i} v_i^{\#}(x_i|t_j) f_i^{\#}(x_i|t_j) dx_i = a_i(t_j) v_i^{\#}(t_i|t_j). \tag{3.31}
$$

Differentiating equation (3.31) with respect to t_i, we have

$$-\frac{f_i^{\#}(t_i|t_j)}{(F_i^{\#}(t_i|t_j))^2}\int_0^{t_i} v_i^{\#}(x_i|t_j)f_i^{\#}(x_i|t_j) + v_i^{\#}(t_i|t_j)\lambda_i^{\#}(t_i|t_j) = a_i(t_j)\frac{\partial}{\partial t_i}v_i^{\#}(t_i|t_j),$$

$$-\lambda_i^{\#}(t_i|t_j)a_i(t_j)v_i^{\#}(t_i|t_j) + \lambda_i^{\#}(t_i|t_j)v_i^{\#}(t_i|t_j) = a_i(t_j)\frac{\partial}{\partial t_i}v_i^{\#}(t_i|t_j),$$

$$\lambda_i^{\#}(t_i|t_j)v_i^{\#}(t_i|t_j)(1-a_i(t_j)) = a_i(t_j)\frac{\partial}{\partial t_i}v_i^{\#}(t_i|t_j),$$

$$\left(1-\frac{\partial}{\partial t_i}v_i^{\#}(t_i|t_j)\right)(1-a_i(t_j)) = a_i(t_j)\frac{\partial}{\partial t_i}v_i^{\#}(t_i|t_j),$$

$$1-\frac{\partial}{\partial t_i}v_i^{\#}(t_i|t_j) - a_i(t_j) + a_i(t_j)\frac{\partial}{\partial t_i}v_i^{\#}(t_i|t_j) = a_i(t_j)\frac{\partial}{\partial t_i}v_i^{\#}(t_i|t_j),$$

$$\frac{\partial}{\partial t_i}v_i^{\#}(t_i|t_j) = 1-a_i(t_j) = l_i(t_j),$$

$$v_i^{\#}(t_i|t_j) = l_i(t_j)t_i + A_i(t_j),$$

which is equivalent to $v_i(t_1,t_2) = l_i(t_j)t_i + A_i(t_j)$. Using the condition $\lim_{t_i\to 0} v_i(t_1,t_2) = 0$, we have $A_i(t_j) = 0$. Hence we have $v_i(t_1,t_2) = l_i(t_j)t_i$. The rest of the proof follows from Theorem 2.1 of Nair and Asha (2008). □

Theorem 3.4.4. *If (X_1, X_2) is a random vector in the support $(-\infty, b_1)\times(-\infty, b_2)$ admitting an absolutely continuous df F, then relationship*

$$C\xi_i^{\#}(t_1,t_2) = v_i^{\#}(t_i|t_j), \quad i,j=1,2, \quad i\neq j, \tag{3.32}$$

holds if and only if (X_1, X_2) is distributed as

$$F(x_1,x_2) = \exp[p_1(x_1-b_1)+p_2(x_2-b_2)+p_3(x_1-b_1)(x_2-b_2)],\ p_i, b_i>0,\ i=1,2. \tag{3.33}$$

Proof. Suppose that (X_1, X_2) follows (3.33). Now using Theorem 3.4.1 we have,

$$C\xi_i^{\#}(t_1,t_2) = \frac{1}{F_i^{\#}(t_i|t_j)}\int_0^{t_i} v_i^{\#}(x_i|t_j)f_i^{\#}(x_i|t_j)dx_i,$$

$$= \frac{1}{F_i^{\#}(t_i|t_j)} \int_0^{t_i} v_i(x_i, t_j) f_i^{\#}(x_i|t_j) dx_i.$$

For $i = 1$, we have

$$\begin{aligned}
C\xi_1^{\#}(t_1, t_2) &= \frac{1}{F_1^{\#}(t_1|t_2)} \int_0^{t_1} v_1(x_1, t_2) f_1^{\#}(x_1|t_2) dx_1, \\
&= \frac{1}{e^{p_1(t_1-b_1)+p_2(t_2-b_2)+p_3(t_1-b_1)(t_2-b_2)}} \\
&\quad \int_{-\infty}^{t_1} \frac{1}{(p_1 + p_3(t_2 - b_2))} e^{p_1(x_1-b_1)+p_2(t_2-b_2)+p_3(x_1-b_1)(t_2-b_2)} dx_1, \\
&= \frac{1}{e^{p_1(t_1-b_1)+p_2(t_2-b_2)+p_3(t_1-b_1)(t_2-b_2)}} \\
&\quad \int_{-\infty}^{t_1} \frac{(p_1 + p_3(t_2 - b_2))}{(p_1 + p_3(t_2 - b_2))} e^{p_1(x_1-b_1)+p_2(t_2-b_2)+p_3(x_1-b_1)(t_2-b_2)} dx_1, \\
&= \frac{1}{e^{p_1(t_1-b_1)+p_2(t_2-b_2)+p_3(t_1-b_1)(t_2-b_2)}} \int_{-\infty}^{t_1} e^{p_1(x_1-b_1)+p_2(t_2-b_2)+p_3(x_1-b_1)(t_2-b_2)} dx_1, \\
&= \frac{1}{(p_1 + p_3(t_2 - b_2))} = v_1^{\#}(t_1|t_2).
\end{aligned}$$

Similarly for $i = 2$, we get

$$C\xi_2^{\#}(t_1, t_2) = v_2^{\#}(t_2|t_1).$$

Conversely, suppose that relationship (3.32) holds. Then

$$\frac{1}{F_i^{\#}(t_i|t_j)} \int_{-\infty}^{t_i} v_i^{\#}(x_i|t_j) f_i^{\#}(x_i|t_j) dx_i = v_i^{\#}(t_i|t_j). \tag{3.34}$$

Differentiating equation (3.34) with respect to t_i, we get

$$-\frac{f_i^{\#}(t_i|t_j)}{(F_i^{\#}(t_i|t_j))^2} \int_{-\infty}^{t_i} v_i^{\#}(x_i|t_j) f_i^{\#}(x_i|t_j) dx_i + \lambda_i^{\#}(t_i|t_j) v_i^{\#}(t_i|t_j) = \frac{\partial}{\partial t_i} v_i^{\#}(t_i|t_j),$$

$$-\lambda_i^{\#}(t_i|t_j) v_i^{\#}(t_i|t_j) + \lambda_i^{\#}(t_i|t_j) v_i^{\#}(t_i|t_j) = \frac{\partial}{\partial t_i} v_i^{\#}(t_i|t_j),$$

which implies $\frac{\partial}{\partial t_i} v_i^{\#}(t_i|t_j) = 0$. Hence $v_i^{\#}(t_i|t_j) = a_i(t_j)$, *i.e.*, $v_i(t_1, t_2) = a_i(t_j)$. Now the proof follows from Theorem 2.3 of Nair and Asha (2008). □

3.5 An application of dynamic cumulative residual entropy

Sometimes we will be faced with the problem of extracting the distribution F from the limited and incomplete information which would be typically available in many real life situations. To produce a model for the data generating distribution, the well known maximum entropy (ME) paradigm can be employed. In the ME procedure, we start with the fact that distribution F is not known. But we may have some information about this distribution based on which we want to derive a model that best approximates the distribution F. We proceed by formulating the partial knowledge about F in terms of a set of information constraints. The distribution that maximizes the entropy subject to the constraints is the least committal with respect to unknown or missing information and hence is least prejudiced. So in this sense, ME distribution is preferred over any other.

For univariate case, we have the dynamic cumulative residual entropy function (3.2), which can also be written as $\xi(t) = \frac{1}{\bar{F}(t)} \int_t^\infty r(x) f(x) dx$, where $r(t) = E(X - t | X > t)$, the mean residual life function. Though variety of research is available for ME principle based residual Shannon entropy, a study of the same using cumulative residual entropy does not appear to have been taken up. Therefore in the following theorem we illustrate the usefulness of the univariate DCRE in identifying a ME model.

In survival analysis and in life testing, many results are based on the assumption that the life of a system is described by an exponential distribution. Following the results in Ebrahimi (2000), we derive exponential distribution as the distribution which maximizes the dynamic cumulative residual entropy function.

Theorem 3.5.1. *The exponential distribution uniquely maximizes the DCRE subject to the constraints 1)* $\int_0^\infty f(x)dx = 1$, *2)* $r(0) = \theta(> 0)$, *3)* $r(x)$ *is decreasing in* x.

Proof. We have

$$\xi(t) = \frac{1}{\bar{F}(t)} \int_t^\infty r(x)f(x)dx.$$

Using conditions (2) and (3) in the statement, we get

$$\xi(t) \leq \frac{1}{\bar{F}(t)} \int_t^\infty r(t)f(x)dx \leq r(0) = \theta. \tag{3.35}$$

From (3.35) it is clear that a density f^* which satisfies the equality maximizes DCRE. So, if $\xi^*(t)$ denotes the DCRE corresponding to f^* we have $\xi^*(t) = \theta$. That is,

$$\frac{1}{\bar{F}^*(t)} \int_t^\infty r^*(x)f^*(x)dx = \theta. \tag{3.36}$$

where $\bar{F}^*$ and r^* are the survival and the mean residual life functions corresponding to the density f^*. Differentiating equation (3.36) with respect to t, we get $\theta h^*(t) - h^*(t)r^*(t) = 0$ where $h^*(t) = \frac{f^*(t)}{\bar{F}^*(t)}$. That is, $h^*(t)(\theta - r^*(t)) = 0$, which implies $r^*(t) = \theta$. This proves the result.

Conversely, let $f^*(x) = \frac{1}{\theta}e^{-\frac{1}{\theta}x}$, $\theta > 0$. Then it is shown that f^* uniquely maximizes $\xi^*(t)$ over all probability densities f satisfying the information constraints from (1) to

(3). We have

$$\begin{aligned}
\xi^*(t) &= \frac{1}{\bar{F}^*(t)} \int_t^\infty r^*(x) f^*(x) dx = \frac{1}{\bar{F}^*(t)} \int_t^\infty \theta f^*(x) dx = \theta = r(0) \\
&= \frac{1}{\bar{F}(t)} \int_t^\infty r(0) f(x) dx \geq \frac{1}{\bar{F}(t)} \int_t^\infty r(t) f(x) dx \geq \frac{1}{\bar{F}(t)} \int_t^\infty r(x) f(x) dx \\
&= \xi(t),
\end{aligned}$$

which completes the proof. □

Chapter 4

Dynamic cumulative residual Renyi's entropy

4.1 Introduction

Cumulative residual entropy (CRE) proposed by Rao et al. (2004) gains its importance over Shannon's entropy as it utilizes cdf which is more regular than pdf. Renyi's entropy is considered as the well known generalized form of Shannon's entropy. Like Shannon's entropy, Renyi's entropy proposed by Renyi uses pdf in its definition. Motivated by CRE, an idea to replace pdf in Renyi's definition of entropy was born. This chapter is the consequence of such thoughts and the studies made in this regard.

In this chapter we introduce a new measure of uncertainty namely cumulative residual Renyi's entropy of order α. Section 4.2 includes the definition of cumulative residual Renyi's entropy and definiton and properties of dynamic cumulative residual Renyi's entropy (DCRRE) and some characterization theorems arising out of it. In Section 4.3,

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we examine DCRRE in the context of weighted and equilibrium distributions and study its various relationships. Finally Section 4.4 introduces DCRRE in the bivariate case and proves certain characterizations based on it.

4.2 Dynamic cumulative residual Renyi's entropy

Motivated with the usefulness of Renyi's entropy of order α and cumulative residual entropy for measuring uncertainty, in the present chapter, we propose a new measure of uncertainty called cumulative residual Renyi's entropy of order α. Analogous to the definition of cumulative residual entropy (3.1) by Rao et al. (2004) we define cumulative residual Renyi's entropy (CRRE) as follows:

Definition 4.2.1. *For a non-negative rv X with an absolutely continuous sf $\bar{F}$, the cumulative residual Renyi's entropy of order α is defined as*

$$\gamma(\alpha) = \frac{1}{1-\alpha} \log \left(\int_0^\infty \bar{F}^\alpha(x) dx \right) \; for \; \begin{matrix} \alpha \neq 1 \\ \alpha > 0 \end{matrix} . \tag{4.1}$$

When $\alpha \to 1$, (4.1) reduces to

$$\gamma(1) = \lim_{\beta \to 1} \gamma(\beta) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x) dx,$$

which is the cumulative residual entropy (3.1) and hence possesses all the properties discussed in Rao et al. (2004). Studying the effects of the age t of an individual or an item, the information about the remaining lifetime is of importance. In such situations, either (3.1) or (4.1) is not suitable and therefore it has to be modified to include the current age into account. Information measures that include the age are functions of t and hence dynamic. Hence we define a dynamic cumulative residual Renyi's entropy

(DCRRE) as follows.

Definition 4.2.2. *For a non-negative rv X with an absolutely continuous sf $\bar{F}$, DCRRE of order α denoted by $\gamma(\alpha;t)$ is defined as*

$$\gamma(\alpha;t) = \frac{1}{1-\alpha}\log\left(\int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)}dx\right) \ \textit{for} \ \begin{array}{c}\alpha \neq 1 \\ \alpha > 0\end{array}, \tag{4.2}$$

which can be written as

$$(1-\alpha)\gamma(\alpha;t) = \log\left(\int_t^\infty \bar{F}^\alpha(x)dx\right) - \alpha\log\bar{F}(t). \tag{4.3}$$

Differentiating (4.3) with respect to t, we have

$$(1-\alpha)\gamma'(\alpha;t) = \alpha h(t) - e^{-(1-\alpha)\gamma(\alpha;t)}, \tag{4.4}$$

where $\gamma'(\alpha;t)$ denotes the derivative of $\gamma(\alpha;t)$ with respect to t and $h(t) = \frac{f(t)}{\bar{F}(t)}$ is the hazard rate of X. Obviously, when a system has completed t units of time, for different values of α, $\gamma(\alpha;t)$ gives Renyi's information for the remaining life of the system. Also, $\gamma(\alpha;0) = \gamma(\alpha)$.

Remark 4.2.1. *The variation of DCRRE of order α can be obtained from the following example.*

Suppose that X follows exponential distribution with mean $\frac{1}{2}$. Then $\gamma(\alpha;t) = \frac{1}{(\alpha-1)}\log 2\alpha$. Clearly, for $\alpha > 1$, $\gamma(\alpha;t)$ is positive while for $0.5 < \alpha < 1$, $\gamma(\alpha;t)$ is negative. When $\alpha = \frac{1}{2}$, $\gamma(\alpha;t)$ is zero.

$\gamma(\alpha;t)$ has been obtained for various distributions in the Table 4.1.

In the following theorem we prove that DCRRE determines $\bar{F}$ uniquely.

Sl. No.	Distributions	$\bar{F}(t)$	$\gamma(\alpha;t)$
1	Uniform	$1-\frac{t}{a};\ 0<t<a,\ a>0$	$\frac{1}{1-\alpha}\log\left(\frac{a-t}{\alpha+1}\right)$
2	Exponential	$e^{-\lambda t};\ \lambda>0,\ t>0$	$\frac{1}{1-\alpha}\log\left(\frac{1}{\lambda\alpha}\right)$
3	Weibull	$e^{-t^p};\ t>0,\ p>0$	$\frac{1}{1-\alpha}\log\left(\frac{\alpha^{-1/p}}{pe^{-\alpha t^p}}\Gamma(\frac{1}{p},\alpha t^p)\right)$
4	Pareto I	$\left(\frac{k}{t}\right)^c;\ t>k,\ c,k>0$	$\frac{1}{1-\alpha}\log\left(\frac{t}{c\alpha-1}\right)$
5	Pareto II	$(1+pt)^{-q};\ p>0,\ q>0,\ t>0$	$\frac{1}{1-\alpha}\log\left(\frac{1+pt}{p(q\alpha-1)}\right)$
6	Beta	$(1-at)^b;\ a>0,\ b>0,\ 0<t<\frac{1}{a}$	$\frac{1}{1-\alpha}\log\left(\frac{1-at}{a(b\alpha+1)}\right)$

Table 4.1:

Theorem 4.2.1. *Let X be a non-negative rv having an absolutely continuous sf $\bar{F}(t)$ and hazard rate $h(t)$ with $\gamma(\alpha;t)<\infty$; $t\geq 0$; $\alpha>0$, $\alpha\neq 1$. Then for each α, $\gamma(\alpha;t)$ uniquely determines $\bar{F}(t)$.*

Proof. Let $\bar{F}_1(t)$ and $\bar{F}_2(t)$ be two sfs with DCRRE $\gamma_1(\alpha;t)$ and $\gamma_2(\alpha;t)$ and failure rates $h_1(t)$ and $h_2(t)$ respectively. Now $\gamma_1(\alpha;t)=\gamma_2(\alpha;t)$ implies that

$$\gamma_1'(\alpha;t)=\gamma_2'(\alpha;t),$$

which is equivalent to

$$(1-\alpha)\gamma_1'(\alpha;t)=(1-\alpha)\gamma_2'(\alpha;t). \tag{4.5}$$

Using (4.4), equation (4.5) becomes

$$\alpha h_1(t)-e^{-(1-\alpha)\gamma_1(\alpha;t)}=\alpha h_2(t)-e^{-(1-\alpha)\gamma_2(\alpha;t)}. \tag{4.6}$$

But $\gamma_1(\alpha;t)=\gamma_2(\alpha;t)$. Equation (4.6) then reduces to

$$\alpha h_1(t)=\alpha h_2(t),$$

which implies that $h_1(t)=h_2(t)$, or equivalently $\bar{F}_1(t)=\bar{F}_2(t)$. □

In the following theorem we show how $\gamma(\alpha; t)$ is affected by an increasing transformation of rv X.

Theorem 4.2.2. *If ϕ is a strictly increasing function, then*

$$\gamma_{\phi(X)}(\alpha; t) = \gamma_X(\alpha, \phi^{-1}(t)).$$

Proof. Using the definition of DCRRE, we have

$$\begin{aligned}
\gamma_{\phi(X)}(\alpha; t) &= \frac{1}{1-\alpha} \log \int_t^{\infty} \frac{\bar{F}^{\alpha}(\phi^{-1}(x))}{\bar{F}^{\alpha}(\phi^{-1}(t))} \frac{dx}{\phi'(\phi^{-1}(x))}, \\
&= \frac{1}{1-\alpha} \log \int_{\phi^{-1}(t)}^{\infty} \frac{\bar{F}^{\alpha}(y)}{\bar{F}^{\alpha}(\phi^{-1}(t))} dy, \\
&= \gamma_X(\alpha, \phi^{-1}(t)).
\end{aligned}$$

□

Example 4.2.1. *Let X be a rv distributed as Pareto I with sf,*

$$\bar{F}_X(t) = \left(\frac{k}{t}\right)^c; \; t > k, c, k > 0,$$

then from the table 4.1 we have

$$\gamma_X(\alpha; t) = \frac{1}{1-\alpha} \log\left(\frac{t}{c\alpha - 1}\right), c\alpha - 1 > 0.$$

Now consider $Y = \phi(X) = X - k$, then Y follows Pareto II distribution with sf

$$\bar{F}_Y(t) = \left(1 + \frac{t}{k}\right)^{-c}; \quad k, c > 0, \; t > 0.$$

Now by definition

$$\gamma_Y(\alpha, t) = \frac{1}{1-\alpha} \log\left(\frac{t+k}{c\alpha - 1}\right), c\alpha - 1 > 0,$$

which can be verified using Theorem 4.2.2. □

Theorem 4.2.3. *Let $X_1, X_2, X_3, \ldots, X_n$ be n iid rvs with common sf $\bar{F}$ and let $X_{(1)}$ denote the first order statistic, then the relationship*

$$\gamma_{X_{(1)}}(\alpha; t) = \left(\frac{1 - n\alpha}{1 - \alpha}\right) \gamma(n\alpha; t)$$

holds for all n except $n = \frac{1}{\alpha}$, where $\gamma_{X_{(1)}}(\alpha, t)$ denote the DCRRE of $X_{(1)}$.

Proof. Let $\bar{F}_{X_{(1)}}$ denote the sf of $X_{(1)}$, then

$$\begin{aligned}
\gamma_{X_{(1)}}(\alpha; t) &= \frac{1}{1-\alpha} \log \int_t^\infty \frac{\left(\bar{F}_{X_{(1)}}(x)\right)^\alpha}{\left(\bar{F}_{X_{(1)}}(t)\right)^\alpha} dx, \\
&= \frac{1}{1-\alpha} \log \int_t^\infty \frac{\left(\bar{F}(x)\right)^{n\alpha}}{\left(\bar{F}(t)\right)^{n\alpha}} dx, \\
&= \left(\frac{1 - n\alpha}{1 - \alpha}\right) \frac{1}{1 - n\alpha} \log \int_t^\infty \frac{\left(\bar{F}(x)\right)^{n\alpha}}{\left(\bar{F}(t)\right)^{n\alpha}} dx, \\
&= \left(\frac{1 - n\alpha}{1 - \alpha}\right) \gamma(n\alpha; t).
\end{aligned}$$

□

Theorem 4.2.4. *For the rv considered in Theorem 4.2.1, the relationship*

$$(1 - \alpha)\gamma'(\alpha; t) = Ch(t), \tag{4.7}$$

where C is a constant, holds if and only if X is distributed as

(a) Pareto II distribution with sf

$$\bar{F}(t) = (1 + pt)^{-q}; \quad p > 0,\ q > 0,\ t > 0, \tag{4.8}$$

(b) exponential distribution with sf

$$\bar{F}(t) = e^{-\lambda t};\ \lambda > 0,\ t > 0, \tag{4.9}$$

(c) finite range distribution with sf

$$\bar{F}(t) = (1 - at)^b;\ a > 0, b > 0, 0 < t < \frac{1}{a}, \tag{4.10}$$

according as $C \gtreqless 0$.

Proof. Assume that the relationship (4.7) holds. Using (4.4), (4.7) becomes

$$\alpha h(t) - e^{-(1-\alpha)\gamma(\alpha;t)} = Ch(t),$$

which is equivalent to

$$(\alpha - C)h(t) = e^{-(1-\alpha)\gamma(\alpha;t)}. \tag{4.11}$$

Using the expression of DCRRE in (4.2), equation (4.11) becomes

$$(\alpha - C)f(t)\int_t^{\infty} \bar{F}^{\alpha}(x)dx = \bar{F}^{\alpha+1}(t). \tag{4.12}$$

Differentiating (4.12) with respect to t, we get

$$(\alpha - C)f'(t)\int_t^{\infty} \bar{F}^{\alpha}(x)dx - (\alpha - C)f(t)\bar{F}^{\alpha}(t) = -(\alpha + 1)\bar{F}^{\alpha}(t)f(t). \tag{4.13}$$

Using (4.12), equation (4.13) becomes

$$f'(t)\frac{\bar{F}^{\alpha+1}(t)}{f(t)} - (\alpha - C)f(t)\bar{F}^{\alpha}(t) = -(\alpha + 1)\bar{F}^{\alpha}(t)f(t). \tag{4.14}$$

Dividing the equation (4.14) by $f(t)\bar{F}^{\alpha}(t)$and simplifying, yield $\frac{d}{dt}\log f(t) = (C+1)\frac{d}{dt}\log\bar{F}(t)$, which implies

$$\frac{d}{dt}\log h(t) = C\frac{d}{dt}\log\bar{F}(t). \tag{4.15}$$

Integrating (4.15) with respect to t, we get

$$\log h(t) = C\log\bar{F}(t) + K, \tag{4.16}$$

where K is the constant of integration. Now differentiating equation (4.16) with respect to t, we obtain $\frac{h'(t)}{h(t)} = -C\frac{f(t)}{\bar{F}(t)}$, or

$$\frac{d}{dt}\left[\frac{1}{h(t)}\right] = C. \tag{4.17}$$

Integrating equation (4.17) with respect to t, we obtain $\frac{1}{h(t)} = Ct + l$, where $l > 0$ is the constant of integration, or equivalently $h(t) = \frac{1}{Ct+l}$. Since the hazard rate uniquely determines sf using the relationship $\bar{F}(t) = \exp\left(-\int_0^t h(x)dx\right)$, the models (4.8), (4.9) and (4.10) follow according as $C \underset{<}{\overset{>}{=}} 0$.

To prove the converse part, first assume that X is distributed as Pareto II with sf (4.8). Now using (4.3), we have

$$(1-\alpha)\gamma(\alpha;t) = \log\left[\frac{1+pt}{p(q\alpha-1)}\right] = \log(1+pt) + log\left[\frac{1}{p(q\alpha-1)}\right],$$

which on differentiation yields

$$(1-\alpha)\gamma'(\alpha;t) = \frac{p}{1+pt} = Ch(t),$$

with

$$h(t) = \frac{pq}{1+pt} \text{ and } C = \frac{1}{q} > 0,$$

(4.7) follows. When X is distributed as exponential with sf (4.9), we have

$$(1-\alpha)\gamma(\alpha;t) = \log\left(\frac{1}{\lambda\alpha}\right),$$

from which (4.7) follows with $C = 0$. When X is distributed as finite range with sf (4.10), we get

$$(1-\alpha)\gamma'(\alpha;t) = -\frac{a}{1-at} = Ch(t),$$

with

$$h(t) = \frac{ab}{(1-at)}$$

and $C = -\frac{1}{b} < 0$, which yields (4.7). □

Theorem 4.2.5. *For a non-negative rv X with an absolutely continuous sf $\bar{F}$ and mean residual life function $r(t) = E(X-t|X>t)$, the relationship*

$$(1-\alpha)\gamma(\alpha;t) = \log[Cr(t)], \tag{4.18}$$

holds if and only if X has sf (4.8), (4.9) *or* (4.10) *according as*

$$\frac{C\alpha - 1}{C(1-\alpha)} \gtreqless 0.$$

Proof. Assume that the relationship (4.18) holds. Then

$$(1-\alpha)\gamma(\alpha;t) = \log C + \log r(t). \tag{4.19}$$

Differentiating (4.19) with respect to t, we get

$$(1-\alpha)\gamma'(\alpha;t) = \frac{r'(t)}{r(t)}. \tag{4.20}$$

Using (4.4), equation (4.20) becomes

$$\frac{r'(t)}{r(t)} = \alpha h(t) - e^{-(1-\alpha)\gamma(\alpha;t)}. \tag{4.21}$$

Using (4.18), equation (4.21) becomes

$$\frac{r'(t)}{r(t)} = \alpha h(t) - \frac{1}{Cr(t)},$$

which is equivalent to

$$Cr'(t) = \alpha Cr(t)h(t) - 1.$$

Now using the relationship between $h(t)$ and $r(t)$, the above expression becomes $Cr'(t) = \alpha C(r'(t)+1) - 1$. Equivalently, $r'(t) = \frac{C\alpha-1}{C(1-\alpha)} = P$, a constant. This implies that $r(t) = Pt + Q$, where Q is the constant of integration, which is a characterization to the models (4.8), (4.9) and (4.10) according as $P \gtreqless 0$. The converse part is quite straightforward. □

Definition 4.2.3. *The sf $\bar{F}(t)$ is said to have increasing (decreasing) α order dynamic cumulative residual Renyi's entropy IDCRRE (DDCRRE) if $\gamma(\alpha;t)$ is increasing (decreasing) in t; $t > 0$. i.e., $\bar{F}(t)$ have IDCRRE (DDCRRE) if $\gamma'(\alpha;t) \geq (\leq) 0$. $\bar{F}(t)$ is both IDCRRE and DDCRRE if $\gamma'(\alpha;t) = 0$.*

Examples:

(a) If X is distributed uniformly on $(0,a)$, then $\bar{F}(t)$ is IDCRRE for $\alpha > 1$ and DDCRRE

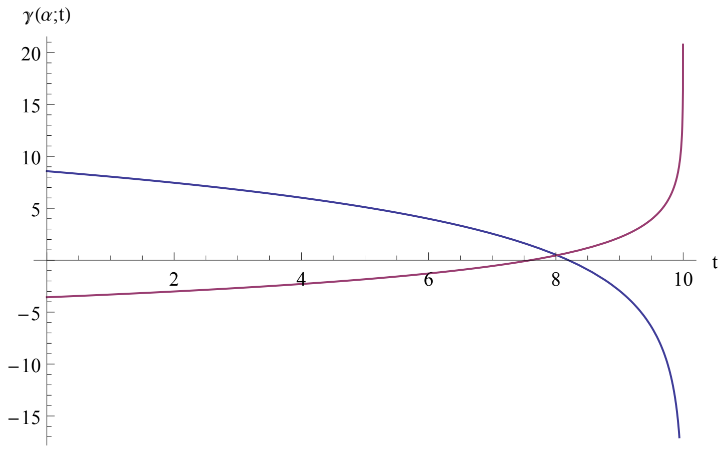

Figure 4.1:

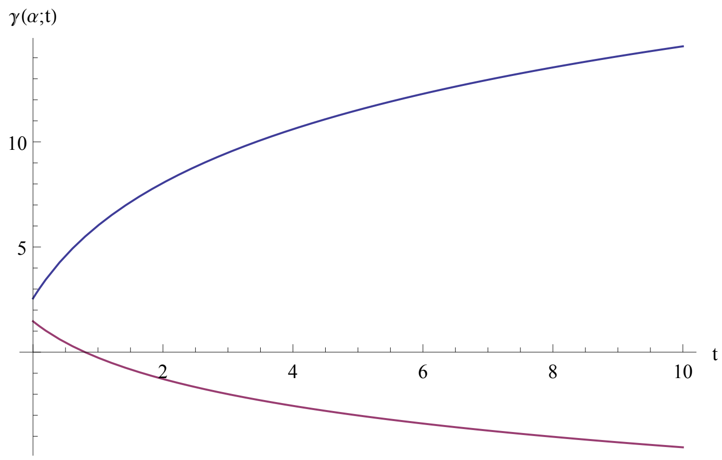

Figure 4.2:

for $0 < \alpha < 1$. Figure 4.1 represents DCRRE for $\alpha = 1.4$ (lower curve) and DCRRE for $\alpha = 0.8$ (upper curve) when $a = 10$.

(b) When X is distributed as Pareto II with sf (4.8), then $\bar{F}(t)$ is IDCRRE for $0 < \alpha < 1$ and DDCRRE for $\alpha > 1$. Figure 4.2 represents DCRRE for $\alpha = 1.4$ (lower curve) and DCRRE for $\alpha = 0.8$ (upper curve) for $p = 1$ and $q = 2$.

Theorem 4.2.6. *$\bar{F}(t)$ is both IDCRRE and DDCRRE if and only if X follows exponential distribution.*

Proof. Suppose $\bar{F}(t)$ is both IDCRRE and DDCRRE then we have $\gamma'(\alpha;t) = 0 \Rightarrow (1-\alpha)\gamma'(\alpha;t) = 0$. Now using Theorem 4.2.4, X follows exponential distribution. □

Corollory 4.2.1. *DCRRE is constant if and only if X is exponentially distributed.*

4.3 Weighted dynamic cumulative residual Renyi's entropy

The concept of weighted distributions is usually considered in connection with modeling statistical data, where the usual practice of employing standard distributions is not found appropriate. A survey of research on weighted distributions in various fields of applications is available in Nair and Sunoj (2003), Di Crescenzo and Longobardi (2006), Sunoj and Maya (2006), Maya and Sunoj (2008) and Sunoj and Sreejith (2012). If X is an absolutely continuous non-negative rv with pdf f and sf $\bar{F}$, then the pdf f^w and the sf $\bar{F}^w$ of the weighted rv X^w associated to X and to a positive real function $w(\cdot)$ are defined by

$$f^w(t) = \frac{w(t)f(t)}{E(w(X))} \tag{4.22}$$

and

$$\bar{F}^w(t) = \frac{E(w(X)|X > t)}{E(w(X))}\bar{F}(t), \tag{4.23}$$

where $E(w(X)) < \infty$. When the weight function is proportional to lengths of units of interest (*i.e.,* $w(t) = t$), then the model (4.22) is known as length-biased model with rv denoted by X^L. Analogous to the definition of DCRRE in (4.2), the weighted dynamic cumulative residual Renyi's entropy denoted by $\gamma_w(\alpha;t)$ is defined as

$$\gamma_w(\alpha;t) = \frac{1}{1-\alpha}\log\left(\int_t^\infty \frac{(\bar{F}^w(x))^\alpha}{(\bar{F}^w(t))^\alpha}dx\right) \text{ for } \begin{array}{c}\alpha \neq 1\\ \alpha > 0\end{array}, \tag{4.24}$$

For the length-biased rv X^L, DCRRE is given by

$$\gamma_L(\alpha;t) = \frac{1}{1-\alpha}\log\left(\int_t^\infty \frac{(\bar{F}^L(x))^\alpha}{(\bar{F}^L(t))^\alpha}dx\right),$$

where $\bar{F}^L(t) = \frac{m(t)}{\mu}\bar{F}(t)$ with $m(t) = E(X|X > t)$ denoting the vitality function.

Theorem 4.3.1. *If $E(w(X)|X > x) \leq E(w(X)|X > t)$ for all $x > t$, then $\gamma_w(\alpha;t) \leq (\geq)\gamma(\alpha;t)$ for $0 < \alpha < 1$ $(\alpha > 1)$. If $E(w(X)|X > x) \geq E(w(X)|X > t)$ for all $x > t$, then $\gamma_w(\alpha;t) \geq (\leq)\gamma(\alpha;t)$ for $0 < \alpha < 1$ $(\alpha > 1)$.*

Proof. If $E(w(X)|X > x) \leq E(w(X)|X > t)$ for all $x > t$, then using (4.23) and (4.24) we have

$$\begin{aligned}\gamma_w(\alpha;t) &= \frac{1}{1-\alpha}\log\left(\int_t^\infty \frac{[E(w(X)|X > x)\bar{F}(x)]^\alpha}{[E(w(X)|X > t)\bar{F}(t)]^\alpha}dx\right),\\ &\leq (\geq)\frac{1}{1-\alpha}\log\left(\int_t^\infty \frac{\bar{F}^\alpha(x)}{\bar{F}^\alpha(t)}dx\right) = \gamma(\alpha;t) \text{ for } 0 < \alpha < 1 \quad (\alpha > 1).\end{aligned}$$ □

Corollory 4.3.1. *If $m(x) \leq m(t)$ for all $x > t$, then $\gamma_L(\alpha;t) \leq (\geq)\gamma(\alpha;t)$ for $0 < \alpha < 1$ $(\alpha > 1)$. If $m(x) \geq m(t)$ for all $x > t$, then $\gamma_L(\alpha;t) \geq (\leq)\gamma(\alpha;t)$ for $0 < \alpha < 1$ $(\alpha > 1)$.*

When the weight function is given by $w(t) = \frac{\bar{F}(t)}{f(t)}$ (also called Mill's ratio), the corresponding weighted distribution is called the equilibrium distribution. The equilibrium distribution arises naturally in renewal theory and it is the distribution of the backward and the forward recurrence time in the limiting case. For a recent survey of research on various applications of equilibrium distribution we refer to Gupta and Sankaran (1998), Gupta (2007), Sunoj and Maya (2008) and Nair and Preeth (2008). Let X^E be a rv corresponding to the equilibrium distribution with pdf

$$f^E(t) = \frac{\bar{F}(t)}{\mu}, \quad t > 0,$$

where $\mu = E(X) < \infty$, then DCRRE of X^E is obtained as

$$\gamma_E(\alpha;t) = \frac{1}{(1-\alpha)} \log\left(\int_t^\infty \frac{(\bar{F}^E(x))^\alpha}{(\bar{F}^E(t))^\alpha} dx\right) \text{ for } \begin{array}{c} \alpha \neq 1 \\ \alpha > 0 \end{array}, \tag{4.25}$$

where

$$\bar{F}^E(t) = \frac{r(t)}{\mu}\bar{F}(t).$$

Equation (4.25) can be equivalently written as

$$\gamma_E(\alpha;t) = \frac{1}{(1-\alpha)} \log\left(\int_t^\infty \frac{r^\alpha(x)\bar{F}^\alpha(x)}{r^\alpha(t)\bar{F}^\alpha(t)} dx\right).$$

Theorem 4.3.2. *If $\bar{F}(t)$ is increasing mean residual life (IMRL), then $\gamma_E(\alpha;t) \geq (\leq)\gamma(\alpha;t)$ for $0 < \alpha < 1$ $(\alpha > 1)$. If $\bar{F}(t)$ is decreasing mean residual life (DMRL), then $\gamma_E(\alpha;t) \leq (\geq)\gamma(\alpha;t)$ for $0 < \alpha < 1$ $(\alpha > 1)$.*

Proof. Since $\bar{F}(t)$ is IMRL (DMRL) we have $r(x) \geq (\leq) r(t)$ for all $x > t$, the remaining part is similar to the proof of Theorem 4.3.1. □

Theorem 4.3.3. *The relationship*

$$(1-\alpha)\gamma_E(\alpha;t) = (1-\alpha)\gamma_L(\alpha;t) = \log(Ct), \tag{4.26}$$

where C (> 0) is a constant, holds if and only if X follows Pareto I distribution with sf

$$\bar{F}(t) = \left(\frac{k}{t}\right)^c, t > k, k > 0, c > 1.$$

Proof. Assume that (4.26) holds, now using (4.25), we obtain

$$\log\left(\int_t^\infty \frac{(\bar{F}^E(x))^\alpha}{(\bar{F}^E(t))^\alpha} dx\right) = \log(Ct),$$

equivalently,

$$\int_t^\infty \frac{(\bar{F}^E(x))^\alpha}{(\bar{F}^E(t))^\alpha} dx = Ct. \tag{4.27}$$

Differentiating (4.27) with respect to t, we get

$$\frac{\alpha h^E(t)}{(\bar{F}^E(t))^\alpha} \int_t^\infty (\bar{F}^E(x))^\alpha dx - 1 = C,$$

where $h^E(t) = \frac{f^E(t)}{\bar{F}^E(t)} = \frac{1}{r(t)}$ is the failure rate of X^E. Substituting (4.27) in the above equation, we get $r(t) = \frac{\alpha C}{C+1} t = Pt$, where P (> 0) is a constant, follows Pareto I. The converse part is quite straightforward. □

4.4 Conditional dynamic cumulative residual Renyi's Entropy

In Sections 1.10 and 1.11 of Chapter 1 we discussed briefly on conditionally specified models (identifies bivariate models by using densities of rvs of the form $(X_i|X_j = t_j)$, $i, j = 1, 2$, $i \neq j$) and conditional survival models (identifies bivariate models by using densities of rvs of the form $(X_i|X_j > t_j)$, $i, j = 1, 2$, $i \neq j$). Characterization of the bivariate density given the forms of the marginal density of $X_1(X_2)$ and the conditional density of X_1 given $X_2 = t_2$ (X_2 given $X_1 = t_1$) for certain classes of distributions, have been considered by Seshadri and Patil (1964), Nair and Nair (1988) and Hitha and Nair (1991). On the other hand, Gourieroux and Monfort (1979) have developed conditions under which the conditional densities determine the joint density f uniquely. For more recent works on conditional densities we refer to Sankaran and Nair (2000), Sunoj and Sankaran (2005) and Kotz et al. (2007). Accordingly in following Sections 4.4.1 and 4.4.2, we consider conditional dynamic cumulative residual Renyi's entropies of X_i given $X_j = t_j$ and X_i given $X_j > t_j$, $i, j = 1, 2$, $i \neq j$ respectively and study some characteristic relationships in the context of reliability modeling.

4.4.1 Conditional dynamic cumulative residual Renyi's entropy for X_i given $X_j = t_j$

Let (X_1, X_2) be a bivariate random vector admitting an absolutely continuous pdf f and cdf F with respect to Lesbegue measure in the positive octant $\mathbb{R}_2^+ = \{(t_1, t_2)|t_i > 0, i = 1, 2\}$ of the two-dimensional Euclidean space $\mathbb{R}_2$. Let $\bar{F}_i(t_i|t_j)$, $i, j = 1, 2$, $i \neq j$ denote the sf of X_i given $X_j = t_j$. Then the conditional dynamic cumulative residual Renyi's entropy (CDCRRE) of X_i given $X_j = t_j$ is defined as

$$\gamma_i(\alpha; t_1, t_2) = \frac{1}{(1-\alpha)} \log \left(\int_{t_i}^{\infty} \frac{\bar{F}_i^{\alpha}(x_i|t_j)}{\bar{F}_i^{\alpha}(t_i|t_j)} dx_i \right), \quad i, j = 1, 2, \ i \neq j, \tag{4.28}$$

which can be written as

$$(1-\alpha)\gamma_i(\alpha; t_1, t_2) = \log \left(\int_{t_i}^{\infty} \bar{F}_i^{\alpha}(x_i|t_j) dx_i \right) - \alpha \log \bar{F}_i(t_i|t_j). \tag{4.29}$$

Differentiating (4.29) with respect to t_i, we have

$$(1-\alpha)\frac{\partial}{\partial t_i}\gamma_i(\alpha; t_1, t_2) = \alpha h_i(t_i|t_j) - e^{-(1-\alpha)\gamma_i(\alpha; t_1, t_2)}, \tag{4.30}$$

where $h_i(t_i|t_j)$, $i, j = 1, 2$, $i \neq j$, is the failure rate of X_i given $X_j = t_j$.

Theorem 4.4.1. *The relationship*

$$(1-\alpha)\gamma_i(\alpha; t_1, t_2) = \log[C r_i(t_i|t_j)], \quad i, j = 1, 2, \ i \neq j, \tag{4.31}$$

holds for all t_i and t_j, where $C(> 0)$ is a constant independent of t_i and t_j, $i \neq j$,

$i, j = 1, 2$ and $r_i(t_i|t_j) = E(X_i - t_i|X_i > t_i, X_j = t_j)$ is the MRLF of X_i given $X_j = t_j$, if and only if (X_1, X_2) follows either bivariate distribution with Pareto conditionals given in Arnold (1987) with pdf

$$f(t_1, t_2) = K_1(1 + a_1t_1 + a_2t_2 + bt_1t_2)^{-c}, a_1, a_2 > 0, b \geq 0, c > 2, K_1 > 0, \text{ the normalizing constant}; \ t_1, t_2 > 0, \quad (4.32)$$

or bivariate distribution with exponential conditionals of Arnold and Strauss (1988) with pdf

$$f(t_1, t_2) = K_2 \exp(-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2), \lambda_1, \lambda_2 > 0, \theta \geq 0, K_2 > 0, \text{ the normalizing constant}; \ t_1, t_2 > 0, \quad (4.33)$$

or bivariate distribution with beta conditionals with pdf

$$f(t_1, t_2) = K_3(1 - p_1t_1 - p_2t_2 + qt_1t_2)^d, \ p_1, p_2, d > 0, q \geq 0, K_3 > 0, \text{ the normalizing constant}; \ 0 < t_1 < \frac{1}{p_1}, \ 0 < t_2 < \frac{1 - p_1t_1}{p_2 - qt_1}, \quad (4.34)$$

according as $P \gtreqless 0$, where $P = \left(\frac{C\alpha - 1}{C(1-\alpha)}\right)$.

Proof. Suppose that (4.31) holds, then for $i = 1$, we have

$$\log[Cr_1(t_1|t_2)] = (1 - \alpha)\gamma_1(\alpha; t_1, t_2),$$

which is equivalent to

$$Cr_1(t_1|t_2) = \int_{t_1}^{\infty} \frac{\bar{F}_1^{\alpha}(x_1|t_2)}{\bar{F}_1^{\alpha}(t_1|t_2)} dx_1. \quad (4.35)$$

Differentiating with respect to t_1, (4.35) becomes

$$C\frac{\partial}{\partial t_1}r_1(t_1|t_2) = \frac{\alpha h_1(t_1|t_2)}{\bar{F}_1^{\alpha}(t_1|t_2)}\int_{t_1}^{\infty}\bar{F}_1^{\alpha}(x_1|t_2)dx_1 - 1. \tag{4.36}$$

Using equation (4.35) and the relationship between failure rate and MRLF, (4.36) reduces to

$$C\frac{\partial}{\partial t_1}r_1(t_1|t_2) = C\alpha\left[\frac{\partial}{\partial t_1}r_1(t_1|t_2) + 1\right] - 1,$$

implies that $\frac{\partial}{\partial t_1}r_1(t_1|t_2) = \frac{C\alpha-1}{C(1-\alpha)}$. Now integrating with respect to t_1, we have

$$r_1(t_1|t_2) = \frac{C\alpha - 1}{C(1-\alpha)}t_1 + B_1(t_2) = At_1 + B_1(t_2),$$

where

$$A = \frac{C\alpha - 1}{C(1-\alpha)}.$$

Similarly, for $i = 2$ we have $r_2(t_2|t_1) = At_2 + B_2(t_1)$. Hence $r_i(t_i|t_j) = At_i + B_i(t_j)$, $i \neq j$, $i, j = 1, 2$, where $B_i(t_j)$ is a function of t_j only. Now using Sankaran and Nair (2000), the proof of the theorem follows, according as $A \gtreqless 0$.

Conversely, when (X_1, X_2) follows (4.32), using (4.28), we get

$$\begin{aligned}(1-\alpha)\gamma_i(\alpha; t_1, t_2) &= \log\left[\frac{(c-2)}{(c\alpha - \alpha - 1)}\frac{(1 + a_1t_1 + a_2t_2 + bt_1t_2)}{(c-2)(a_i + bt_j)}\right], \\ &= \log[Cr_i(t_i|t_j)], \quad i \neq j,\ j = 1, 2,\end{aligned}$$

with

$$C = \frac{(c-2)}{(c\alpha - \alpha - 1)} \text{ such that } \frac{C\alpha - 1}{C(1-\alpha)} > 0.$$

When (X_1, X_2) follows (4.33), we have

$$(1-\alpha)\gamma_i(\beta; t_1, t_2) = \log\left(\frac{1}{\beta(\lambda_i + \theta t_j)}\right) = \log[Cr_i(t_i|t_j)], \quad i \neq j,\ j = 1, 2$$

with

$$C = \frac{1}{\alpha} \text{ so that } \frac{C\alpha - 1}{C(1-\alpha)} = 0.$$

Similarly, when (X_1, X_2) follows (4.34) we have

$$\begin{aligned}(1-\alpha)\gamma_i(\alpha; t_1, t_2) &= \log\left[\frac{(d+2)}{(d\alpha + \alpha + 1)} \frac{(1 - p_1 t_1 - p_2 t_2 + q t_1 t_2)}{(d+2)(p_i - q t_j)}\right], \\ &= \log[Cr_i(t_i|t_j)], \quad i \neq j,\ i, j = 1, 2,\end{aligned}$$

with

$$C = \frac{d+2}{d\alpha + \alpha + 1} \text{ implies that } \frac{C\alpha - 1}{C(1-\alpha)} < 0,$$

proves the theorem. □

Theorem 4.4.2. *The relationship*

$$(1-\alpha)\frac{\partial}{\partial t_i}\gamma_i(\alpha; t_1, t_2) = Ch_i(t_i|t_j), \tag{4.37}$$

for all t_i and t_j, where C is a constant independent of t_i and t_j, $i \neq j$, $i, j = 1, 2$ holds if and only if (X_1, X_2) is distributed as (4.32) *when $C > 0$,* (4.33) *when $C = 0$ and* (4.34) *when $-1 < C < 0$.*

Proof. Suppose (4.37) holds, now using (4.30), we get

$$\alpha h_i(t_i|t_j) - e^{-(1-\alpha)\gamma_i(\alpha; t_1, t_2)} = Ch_i(t_i|t_j), \quad i \neq j, i, j = 1, 2.$$

From the definition of CDCRRE (4.28), the above expression becomes

$$(\alpha - C)h_i(t_i|t_j) = \frac{\bar{F}_i^{\alpha}(t_i|t_j)}{\int_{t_i}^{\infty} \bar{F}_i^{\alpha}(x_i|t_j)}dx_i.$$

Equivalently,

$$(\alpha - C)\int_{t_i}^{\infty} \bar{F}_i^{\alpha}(x_i|t_j)dx_i = \frac{\bar{F}_i^{\alpha+1}(t_i|t_j)}{f_i(t_i|t_j)}, \tag{4.38}$$

$$(\alpha - C)f_i(t_i|t_j)\int_{t_i}^{\infty} \bar{F}_i^{\alpha}(x_i|t_j)dx_i = \bar{F}_i^{\alpha+1}(t_i|t_j). \tag{4.39}$$

Differentiating (4.39) with respect to t_i and using (4.38), we have

$$\frac{\partial}{\partial t_i} f_i(t_i|t_j)\frac{\bar{F}_i^{\alpha+1}(t_i|t_j)}{f_i(t_i|t_j)} - (\alpha - C)f_i(t_i|t_j)\bar{F}_i^{\alpha}(t_i|t_j) = -(\alpha+1)\bar{F}_i^{\alpha}(t_i|t_j)f_i(t_i|t_j). \tag{4.40}$$

Dividing equation (4.40) by $\bar{F}_i^{\alpha}(t_i|t_j)f_i(t_i|t_j)$ and simplifying, we obtain

$$\frac{\partial}{\partial t_i}\log f_i(t_i|t_j) = (C+1)\frac{\partial}{\partial t_i}\log \bar{F}_i(t_i|t_j),$$

Equivalently,

$$\frac{\partial}{\partial t_i}\log h_i(t_i|t_j) = C\frac{\partial}{\partial t_i}\log \bar{F}_i(t_i|t_j). \tag{4.41}$$

Integrating (4.41) with respect to t_i, we get

$$\log h_i(t_i|t_j) = C\log \bar{F}_i(t_i|t_j) + K_i(t_j).$$

Differentiating the above equation with respect to t_i and rearranging, the above equation becomes

$$\frac{\partial}{\partial t_i}\left[\frac{1}{h_i(t_i|t_j)}\right] = C,$$

which on integration with respect to t_i, gives

$$\frac{1}{h_i(t_i|t_j)} = Ct_i + D_i(t_j). \tag{4.42}$$

From the definition of $h_i(t_i|t_j) = \frac{f_i(t_i|t_j)}{\bar{F}_i(t_i|t_j)} = -\frac{f(t_1,t_2)}{\frac{\partial}{\partial t_j}\bar{F}(t_1,t_2)}$, (4.42) becomes

$$-\frac{\partial}{\partial t_j}\bar{F}(t_1,t_2) = f(t_1,t_2)[Ct_i + D_i(t_j)].$$

Differentiating with respect to t_i and simplifying, we get

$$\frac{\partial}{\partial t_i}\log f(t_1,t_2) = -\frac{(C+1)}{[Ct_i + D_i(t_j)]}.$$

Now on integrating with respect to t_i, we have

$$\log f(t_1,t_2) = -\frac{(C+1)}{C}\log[Ct_i + D_i(t_j)] + \log m_i(t_j).$$

Equivalently,

$$f(t_1,t_2) = m_i(t_j)[Ct_i + D_i(t_j)]^{-\frac{(C+1)}{C}}, \quad C \neq 0,\ i \neq j,\ i,j = 1,2. \tag{4.43}$$

Applying for $i = 1,2$ and equating, we obtain

$$m_1(t_2)[Ct_1 + D_1(t_2)]^{-\frac{(C+1)}{C}} = m_2(t_1)[Ct_2 + D_2(t_1)]^{-\frac{(C+1)}{C}}. \tag{4.44}$$

As $t_1 \to 0$, (4.44) becomes

$$m_1(t_2) = \frac{m_2(0)[Ct_2 + D_2(0)]^{-\frac{(C+1)}{C}}}{[D_1(t_2)]^{-\frac{(C+1)}{C}}}.$$

Similarly, as $t_2 \to 0$, (4.44) becomes

$$m_2(t_1) = \frac{m_1(0)[Ct_1 + D_1(0)]^{-\frac{(C+1)}{C}}}{[D_2(t_1)]^{-\frac{(C+1)}{C}}}.$$

Substituting for $m_1(t_2)$ and $m_2(t_1)$, (4.44) becomes

$$\frac{m_2(0)[Ct_2 + D_2(0)]^{-\frac{(C+1)}{C}}}{[D_1(t_2)]^{-\frac{(C+1)}{C}}}[Ct_1 + D_1(t_2)]^{-\frac{(C+1)}{C}} = \frac{m_1(0)[Ct_1 + D_1(0)]^{-\frac{(C+1)}{C}}}{[D_2(t_1)]^{-\frac{(C+1)}{C}}}[Ct_2 + D_2(t_1)]^{-\frac{(C+1)}{C}}. \quad (4.45)$$

For $i = 1$ in (4.43) and as $t_1 \to 0$, we get

$$\lim_{t_1 \to 0} f(t_1, t_2) = m_1(t_2)[D_1(t_2)]^{-\frac{(C+1)}{C}}. \quad (4.46)$$

Similarly for $i = 2$ and as $t_1 \to 0$, we have

$$\lim_{t_1 \to 0} f(t_1, t_2) = m_2(0)[Ct_2 + D_2(0)]^{-\frac{(C+1)}{C}}. \quad (4.47)$$

Equating (4.46) and (4.47), we obtain

$$m_1(t_2)[D_1(t_2)]^{-\frac{(C+1)}{C}} = m_2(0)[Ct_2 + D_2(0)]^{-\frac{(C+1)}{C}}. \quad (4.48)$$

As $t_2 \to 0$, (4.48) becomes

$$\frac{m_1(0)}{m_2(0)} = \frac{[D_2(0)]^{-\frac{(C+1)}{C}}}{[D_1(0)]^{-\frac{(C+1)}{C}}}.$$

Then equation (4.45) becomes

$$\frac{[Ct_2 + D_2(0)]^{-\frac{(C+1)}{C}}[Ct_1 + D_1(t_2)]^{-\frac{(C+1)}{C}}}{[D_1(t_2)]^{-\frac{(C+1)}{C}}}$$

$$= \frac{[D_2(0)]^{-\frac{(C+1)}{C}}}{[D_1(0)]^{-\frac{(C+1)}{C}}} \frac{[Ct_1 + D_1(0)]^{-\frac{(C+1)}{C}} [Ct_2 + D_2(t_1)]^{-\frac{(C+1)}{C}}}{[D_2(t_1)]^{-\frac{(C+1)}{C}}}.$$

Equivalently, we get

$$\frac{1}{t_1 D_2(t_1)} - \frac{1}{t_1 D_2(0)} + \frac{C}{D_2(t_1) D_1(0)} = \frac{1}{t_2 D_1(t_2)} - \frac{1}{t_2 D_1(0)} + \frac{C}{D_1(t_2) D_2(0)}. \tag{4.49}$$

Since (4.49) is true for all $t_1, t_2 \geq 0$, we may take both sides of (4.49) equal to n, where n is a constant. Using the expression of $m_1(t_2)$, the joint pdf $f(t_1, t_2)$ in (4.43) for $i = 1$ becomes,

$$f(t_1, t_2) = \frac{m_2(0)[Ct_2 + D_2(0)]^{-\frac{(C+1)}{C}}}{[D_1(t_2)]^{-\frac{(C+1)}{C}}} [Ct_1 + D_1(t_2)]^{-\frac{(C+1)}{C}}$$

or

$$f(t_1, t_2) = m_2(0)(D_2(0))^{-\frac{(C+1)}{C}} \left(1 + \frac{Ct_2}{D_2(0)}\right)^{-\frac{(C+1)}{C}} \left(1 + \frac{Ct_1}{D_1(t_2)}\right)^{-\frac{(C+1)}{C}}. \tag{4.50}$$

Now using (4.49) and substituting for $1 + \frac{Ct_1}{D_1(t_2)}$, the joint pdf $f(t_1, t_2)$ in (4.50) becomes

$$f(t_1, t_2) = m_2(0)[D_2(0)]^{-\frac{(C+1)}{C}} \left[1 + \frac{Ct_1}{D_1(0)} + \frac{Ct_2}{D_2(0)} + nCt_1 t_2\right]^{-\frac{(C+1)}{C}}, \tag{4.51}$$

which is of the form (4.32) with $K_1 = m_2(0)[D_2(0)]^{-\frac{(C+1)}{C}}$, $a_1 = \frac{C}{D_1(0)}$, $a_2 = \frac{C}{D_2(0)}$, $b = nC$ and $c = \frac{C+1}{C}$. If $C > 0$, since $D_i(t_j)$ is a non-negative function of t_j we have $K_1, a_1, a_2, b > 0$. Similarly, if $-1 < C < 0$, equation (4.51) takes the form (4.34) with $K_3, p_1, p_2 > 0$, $d > 0$, $0 < t_1 < \frac{1}{p_1}$ and $0 < t_2 < \frac{1 - p_1 t_1}{p_2 - q t_1}$. When $C = 0$ from (4.42), we get

$$h_i(t_i|t_j) = \frac{1}{D_i(t_j)},$$

following the similar steps, we obtain

$$-\log f(t_1, t_2) = \frac{t_i}{D_i(t_j)} + Q_i(t_j),$$

where $Q_i(t_j)$ is a function of t_j only, $i \neq j$, $i, j = 1, 2$. Equivalently, we have

$$f(t_1, t_2) = e^{-\left[\frac{t_i}{D_i(t_j)} + Q_i(t_j)\right]}, i \neq j,\ i, j = 1, 2. \tag{4.52}$$

For $i = 1, 2$ and equating, (4.52) becomes a functional equation

$$\frac{t_1}{D_1(t_2)} + Q_1(t_2) = \frac{t_2}{D_2(t_1)} + Q_2(t_1),$$

which gives the solution as $D_1(t_2) = \frac{1}{\lambda_1 + \theta t_2}$ and $D_2(t_1) = \frac{1}{\lambda_2 + \theta t_1}$. Then $Q_1(t_2) = Q_2 + \lambda_2 t_2$ and $Q_2(t_1) = Q_1 + \lambda_1 t_1$, where $\lambda_1, \lambda_2, \theta$ are non-negative constants and $Q_i = Q_i(0)$, $i = 1, 2$. Substituting these in (4.52), we have (4.33). The converse part is straightforward. □

Theorem 4.4.3. $\gamma_i(\alpha; t_1, t_2)$, $i = 1, 2$ *is locally constant (i.e.,* $\gamma_i(\alpha; t_1, t_2)$ *is a function of* t_j *only) if and only if* (X_1, X_2) *follows bivariate distribution with exponential conditionals of Arnold and Strauss (1988) with pdf* (4.33).

Proof. Let $\gamma_i(\alpha; t_1, t_2)$, $i = 1, 2$ be locally constant. This implies that $\frac{\partial}{\partial t_i}\gamma_i(\alpha; t_1, t_2) = 0$, or $(1 - \alpha)\frac{\partial}{\partial t_i}\gamma_i(\alpha; t_1, t_2) = 0$. Now rest of the proof follows from Theorem 4.4.2. □

4.4.2 Conditional dynamic cumulative residual Renyi's entropy for X_i given $X_j > t_j$

Let (X_1, X_2) be a bivariate random vector admitting an absolutely continuous pdf f and cdf F with respect to Lesbegue measure in the positive octant $\mathbb{R}_2^+ = \{(t_1, t_2) | t_i >$

$0,\ i = 1,2\}$ of the two dimensional Euclidean space $\mathbb{R}_2$. Let the sf of X_i given $X_j > t_j$ be $\bar{F}_i^*(t_i|t_j),\ i,j = 1,2,\ i \neq j$. Now using (4.2) the CDCRRE (Conditional dynamic cumulative residual Renyi's entropy) of X_i given $X_j > t_j$ turns out to be

$$\gamma_i^*(\alpha; t_1, t_2) = \frac{1}{1-\alpha} \log \left(\int_{t_i}^{\infty} \frac{(\bar{F}_i^*(x_i|t_j))^{\alpha}}{(\bar{F}_i^*(t_i|t_j))^{\alpha}} dx_i \right), \tag{4.53}$$

which can be written as

$$(1-\alpha)\gamma_i^*(\alpha; t_1, t_2) = \log \left(\int_{t_i}^{\infty} (\bar{F}_i^*(x_i|t_j))^{\alpha} dx_i \right) - \alpha \log \bar{F}_i^*(t_i|t_j). \tag{4.54}$$

Differentiating with respect to t_i, (4.54) becomes

$$(1-\alpha)\frac{\partial}{\partial t_i}\gamma_i^*(\alpha; t_1, t_2) = \alpha h_i^*(t_i|t_j) - e^{-(1-\alpha)\gamma_i^*(\alpha; t_1, t_2)},$$

where

$$h_i^*(t_i|t_j) = -\frac{\partial}{\partial t_i} \log \bar{F}_i^*(t_i|t_j) = -\frac{\partial}{\partial t_i} \log \bar{F}(t_1, t_2) = h_i(t_1, t_2),$$

$i, j = 1,2,\ i \neq j,\ i^{th}$ component of the vector-valued failure rate due to Johnson and Kotz (1975).

$\gamma_i^*(\alpha; t_1, t_2)$ is obtained for bivariate Pareto I and bivariate Weibull is the following examples.

Examples.

(a) If (X_1, X_2) is distributed as bivariate Pareto I with joint sf $\bar{F}(t_1, t_2) = t_1^{-\rho_1} t_2^{-\rho_2} t_1^{-\theta \log t_2}$; $\rho_1, \rho_2, \theta > 0,\ t_1, t_2 > 1$, then

$$\gamma_i^*(\alpha; t_1, t_2) = \frac{1}{1-\alpha}\left[\log t_i - \log(\alpha(\rho_i + \theta \log t_j) - 1)\right],$$

$i, j = 1, 2,\ i \neq j.$

(b) If (X_1, X_2) follows bivariate Weibull with joint sf $\bar{F}(t_1, t_2) = e^{-l_1 t_1^a - l_2 t_2^a - \theta t_1^a t_2^a}$; $t_1, t_2 > 0$, $l_1, l_2, \theta,\ a > 0$, then

$$\gamma_i^*(\alpha; t_1, t_2) = \frac{1}{1-\alpha}\left[\log(\frac{1}{a}(\alpha(l_i + \theta t_j^a))^{-1/a}\Gamma(\frac{1}{a}, \alpha(l_i + \theta t_j^\alpha)t_i^a)) - \alpha(l_i + \theta t_j^a)t_i^a\right],$$

$i, j = 1, 2,\ i \neq j.$

Theorem 4.4.4. *The relationship*

$$(1-\alpha)\gamma_i^*(\alpha; t_1, t_2) = \log[C^* r_i^*(t_i|t_j)], \tag{4.55}$$

holds for all t_i *and* t_j*, where* $C^*(>0)$ *is a constant independent of* t_i *and* t_j*,* $i \neq j$*,* $i, j = 1, 2$ *and*

$$r_i^*(t_i|t_j) = E(X_i - t_i|X_i > t_i, X_j > t_j) = r_i(t_1, t_2)$$

is the i^{th} *component of vector-valued MRLF in the bivariate case, if and only if* (X_1, X_2) *follows either bivariate Pareto II with joint sf*

$$\bar{F}(t_1, t_2) = (1 + a_1 t_1 + a_2 t_2 + b t_1 t_2)^{-c};\ a_1, a_2 > 0, c > 1,$$
$$0 \leq b \leq (c+1)a_1 a_2;\ t_1, t_2 > 0, \tag{4.56}$$

or Gumbel's bivariate exponential with joint sf

$$\bar{F}(t_1, t_2) = e^{-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2};\ \lambda_1, \lambda_2 > 0, 0 \leq \theta < \lambda_1\lambda_2;\ t_1, t_2 > 0, \tag{4.57}$$

or bivariate finite range with joint sf

$$\bar{F}(t_1,t_2) = (1 - p_1t_1 - p_2t_2 + qt_1t_2)^d;\ p_1,p_2,d > 0,$$
$$1 - d \le \frac{q}{p_1p_2} \le 1;\ 0 < t_1 < \frac{1}{p_1},\ 0 < t_2 < \frac{1-p_1t_1}{p_2 - qt_1}, \quad (4.58)$$

according as $P^* \gtreqless 0$ *where* $P^* = \frac{C^*\alpha - 1}{C^*(1-\alpha)}$.

Proof. Assume that (4.55) holds, then using (4.53) and applying the similar steps as in Theorem 4.4.1, we obtain

$$r_i^*(t_i|t_j) = \frac{C^*\alpha - 1}{C^*(1-\alpha)}t_i + B_i(t_j) = At_i + B_i(t_j),$$

where $A = \frac{C^*\alpha-1}{C^*(1-\alpha)}$ and $B_i(t_j)$ is a function of t_j only, $i \neq j,\ i,j = 1,2$. Now using a characterization theorem in Sankaran and Nair (1993a), (X_1, X_2) follows bivariate Pareto II with sf (4.56) when $A > 0$, Gumbel's exponential with sf (4.57) when $A = 0$, and bivariate finite range with sf (4.58) when $A < 0$.

Conversely, when (X_1, X_2) follows bivariate Pareto II with sf (4.56), using (4.53), we have

$$\begin{aligned}(1-\alpha)\gamma_i^*(\alpha;t_1,t_2) &= \log\left(\int_{t_i}^{\infty} \frac{(1 + a_ix_i + a_jt_j + bx_it_j)^{-c\alpha}}{(1 + a_1t_1 + a_2t_2 + bt_1t_2)^{-c\alpha}}dx_i\right), \\ &= \log\left[\frac{(c-1)}{(c\alpha-1)}\frac{(1 + a_1t_1 + a_2t_2 + bt_1t_2)}{(c-1)(a_i + bt_j)}\right] = \log(C^* r_i^*(t_i|t_j)),\end{aligned}$$

we have $C^* = \frac{c-1}{c\alpha-1}$, so that $P^* = \frac{C^*\alpha-1}{C^*(1-\alpha)} > 0$. When (X_1, X_2) follows Gumbel's exponential with sf (4.57), then

$$(1-\alpha)\gamma_i^*(\alpha;t_1,t_2) = \log\left(\int_{t_i}^{\infty}\frac{(e^{-\lambda_i x_i-\lambda_j t_j-\theta x_i t_j})^{\alpha}}{(e^{-\lambda_1 t_1-\lambda_2 t_2-\theta t_1 t_2})^{\alpha}}dx_i\right)$$
$$= \log\left(\frac{1}{\alpha(\lambda_i+\theta t_j)}\right) = \log(C^* r_i^*(t_i|t_j)),$$

where $C^* = \frac{1}{\alpha}$, such that

$$P^* = \frac{C^*\alpha - 1}{C^*(1-\alpha)} = 0.$$

Finally, when (X_1, X_2) follows bivariate finite range with sf (4.58), we have

$$(1-\alpha)\gamma_i^*(\alpha;t_1,t_2) = \log\left(\int_{t_i}^{\frac{1-p_j t_j}{p_i - q t_j}}\frac{(1-p_i x_i - p_j t_j + q x_i t_j)^{d\alpha}}{(1-p_1 t_1 - p_2 t_2 + q t_1 t_2)^{d\alpha}}dx_i\right),$$
$$= \log\left[\frac{(d+1)}{(d\alpha+1)}\frac{(1-p_1 t_1 - p_2 t_2 + q t_1 t_2)}{(d+1)(p_i - q t_j)}\right] = \log(C^* r_i^*(t_i|t_j)),$$

where $C^* = \frac{d+1}{d\alpha+1}$ such that $P^* = \frac{C^*\alpha-1}{C^*(1-\alpha)} < 0$, proves the theorem. □

Theorem 4.4.5. *The relationship*

$$(1-\alpha)\frac{\partial}{\partial t_i}\gamma_i^*(\alpha;t_1,t_2) = C h_i^*(t_i|t_j), \tag{4.59}$$

for all t_i and t_j, where C is a constant independent of t_i and t_j, $i \neq j$, $i,j = 1,2$ holds if and only if (X_1, X_2) is distributed as bivariate Pareto II with sf (4.56) when $C > 0$, Gumbel's exponential with sf (4.57) when $C = 0$ and bivariate finite range with sf (4.58) when $C < 0$.

Proof. Assume that (4.59) holds, then using (4.53) and applying the similar steps as in Theorem 4.4.2, we get

$$h_i^*(t_i|t_j) = \frac{1}{Ct_i + D_i(t_j)}, \text{ or } h_i(t_1,t_2) = \frac{1}{Ct_i + D_i(t_j)}.$$

Now characterization to (4.56), (4.57) and (4.58) follows from Roy (1989). The converse of the theorem can be easily proved. □

Theorem 4.4.6. $\gamma_i^*(\alpha; t_1, t_2)$ *is locally constant if and only if* (X_1, X_2) *follows Gumbel's bivariate exponential with sf* (4.57).

Proof. Since $\gamma_i^*(\alpha; t_1, t_2)$ is locally constant,

$$(1-\alpha)\frac{\partial}{\partial t_i}\gamma_i^*(\alpha; t_1, t_2) = 0.$$

Rest of the proof follows from Theorem 4.4.5. □

Chapter 5

Characterizations of bivariate models using certain dynamic information measures

5.1 Introduction

In the previous chapters we came across some important information measures like Kullback-Leibler Divergence measure, Renyi's divergence measure and Kerridge's inaccuracy. Studies on these measures and their dynamic forms have been done by several researchers in the past decades, but a little work could be found on these measures in the bivariate conditional set up. In two-component reliability systems, where the operational status of one is known in advance, there comes the importance of conditionally specified and conditional survival models. A brief discussion on these models is available

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2. *Navarro, J., Sunoj, S. M. and Linu, M. N. (2014). "Characterizations of bivariate models using some dynamic conditional information divergence measures", Communications in Statistics-Theory and Methods, USA, 43(9), 1939–1948.*

in the Sections 1.10 and 1.11 of Chapter 1. Motivated by the usefulness of these conditional models, the dynamic versions of information measures *viz.* Kullback-Leibler divergence, Renyi's divergence and Kerridge's inaccuracy are extended to conditionally specified and conditional survival models, and studied its usefulness in identifying bivariate distributions and to obtain some bounds for these measures using likelihood ratio ordering.

5.2 Conditional Kullback-Leibler discrimination of type 1

In this section dynamic Kullback-Leibler divergence measure proposed by Ebrahimi and Kirmani (1996b) is extended to conditionally specified models called conditional Kullback-Leibler discrimination of type 1 (CKLD_1), the definition to which is as follows:

Definition 5.2.1. *Let (X_1, X_2) and (Y_1, Y_2) be two bivariate random vectors with joint pdfs f and g, joint cdfs F and G, joint sfs $\bar{F}$ and $\bar{G}$ respectively. Let us assume that the common support is $S = (l, \infty) \times (l, \infty)$ for $l \geq 0$. Also let $f_i(t_i|t_j)$ and $g_i(t_i|t_j)$, $\bar{F}_i(t_i|t_j)$ and $\bar{G}_i(t_i|t_j)$ denote the pdfs and the sfs of $(X_i|X_j = t_j)$ and $(Y_i|Y_j = t_j)$ respectively for $i, j = 1, 2, i \neq j$. Then we define the conditional Kullback-Leibler discrimination of type 1 ($CKLD_1$) information function as*

$$I_{X_i,Y_i}(t_1, t_2) = \int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log \frac{f_i(x_i|t_j)\bar{G}_i(t_i|t_j)}{g_i(x_i|t_j)\bar{F}_i(t_i|t_j)} dx_i,$$

for $i, j = 1, 2,\ i \neq j$ and $t_1, t_2 \geq l$.

Note that,

$$I_{X_i,Y_i}(t_1, t_2) = I_{(X_i|X_j=t_j),(Y_i|Y_j=t_j)}(t_i), \tag{5.1}$$

for $i, j = 1, 2,\ i \neq j$ and $t_1, t_2 \geq l$. Hence $I_{X_i,Y_i}(t_1, t_2)$ is the dynamic Kullback-Leibler discrimination measure at time t_i defined by Ebrahimi and Kirmani (1996b) but applied to the conditional rvs $(X_i|X_j = t_j)$ and $(Y_i|Y_j = t_j)$ for $i = 1, 2,\ i \neq j$. As in the univariate case, this function measures the information distance between the conditional distributions of the residual lifetimes of the two random vectors. In the bivariate case, other interesting options are also available (see Ebrahimi et al. (2007)).

In survival studies, the most widely used semi-parametric model is the Cox proportional hazard rate (PHR) model. Let X and Y be two rvs with the same support S and with the hazard rate functions $h_X = f/\bar{F}$ and $h_Y = g/\bar{G}$, respectively. Then X and Y satisfy the PHR model when, for $\theta > 0$,

$$h_Y(t) = \theta h_X(t),$$

for all $t \in S$. This relationship is also equivalent to

$$\bar{G}(t) = [\bar{F}(t)]^{\theta},$$

for all t (see Cox (1959)). Ebrahimi and Kirmani (1996a) obtained the following result.

Theorem 5.2.1. *(Ebrahimi and Kirmani (1996a)) The function $I_{X,Y}(t)$ is a constant if and only if X and Y satisfy PHR model.*

In a similar way the random vectors (X_1, X_2) and (Y_1, Y_2) satisfy conditional proportional hazard rate (CPHR) model (see Sankaran and Sreeja (2007)) when their respective conditional hazard rate functions satisfy

$$h_{(Y_i|Y_j)}(t_i|t_j) = \theta_i(t_j) h_{(X_i|X_j)}(t_i|t_j), \tag{5.2}$$

for $i, j = 1, 2$, $i \neq j$, where $\theta_i(t_j)$ is a non-negative function of t_j. Then we can state the result as follows.

Theorem 5.2.2. *For $i, j = 1, 2$, $i \neq j$, the function $I_{X_i,Y_i}(t_1, t_2)$ depends only on t_j if and only if $(Y_i|Y_j = t_j)$ and $(X_i|X_j = t_j)$ satisfy the CPHR model* (5.2).

Proof. The proof is obtained from Theorem 5.2.1 and (5.1). □

Now let us consider the random vector (X_1^w, X_2^w) which has the bivariate weighted distribution associated to (X_1, X_2) and to two non-negative real functions w_1 and w_2, *i.e.,* its joint pdf is

$$f^w(x_1, x_2) = \frac{w_1(x_1)w_2(x_2)f(x_1, x_2)}{E(w_1(X_1)w_2(X_2))}, \tag{5.3}$$

where f is the joint pdf of (X_1, X_2) and $E(w_1(X_1)w_2(X_2)) < \infty$.
Integrating (5.3) with respect to x_j over (l, ∞) yields

$$\begin{aligned} f_i^w(x_i) &= \int_l^\infty \frac{w_i(x_i)w_j(x_j)f(x_1, x_2)}{E(w_1(X_1)w_2(X_2))} dx_j, \\ &= \frac{w_i(x_i)E(w_j(X_j)|X_i = x_i)f_i(x_i)}{E(w_1(X_1)w_2(X_2))}. \end{aligned}$$

i.e., the marginal rv X_i^w has the univariate weighted distribution associated to X_i and $w_i^\bullet(x_i) = w_i(x_i)E(w_j(X_j)|X_i = x_i)$ for $i, j = 1, 2, i \neq j$. Analogously it is easy to prove that $X_i^w|X_j^w = x_j$ has (univariate) weighted distribution associated to $X_i|X_j = x_j$ and $w_i(x_i)$ for $i, j = 1, 2, i \neq j$. Particularly, when $w_1(x_1) = x_1$ and $w_2(x_2) = x_2$, the random vector (X_1^w, X_2^w) is called length-biased random vector. There are other options in defining the bivariate weighted distribution which can be found in Navarro et al. (2006). Now we can state the main result of this section as follows.

Theorem 5.2.3. *Let (X_1^w, X_2^w) be a random vector which has the bivariate weighted distribution associated to (X_1, X_2) and to two non-negative and differentiable functions*

w_1 and w_2. Let us assume that the support of (X_1, X_2) is $S = (l, \infty) \times (l, \infty)$ for $l \geq 0$. Then the following conditions are equivalent:

(a) (X_1^w, X_2^w) and (X_1, X_2) satisfy the CPHR model (5.2).

(b) $I_{X_i, X_i^w}(t_1, t_2)$ depends only on t_j for $i, j = 1, 2,\ i \neq j$.

(c) The conditional reliability functions of (X_1, X_2) satisfy

$$\log \bar{F}_i(t_i|t_j) = \frac{\log(w_i(t_i)/w_i(l))}{\theta_i(t_j) - 1},$$

for $i, j = 1, 2,\ i \neq j$.

(d) (X_1, X_2) has the following pdf

$$f(x_1, x_2) = ca_1a_2\frac{w_1'(x_1)w_2'(x_2)}{w_1^{a_1+1}(x_1)w_2^{a_2+1}(x_2)}\exp\left(-\phi a_1a_2\left(\log\frac{w_1(x_1)}{w_1(l)}\right)\left(\log\frac{w_2(x_2)}{w_2(l)}\right)\right),$$

for $x_1, x_2 \geq l$, where $c > 0,\ \phi \geq 0$ and $a_i > 1$ or $a_i < 0$ for $i = 1, 2$.

Proof. The equivalence between (a) and (b) is the consequence of Theorem 5.2.2. Let us prove that (a) implies (c). So let us assume that (X_1^w, X_2^w) and (X_1, X_2) satisfy the CPHR model (5.2) for $i, j = 1, 2,\ i \neq j$. From the expression of the pdf of (X_1^w, X_2^w) given in (5.3), it is easy to prove that the pdf of $(X_i^w|X_j^w = t_j)$ is given by

$$f_i^w(t_i|t_j) = \frac{w_i(t_i)f_i(t_i|t_j)}{E(w_i(X_i)|X_j = t_j)},$$

for $i, j = 1, 2,\ i \neq j$, where $f_i(t_i|t_j)$ is the pdf of $(X_i|X_j = t_j)$. Then the hazard rate $h_{X_i^w|X_j^w}(t_i|t_j)$ of $(X_i^w|X_j^w = t_j)$ is given by

$$h_{X_i^w|X_j^w}(t_i|t_j) = \frac{w_i(t_i)f_i(t_i|t_j)}{\int_{t_i}^{\infty} w_i(x_i)f_i(x_i|t_j)dx_i}. \tag{5.4}$$

Moreover, from (5.2), we have

$$h_{X_i^w|X_j^w}(t_i|t_j) = \theta_i(t_j) h_{X_i|X_j}(t_i|t_j)$$

and hence

$$\frac{w_i(t_i) f_i(t_i|t_j)}{\int_{t_i}^{\infty} w_i(x_i) f_i(x_i|t_j) dx_i} = \theta_i(t_j) \frac{f_i(t_i|t_j)}{\bar{F}_i(t_i|t_j)}.$$

Therefore,

$$\frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} w_i(x_i) f_i(x_i|t_j) dx_i = \frac{1}{\theta_i(t_j)} w_i(t_i).$$

Then differentiating both sides with respect to t_i, we obtain

$$-w_i(t_i) f_i(t_i|t_j) = \frac{1}{\theta_i(t_j)} \left(w_i'(t_i) \bar{F}_i(t_i|t_j) - w_i(t_i) f_i(t_i|t_j) \right),$$

that is,

$$h_{X_i|X_j}(t_i|t_j) = \frac{w_i'(t_i)}{(1 - \theta_i(t_j)) w_i(t_i).}$$

Hence

$$\log \bar{F}_i(t_i|t_j) = -\int_{l}^{t_i} h_{X_i|X_j}(t_i|t_j) dx_i = \frac{1}{(\theta_i(t_j) - 1)} \log \frac{w_i(t_i)}{w_i(l)},$$

for $i, j = 1, 2$, $i \neq j$ and (c) holds.

Let us prove that (c) is equivalent to (d). The expression given in (c) is equivalent to

$$h_{X_i|X_j}(t_i|t_j) = \frac{w_i'(t_i)/w_i(t_i)}{1 - \theta_i(t_j)},$$

for $i, j = 1, 2, i \neq j$, *i.e.*, $(X_1|X_2 = t_2)$ and $(X_2|X_1 = t_1)$ satisfy the conditional proportional hazard rate model considered by Arnold and Strauss (1991) which is equivalent to (d).

Finally, let us prove that (d) implies (a). From the expression of the joint pdf given in (d) it is easy to prove that the conditional hazard rate functions are given by

$$h_{X_i|X_j}(t_i|t_j) = a_i\left(1 - \phi a_j \log\frac{w_j(t_j)}{w_i(l)}\right)\frac{w_i'(t_i)}{w_i(t_i)},$$

for $i, j = 1, 2,\ i \neq j$. Moreover, the weighted version associated to w_1 and w_2 has the following joint pdf

$$f^w(x_1, x_2) = c\frac{w_1'(x_1)w_2'(x_2)}{w_1^{a_1}(x_1)w_2^{a_2}(x_2)}\exp\left(-\phi a_1 a_2\left(\log\frac{w_1(x_1)}{w_1(l)}\right)\left(\log\frac{w_2(x_2)}{w_2(l)}\right)\right),$$

which is also a model included in the type of the pdf given in (d) with parameters $a_1 - 1$ and $a_2 - 1$. Therefore, its hazard rate functions are

$$h_{X_i^w|X_j^w}(t_i|t_j) = (a_i - 1)\left(1 - \phi'(a_j - 1)\log\frac{w_j(t_j)}{w_j(l)}\right)\frac{w_i'(t_i)}{w_i(t_i)},$$

for $i, j = 1, 2,\ i \neq j$. Hence (a) holds.

Note that the condition (c) given in Theorem 5.2.3 implies that either $\log(w_i(t_i)/w_i(l))$ or $-\log(w_i(t_i)/w_i(l))$ should be cumulative hazard rate functions, *i.e.,* they should be non-negative, increasing and they should go to ∞ when t_i goes to ∞. In the first case, w_i should be increasing in $[l, \infty)$ with $w_i(l) > 0$ and $w_i(\infty) = \infty$. In the second case, w_i should be decreasing, with $w_i(t_i) > 0$ for $t_i \in [l, \infty)$ and $w_i(\infty) = 0$. These conditions can also be written as

$$h_{X_i|X_j}(t_i|t_j) = \frac{w_i'(t_i)/w_i(t_i)}{1 - \theta_i(t_j)},$$

for $i, j = 1, 2,\ i \neq j,$ *i.e.,* $(X_1|X_2 = t_2)$ and $(X_2|X_1 = t_1)$ satisfy the conditional proportional hazard rate model considered by Arnold and Strauss (1991). The reliability properties of this semi-parametric model can be seen in Navarro and Sarabia (2013). Actually, the model in (d) is just a truncated version of Arnold and Strauss model in the support $S = (l, \infty) \times (l, \infty)$ and when $l = 0$ both models coincide. Again we have two options, in the first one, $\lambda_i(t_i) = w_i'(t_i)/w_i(t_i)$ is a proper hazard rate function and, in the second one, $\lambda_i(t_i) = -w_i'(t_i)/w_i(t_i)$ is a proper hazard rate function. In the first option, we need $a_i > 1$ and in the second one $a_i < 0$, for $i, j = 1, 2,\ i \neq j$. The model in (d) contains several parametric models. In particular, when $l = 1$ and $w_1(x) = w_2(x) = x$ for $x > 1$, from Theorem 5.2.3, we can characterize the bivariate Pareto model with the following joint pdf

$$f(x_1, x_2) = \frac{c a_1 a_2}{x_1^{a_1+1} x_2^{a_2+1}} \exp(-\phi a_1 a_2 (\log x_1)(\log x_2)),$$

for $x_1, x_2 \geq 1$, where $c > 0$, $a_1, a_2 > 1$ and $\phi \geq 0$. □

We end this section by obtaining some bounds for the CKLD_1 functions using the likelihood ratio (LR) order. $X_i|X_j = t_j$ is said to be smaller than $Y_i|Y_j = t_j$ in likelihood ratio $((X_i|X_j = t_j) \leq_{LR} (Y_i|Y_j = t_j))$ if $\frac{f_i(x_i|t_j)}{g_i(x_i|t_j)}$ is decreasing in x_i, $\forall t_j$.

Theorem 5.2.4. *For $i = 1, 2,\ i \neq j$ if $(X_i|X_j = t_j) \leq_{LR} (Y_i|Y_j = t_j)$, then*

$$I_{X_i,Y_i}(t_i, t_j) \leq \log \frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}.$$

Proof. Since $X_i|X_j = t_j \leq_{\text{LR}} Y_i|Y_j = t_j$ we have $\frac{f_i(x_i|t_j)}{g_i(x_i|t_j)}$ is decreasing in x_i, $\forall\ t_j$

$$i.e., \quad \frac{f_i(x_i|t_j)}{g_i(x_i|t_j)} \leq \frac{f_i(t_i|t_j)}{g_i(t_i|t_j)}, \quad \forall x_i > t_i,\ \forall t_j.$$

$$\text{Now} \quad I_{X_i,Y_i}(t_1,t_2) = \frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} f_i(x_i|t_j) \log\left[\frac{f_i(x_i|t_j)\bar{G}_i(t_i|t_j)}{\bar{F}_i(t_i|t_j) g_i(x_i|t_j)}\right] dx_i,$$
$$\leq \frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} f_i(x_i|t_j) \log\left[\frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}\right] dx_i,$$
$$= \log\left[\frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}\right]. \qquad \square$$

Corollary 5.2.1. *For $i, j = 1, 2,\ i \neq j$, if $X_i|X_j = t_j \leq_{LR} X_i^w|X_j^w = t_j$ then*

$$I_{X_i,X_i{}^w}(t_1,t_2) \leq \log\left[\frac{E(w_i(X_i)|X_i > t_i, X_j = t_j)}{w_i(t_i)}\right].$$

Theorem 5.2.5. *For $i, j = 1, 2,\ i \neq j$, if w_i is increasing, then*

$$I_{X_i,X_i{}^w}(t_1,t_2) \leq \log\left[\frac{E(w_i(X_i)|X_i > t_i, X_j = t_j)}{w_i(t_i)}\right].$$

Proof. Since $w_i(x_i)$ is increasing in x_i we have $w_i(x_i) \geq w_i(t_i),\ \forall\ x_i > t_i$

$$I_{X_i,X_i{}^w}(t_1,t_2) = \frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} f_i(x_i|t_j) \log\left[\frac{f_i(x_i|t_j)\bar{F}_i^w(t_i|t_j)}{\bar{F}_i(t_i|t_j) f_i^w(x_i|t_j)}\right] dx_i,$$
$$= \frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} f_i(x_i|t_j) \log\left[\frac{E(w_i(X_i)|X_i > t_i, X_j = t_j}{w_i(x_i)}\right] dx_i,$$
$$\leq \frac{1}{\bar{F}_i(t_i|t_j)} \int_{t_i}^{\infty} f_i(x_i|t_j) \log\left[\frac{E(w_i(X_i))|X_i > t_i, X_j = t_j}{w_i(t_i)}\right] dx_i,$$
$$= \log\left[\frac{E(w_i(X_i)|X_i > t_i, X_j = t_j)}{w_i(t_i)}\right]. \qquad \square$$

Theorem 5.2.6. *Let (X_1, X_2), (Y_1, Y_2), and (Z_1, Z_2) be three non-negative bivariate random vectors. Let $f_i(t_i|t_j)$, $g_i(t_i|t_j)$ and $q_i(t_i|t_j)$ be the pdfs and $\bar{F}_i(t_i|t_j)$, $\bar{G}_i(t_i|t_j)$ and $\bar{Q}_i(t_i|t_j)$ be the sfs of $X_i|X_j = t_j$, $Y_i|Y_j = t_j$ and $Z_i|Z_j = t_j$ respectively. If $Y_i|Y_j = t_j) \leq_{LR} (Z_i|Z_j = t_j)$, then*

$$I_{X_i,Y_i}(t_1,t_2) \geq I_{X_i,Z_i}(t_1,t_2) + \log\left(\frac{h_{Z_i|Z_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}\right).$$

Proof. Since $Y_i|Y_j = t_j \leq_{\text{LR}} Z_i|Z_j = t_j$, $\frac{g_i(x_i|t_j)}{q_i(x_i|t_j)}$ is decreasing in x_i, $\forall t_j$

$$\begin{aligned} I_{X_i,Y_i}(t_1,t_2) &= \frac{1}{\bar{F}_i(t_i|t_j)}\int_{t_i}^{\infty} f_i(x_i|t_j)\log\left[\frac{f_i(x_i|t_j)\bar{G}_i(t_i|t_j)}{\bar{F}_i(t_i|t_j)g_i(x_i|t_j)}\right]dx_i, \\ &= \frac{1}{\bar{F}_i(t_i|t_j)}\int_{t_i}^{\infty} f_i(x_i|t_j)\log\left[\frac{f_i(x_i|t_j)\bar{Q}_i(t_i|t_j)q_i(x_i|t_j)\bar{G}_i(t_i|t_j)}{\bar{F}_i(t_i|t_j)q_i(x_i|t_j)g_i(x_i|t_j)\bar{Q}_i(t_i|t_j)}\right]dx_i, \\ &\geq \frac{1}{\bar{F}_i(t_i|t_j)}\int_{t_i}^{\infty} f_i(x_i|t_j)\log\left[\frac{f_i(x_i|t_j)\bar{Q}_i(t_i|t_j)h_{Z_i|Z_j}(t_i|t_j)}{\bar{F}_i(t_i|t_j)q_i(x_i|t_j)h_{Y_i|Y_j}(t_i|t_j)}\right]dx_i, \\ &= I_{X_i,Z_i}(t_1,t_2) + \log\left(\frac{h_{Z_i|Z_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}\right). \end{aligned}$$

□

5.3 Conditional Kullback-Leibler discrimination of type 2

In this section dynamic Kullback-Leibler divergence measure proposed by Ebrahimi and Kirmani (1996b) is extended to conditional survival models called conditional Kullback-Leibler discimination of type 2 (CKLD_2), defined as follows:

Definition 5.3.1. *Let (X_1, X_2) and (Y_1, Y_2) be two bivariate random vectors having joint pdf f and g, joint cdf $F$and G, joint sf $\bar{F}$ and $\bar{G}$ respectively. Let their common support be $S = (l,\infty)\times(l,\infty)$ for $l \geq 0$. Also let $f_i^*(t_i|t_j)$ and $g_i^*(t_i|t_j)$, $\bar{F}_i^*(t_i|t_j)$ and $\bar{G}_i^*(t_i|t_j)$ denote the pdf and the sf of $(X_i|X_j > t_j)$ and $(Y_i|Y_j > t_j)$, respectively for $i,j = 1,2, i \neq j$. Then we define the conditional Kullback-Leibler discrimination of type 2 (CKLD2) information function as*

$$I^*_{X_i,Y_i}(t_1,t_2) = \int_{t_i}^{\infty}\frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)}\log\frac{f_i^*(x_i|t_j)\bar{G}_i^*(t_i|t_j)}{g_i^*(x_i|t_j)\bar{F}_i^*(t_i|t_j)}dx_i.$$

Definition 5.3.2. *Let (X_1, X_2) and (Y_1, Y_2) be two non-negative bivariate random vec-*

tors. Then the conditional survival rvs $X_i|X_j > t_j$ and $Y_i|Y_j > t_j$ satisfy CPHM of type 2 if

$$h^*_{Y_i|Y_j}(t_i|t_j) = \theta_i(t_j) h^*_{X_i|X_j}(t_i|t_j), \tag{5.5}$$

where $h^*_{X_i|X_j}(t_i|t_j)$ and $h^*_{Y_i|Y_j}(t_i|t_j)$ are the hazard rates of $X_i|X_j > t_j$ and $Y_i|Y_j > t_j$ respectively and $\theta_i(t_j)$ is a function of t_j only.

From (5.5) it follows that,

$$\begin{aligned}
\int_0^{t_i} h^*_{Y_i|Y_j}(x_i|t_j)dx_i &= \theta_i(t_j)\int_0^{t_i} h^*_{X_i|X_j}(x_i|t_j)dx_i, \\
e^{-\int_0^{t_i} h^*_{Y_i|Y_j}(x_i|t_j)dx_i} &= \left(e^{-\int\limits_0^{t_i} h^*_{X_i|X_j}(x_i|t_j)dx_i}\right)^{\theta_i(t_j)}, \\
\bar{G}^*_i(t_i|t_j) &= \left[\bar{F}^*_i(t_i|t_j)\right]^{\theta_i(t_j)}.
\end{aligned}$$

Theorem 5.3.1. *For $i,j = 1,2, i \neq j$, $X_i|X_j > t_j$ and $X_i^w|X_j^w > t_j$ satisfy CPHR model* (5.5) *if and only if $w_i(t_i) = [\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)-1}$, where $\theta_i(t_j)$ is a function of t_j only.*

Proof. Let $X_i|X_j > t_j$ and $X_i^w|X_j^w > t_j$ satisfy (5.5), then

$$\begin{aligned}
\frac{h^*_{X_i^w|X_j^w}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)} &= \theta_i(t_j), \\
\frac{w_i(t_i)}{E(w_i(X_i)|X_i > t_i, X_j > t_j)} &= \theta_i(t_j),
\end{aligned}$$

i.e.,

$$w_i(t_i) = \frac{\theta_i(t_j)}{\bar{F}^*_i(t_i|t_j)}\int_{t_i}^{\infty} w_i(x_i) f^*_i(t_i|t_j)dx_i. \tag{5.6}$$

Differentiating (5.6) with respect to t_i,

$$w_i'(t_i) = \frac{\theta_i(t_j) f^*_i(t_i|t_j)}{\left[\bar{F}_i(t_i|t_j)\right]^2}\int_{t_i}^{\infty} w_i(x_i) f^*_i(x_i|t_j)dx_i - \theta_i(t_j) w_i(t_i) h^*_{X_i|X_j}(t_i|t_j),$$

$$w_i'(t_i) = h^*_{X_i|X_j}(t_i|t_j)w_i(t_i) - \theta_i(t_j)w_i(t_i)h^*_{X_i|X_j}(t_i|t_j),$$

$$w_i'(t_i) = [1-\theta_i(t_j)]w_i(t_i)h^*_{X_i|X_j}(t_i|t_j),$$
$$\frac{w_i'(t_i)}{w_i(t_i)} = [1-\theta_i(t_j)]\left[-\frac{\partial}{\partial t_i}\log \bar{F}^*_i(t_i|t_j)\right],$$
$$\frac{\partial}{\partial t_i}\log w_i(t_i) = \frac{\partial}{\partial t_i}\log[\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)-1},$$

which on integration and taking exponentials yield, $w_i(t_i) = [\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)-1}$.

Conversely suppose that $w_i(t_i) = [\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)-1}$ then

$$\begin{aligned}
\frac{h^*_{X_i^w|X_j^w}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)} &= \frac{w_i(t_i)}{E(w_i(X_i)|X_i>t_i, X_j>t_j)},\\
&= \frac{w_i(t_i)\bar{F}^*_i(t_i|t_j)}{\int_{t_i}^{\infty} w_i(x_i)f^*_i(x_i|t_j)dx_i},\\
&= \frac{[\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)-1}\bar{F}^*_i(t_i|t_j)}{\int_{t_i}^{\infty}[\bar{F}^*_i(x_i|t_j)]^{\theta_i(t_j)-1}f^*_i(x_i|t_j)dx_i},\\
&= \frac{[\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)}\theta_i(t_j)}{\int_{t_i}^{\infty}\theta_i(t_i)\left[\bar{F}^*_i(x_i|t_j)\right]^{\theta_i(t_j)-1} f^*_i(x_i|t_j)\,dx_i},\\
&= \frac{[\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)}\theta_i(t_j)}{[\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)}},\\
&= \theta_i(t_j),
\end{aligned}$$

which implies $h^*_{X_i^w|X_j^w}(t_i|t_j) = \theta_i(t_j)h^*_{X_i|X_j}(t_i|t_j)$. □

Example 5.3.1. *Let* (X_1, X_2) *follow Gumbel's bivariate exponential distribution with joint sf* $\bar{F}(t_1,t_2) = e^{-\lambda_1 t_1-\lambda_2 t_2-\theta t_1 t_2}, \lambda_1, \lambda_2 > 0, \theta \geq 0, t_1, t_2 > 0$. *We have* $h^*_{X_i|X_j}(t_i|t_j) = (\lambda_i + \theta t_j)$. *Now consider* $\theta_i(t_j) = t_j$, *a non-negative function of* t_j *such that* $w_i(t_i) = [\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)-1} = e^{-(\lambda_i+\theta t_j)t_i(t_j-1))}$. *So we have* $h^*_{X_i^w|X_j^w}(t_i|t_j) = t_j(\lambda_i+\theta t_j) = \theta_i(t_j)$ $h^*_{X_i|X_j}(t_i|t_j)$.

Conversely, let

$$\frac{h^*_{X_i^w|X_j^w}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)} = t_j,$$

$$i.e., \quad w_i(t_i) = t_j \int_{t_i}^{\infty} \frac{w_i(x_i) f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)} dx_i,$$

which on simplification yields $w_i(t_i) = [\bar{F}_i^*(t_i|t_j)]^{t_j - 1}$. □

Theorem 5.3.2. *For* $i, j = 1, 2,\ i \neq j$, $I^*_{X_i,Y_i}(t_1, t_2)$ *is locally constant if and only if* $X_i|X_j > t_j$ *and* $Y_i|Y_j > t_j$ *satisfy CPHR model* (5.5).

Proof. Assume that $X_i|X_j > t_j$ and $Y_i|Y_j > t_j$ satisfy (5.5). So we have

$$\begin{aligned}
\bar{G}_i^*(t_i|t_j) &= \left[\bar{F}_i^*(t_i|t_j)\right]^{\theta_i(t_j)}. \\
I^*_{X_i,Y_i}(t_1, t_2) &= \frac{1}{\bar{F}_i^*(t_i|t_j)} \int_{t_i}^{\infty} f_i^*(x_i|t_j) \log \left[\frac{f_i^*(x_i|t_j)[\bar{F}_i^*(t_i|t_j)]^{\theta_i(t_j)}}{\bar{F}_i^*(t_i|t_j)\theta_i(t_j)[\bar{F}_i^*(x_i|t_j)]^{\theta_i(t_j)-1} f_i^*(x_i|t_j)} \right] dx_i, \\
&= \frac{1}{\bar{F}_i^*(t_i|t_j)} \int_{t_i}^{\infty} f_i^*(x_i|t_j) \log \left[\frac{[\bar{F}_i^*(t_i|t_j)]^{\theta_i(t_j)-1}}{\theta_i(t_j)[\bar{F}_i^*(x_i|t_j)]^{\theta_i(t_j)-1}} \right] dx_i, \\
&= \frac{1}{\bar{F}_i^*(t_i|t_j)} \int_{t_i}^{\infty} f_i^*(x_i|t_j) \Big[(\theta_i(t_j) - 1) \log \bar{F}_i^*(t_i|t_j) \\
&\qquad - \log \theta_i(t_j) - (\theta_i(t_j) - 1) \log \bar{F}_i^*(x_i|t_j) \Big] dx_i, \\
&= (\theta_i(t_j) - 1) \log \bar{F}_i^*(t_i|t_j) - \log \theta_i(t_j) \\
&\qquad - \frac{(\theta_i(t_j) - 1)}{\bar{F}_i^*(t_i|t_j)} \int_{t_i}^{\infty} f_i^*(x_i|t_j) \log \bar{F}_i^*(x_i|t_j) dx_i, \\
&= (\theta_i(t_j) - 1) \log \bar{F}_i^*(t_i|t_j) - \log \theta_i(t_j) \\
&\qquad - \frac{(\theta_i(t_j) - 1)}{\bar{F}_i^*(t_i|t_j)} [\bar{F}_i^*(t_i|t_j) \log \bar{F}_i^*(t_i|t_j) - \bar{F}_i^*(t_i|t_j)], \\
&= (\theta_i(t_j) - 1) - \log \theta_i(t_j), \text{ locally constant.}
\end{aligned}$$

Conversely suppose that $I^*_{X_i,Y_i}(t_1, t_2)$ is locally constant.

i.e., $I^*_{X_i,Y_i}(t_1, t_2) = c_i(t_j)$ (say).

i.e.,

$$\frac{1}{\bar{F}_i^*(t_i|t_j)} \int_{t_i}^{\infty} f_i^*(x_i|t_j) \log \left[\frac{f_i^*(x_i|t_j)}{g_i^*(x_i|t_j)} \right] dx_i + \log \bar{G}_i^*(t_i|t_j) - \log \bar{F}_i^*(t_i|t_j) = c_i(t_j). \quad (5.7)$$

Differentiating (5.7) with respect to t_i , we get

$$\frac{f_i^*(t_i|t_j)}{[\bar{F}_i^*(t_i|t_j)]^2}\int_{t_i}^{\infty} f_i^*(x_i|t_j)\log\left[\frac{f_i^*(x_i|t_j)}{g_i^*(x_i|t_j)}\right]dx_i - h^*_{X_i|X_j}(t_i|t_j)\log\left[\frac{f_i^*(t_i|t_j)}{g_i^*(t_i|t_j)}\right] \\ - h^*_{Y_i|Y_j}(t_i|t_j) + h^*_{X_i|X_j}(t_i|t_j) = 0.$$

Using (5.7) we have,

$$h^*_{X_i|X_j}(t_i|t_j)[c_i(t_j) - \log\bar{G}_i^*(t_i|t_j) + \log\bar{F}_i^*(t_i|t_j)] - h^*_{X_i|X_j}(t_i|t_j)\log\left[\frac{f_i^*(t_i|t_j)}{g_i^*(t_i|t_j)}\right] \\ - h^*_{Y_i|Y_j}(t_i|t_j) + h^*_{X_i|X_j}(t_i|t_j) = 0. \quad (5.8)$$

Dividing (5.8) with $h^*_{X_i|X_j}(t_i|t_j)$, we get

$$c_i(t_j) + \log\left[\frac{\bar{F}_i^*(t_i|t_j)}{\bar{G}_i^*(t_i|t_j)}\right] - \log\left[\frac{f_i^*(t_i|t_j)}{g_i^*(t_i|t_j)}\right] - \frac{h^*_{Y_i|Y_j}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)} + 1 = 0,$$

$$c_i(t_j) + 1 + \log\left[\frac{g_i^*(t_i|t_j)\bar{F}_i^*(t_i|t_j)}{f_i^*(t_i|t_j)\bar{G}_i^*(t_i|t_j)}\right] - \frac{h^*_{Y_i|Y_j}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)} = 0,$$

$$i.e., \quad \frac{h^*_{Y_i|Y_j}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)} - \log\left[\frac{h^*_{Y_i|Y_j}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)}\right] = c_i(t_j) + 1. \quad (5.9)$$

Putting $\frac{h^*_{Y_i|Y_j}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)} = \psi_i(t_1,t_2)$, (5.9) becomes

$$\psi_i(t_1,t_2) - \log\psi_i(t_1,t_2) = 1 + c_i(t_j). \quad (5.10)$$

Now differentiating (5.10) with respect to t_i, we get

$$\frac{\partial}{\partial t_i}\psi_i(t_1,t_2) - \frac{\frac{\partial}{\partial t_i}\psi_i(t_1,t_2)}{\psi_i(t_1,t_2)} = 0$$

or

$$\frac{\partial}{\partial t_i}\psi_i(t_1,t_2)\left[1-\frac{1}{\psi_i(t_1,t_2)}\right]=0,$$

which implies $\psi_i(t_1,t_2)=1$ or $\psi_i(t_1,t_2)=\theta_i(t_j)$ (say).

$$i.e., \quad \frac{h^*_{Y_i|Y_j}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)}=\theta_i(t_j).$$

i.e., $X_i|X_j>t_j$ and $Y_i|Y_j>t_j$ satisfy CPHR model (5.5). □

Corollory 5.3.1. *For $i,j=1,2,\ i\neq j,\ I^*_{X_i,X_i^w}(t_1,t_2)$ is locally constant if and only if $X_i|X_j>t_j$ and $X_i^w|X_j^w>t_j$ satisfy CPHR model (5.5).*

Corollory 5.3.2. *For $i,j=1,2,\ i\neq j,\ I^*_{X_i,X_i^w}(t_1,t_2)$ is locally constant if and only if $w_i(t_i)=[\bar{F}_i^*(t_i|t_j)]^{\theta_i(t_j)-1}$ where $\theta_i(t_j)$ is a function of t_j only.*

Theorem 5.3.3. *Let (X_1^s,X_2^s) be the size-biased random vector having pdf (5.3) where $w_i(x)=x^\beta,\ i=1,2$, then $I^*_{X_i,X_i^s}(t_1,t_2)$ is locally constant if and only if (X_1,X_2) is distributed as bivariate Pareto I specified by the sf*

$$\bar{F}(x_1,x_2)=x_1^{-\alpha_1}x_2^{-\alpha_2}x_1^{-\theta\log x_2};\quad x_1,x_2>1,\ \alpha_1,\alpha_2,\theta>0.$$

Proof. Assume that X is distributed as bivariate Pareto I then, for $i=1$, we have

$$\begin{aligned}
I^*_{X_1,X_1^s}(t_1,t_2)&=\frac{1}{\bar{F}(t_1,t_2)}\int_{t_1}^{\infty}-\frac{\partial}{\partial x_1}\bar{F}(x_1,t_2)\log\left[\frac{E(x_1^\beta|X_1>t_1,X_2>t_2)}{x_1^\beta}\right]dx_1,\\
&=\frac{1}{t_1^{-\alpha_1}t_2^{-\alpha_2}t_1^{-\theta\log t_2}}\int_{t_1}^{\infty}(\alpha_1+\theta\log t_2)x_1^{-(\alpha_1+\theta\log t_2)-1}t_2^{-\alpha_2}\\
&\quad\log\left[\frac{(\alpha_1+\theta\log t_2)t_1^\beta}{(\alpha_1+\theta\log t_2-\beta)x_1^\beta}\right]dx_1,\\
&=\log\left[\frac{(\alpha_1+\theta\log t_2)t_1^\beta}{(\alpha_1+\theta\log t_2-\beta)}\right]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(\alpha_1+\theta\log t_2)}{t_1^{-(\alpha_1+\theta\log t_2)}}\int_{t_1}^{\infty} x_1^{-(\alpha_1+\theta\log t_2)-1}\log(x_1^{\beta})dx_1,\\
&= \log\left[\frac{(\alpha_1+\theta\log t_2)t_1^{\beta}}{(\alpha_1+\theta\log t_2-\beta)}\right]\\
&- \frac{\beta(\alpha_1+\theta\log t_2)}{t_1^{-(\alpha_1+\theta\log t_2)}}\int_{t_1}^{\infty} x_1^{-(\alpha_1+\theta\log t_2)-1}\log(x_1)dx_1,\\
&= \log\left[\frac{(\alpha_1+\theta\log t_2)t_1^{\beta}}{(\alpha_1+\theta\log t_2-\beta)}\right]\\
&- \frac{\beta(\alpha_1+\theta\log t_2)}{t_1^{-(\alpha_1+\theta\log t_2)}}\left[\frac{t_1^{-(\alpha_1+\theta\log t_2)}\log t_1}{(\alpha_1+\theta\log t_2)}+\frac{t_1^{-(\alpha_1+\theta\log t_2)}}{(\alpha_1+\theta\log t_2)^2}\right],\\
&= \log\left[\frac{(\alpha_1+\theta\log t_2)}{(\alpha_1+\theta\log t_2-\beta)}\right]-\frac{\beta}{\alpha_1+\theta\log t_2}.
\end{aligned}
$$

Similarly with $i=2$ we have

$$
I^*_{X_2,X_2^s}(t_1,t_2) = \log\left[\frac{(\alpha_2+\theta\log t_1)}{(\alpha_2+\theta\log t_1-\beta)}\right]-\frac{\beta}{\alpha_2+\theta\log t_1},
$$

$$
i.e., \quad I^*_{X_i,X_i^s}(t_1,t_2) = \log\left[\frac{(\alpha_i+\theta\log t_j)}{(\alpha_i+\theta\log t_j-\beta)}\right]-\frac{\beta}{\alpha_i+\theta\log t_j}, \text{locally constant.}
$$

Conversely suppose that $I^*_{X_i,X_i^s}(t_1,t_2)$ is locally constant. Now using Corollary 5.3.1. we have

$$
h^*_{X_i^s|X_j^s}(t_i|t_j) = K_i(t_j)h^*_{X_i|X_j}(t_i|t_j)
$$

or

$$
t_i^{\beta} = K_i(t_j)E(X_i^{\beta}|X_i>t_i,X_j>t_j),
$$

$$
t_i^{\beta} = K_i(t_j)\frac{1}{\bar{F}_i^*(t_i|t_j)}\int_{t_i}^{\infty} x_i^{\beta}f_i^*(x_i|t_j)dx_i. \tag{5.11}
$$

Differentiating (5.11) with respect to t_i,

$$
\beta t_i^{\beta-1} = K_i(t_j)\frac{f_i^*(t_i|t_j)}{\left[\bar{F}_i^*(t_i|t_j)\right]^2}\int_{t_i}^{\infty} x_i^{\beta}f_i^*(x_i|t_j)dx_i - \frac{K_i(t_j)}{\bar{F}_i^*(t_i|t_j)}t_i^{\beta}f_i^*(t_i|t_j). \tag{5.12}
$$

Using (5.11) and then dividing with t_i^{β}, (5.12) becomes

$$\frac{\beta}{t_i} = h^*_{X_i|X_j}(t_i|t_j) - K_i(t_j)h^*_{X_i|X_j}(t_i|t_j)$$

or

$$h^*_{X_i|X_j}(t_i|t_j) = \left[\frac{\beta}{1-K_i(t_j)}\right]\frac{1}{t_i} = \frac{P_i(t_j)}{t_i} \text{ , where } P_i(t_j) = \frac{\beta}{1-K_i(t_j)},$$

$$i.e., \ -\frac{\frac{\partial}{\partial t_i}\bar{F}(t_1,t_2)}{\bar{F}(t_1,t_2)} = \frac{P_i(t_j)}{t_i},$$
$$\frac{\partial}{\partial t_i}\log\bar{F}(t_1,t_2) = -\frac{P_i(t_j)}{t_i},$$

which, on integration with respect to t_i, gives

$$\log\bar{F}(t_1,t_2) = -P_i(t_j)\log t_i + \log A_i(t_j),$$

where $\log A_i(t_j)$ is the constant of integration.

Taking exponentials,

$$\bar{F}(t_1,t_2) = A_i(t_j)t_i^{-P_i(t_j)}.$$

For $i = 1$, we have

$$\bar{F}(t_1,t_2) = A_1(t_2)t_1^{-P_1(t_2)}. \tag{5.13}$$

As $t_1 \to 1^+$ in (5.13) we get,

$$\bar{F}_2(t_2) = A_1(t_2).$$

Then (5.13) becomes

$$\bar{F}(t_1,t_2) = \bar{F}_2(t_2)t_1^{-P_1(t_2)}. \tag{5.14}$$

Similarly we have

$$\bar{F}(t_1, t_2) = \bar{F}_1(t_1) t_2^{-P_2(t_1)}. \tag{5.15}$$

When $t_2 \to 1^+$, (5.14) becomes

$$\bar{F}_1(t_1) = t_1^{-P_1(1)} = t_1^{-\alpha_1}, \text{ where } \alpha_1 = \lim_{t_2 \to 1^+} P_1(t_2).$$

Substituting this in (5.15), we get

$$\bar{F}(t_1, t_2) = t_1^{-\alpha_1} t_2^{-P_2(t_1)}. \tag{5.16}$$

Similarly we get

$$\bar{F}(t_1, t_2) = t_2^{-\alpha_2} t_1^{-P_1(t_2)}. \tag{5.17}$$

Equating (5.16) and (5.17),

$$t_1^{-\alpha_1} t_2^{-P_2(t_1)} = t_2^{-\alpha_2} t_1^{-P_1(t_2)}$$

or

$$t_1^{P_1(t_2) - \alpha_1} = t_2^{P_2(t_1) - \alpha_2}.$$

Taking logarithms,

$$(P_1(t_2) - \alpha_1) \log t_1 = (P_2(t_1) - \alpha_2) \log t_2,$$

or

$$\frac{P_1(t_2) - \alpha_1}{\log t_2} = \frac{P_2(t_1) - \alpha_2}{\log t_1} = \theta,$$

which implies $P_i(t_j) = \alpha_i + \theta \log t_j$, $i \neq j$, $i, j = 1, 2$.

Substituting $P_1(t_2)$ in (5.17) we have

$$\bar{F}(t_1, t_2) = t_1^{-\alpha_1} t_2^{-\alpha_2} t_1^{-\theta \log t_2},$$

which completes the proof. □

Now we end this section by obtaining some bounds to CKLD$_2$ which uses likelihood ratio ordering of conditional survival rvs. $X_i|X_j > t_j$ is said to be smaller than $Y_i|Y_j > t_j$ in likelihood ratio $((X_i|X_j > t_j) \leq_{LR} (Y_i|Y_j > t_j))$ if $\frac{f_i^*(x_i|t_j)}{g_i^*(x_i|t_j)}$ is decreasing in x_i, $\forall t_j$.

Theorem 5.3.4. *If $X_i|X_j > t_j \leq_{LR} Y_i|Y_j > t_j$ then*

$$I^*_{X_i,Y_i}(t_1, t_2) \leq \log \left[\frac{h^*_{X_i|X_j}(t_i|t_j)}{h^*_{Y_i|Y_j}(t_i|t_j)} \right].$$

Proof. The proof is similar to that of Theorem 5.2.4. □

Corollory 5.3.3. *If $X_i|X_j > t_j \leq_{LR} X_i^w|X_j^w > t_j$ then*

$$I^*_{X_i,X_i^w}(t_1, t_2) \leq \log \left[\frac{E(w_i(X_i)|X_i > t_i, X_j > t_j)}{w_i(t_i)} \right].$$

Theorem 5.3.5. *If w_i is increasing, then*

$$I^*_{X_i,X_i^w}(t_1, t_2) \leq \log \left[\frac{E(w_i(X_i)|X_i > t_i, X_j > t_j)}{w_i(t_i)} \right].$$

Proof. The proof is similar to that of Theorem 5.2.5. □

Theorem 5.3.6. *Let (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2) be three non-negative bivariate random vectors. Let $f_i^*(t_i|t_j)$, $g_i^*(t_i|t_j)$, and $q_i^*(t_i|t_j)$ be the pdfs and $\bar{F}_i^*(t_i|t_j), \bar{G}_i^*(t_i|t_j)$ and $\bar{Q}_i^*(t_i|t_j)$ be the sfs of $X_i|X_j > t_j, Y_i|Y_j > t_j$, and $Z_i|Z_j > t_j$ respectively. If*

$Y_i|Y_j > t_j \leq_{LR} Z_i|Z_j > t_j$, *then*

$$I^*_{X_i,Y_i}(t_1,t_2) \geq \log\left(\frac{h^*_{Z_i|Z_j}(t_i|t_j)}{h^*_{Y_i|Y_j}(t_i|t_j)}\right) + I^*_{X_i,Z_i}(t_1,t_2).$$

Proof. The proof is similar to that of Theorem 5.2.6. □

5.4 Conditional Renyi's discrimination information of type 1

In this section dynamic Renyi's discrimination function given in (2.2) is extended to conditionally specified models called conditional Renyi's discrimination information of type 1 (CRDI_1) function, the definition of which is as follows:

Definition 5.4.1. *Let (X_1, X_2) and (Y_1, Y_2) be two bivariate random vectors with common support $(l,\infty)\times(l,\infty)$ for $l \geq 0$. The joint pdf and sf of (X_1, X_2) are denoted by f and $\bar{F}$ and that of (Y_1, Y_2) by g and $\bar{G}$ respectively. Consider the conditionally specified rvs $(X_i|X_j = t_j)$ and $(Y_i|Y_j = t_j)$ for $i, j = 1, 2, i \neq j$. Their pdf and sf are denoted by $f_i(t_i|t_j)$, $\bar{F}_i(t_i|t_j)$, $g_i(t_i|t_j)$, $\bar{G}_i(t_i|t_j)$ respectively for $i, j = 1, 2, i \neq j$. Then the conditional Renyi's discrimination information of type 1 ($CRDI_1$) function is defined as*

$$I_{X_i,Y_i}(\alpha;t_1,t_2) = \frac{1}{\alpha-1}\log\int_{t_i}^{\infty}\frac{f_i^{\alpha}(x_i|t_j)}{\bar{F}_i^{\alpha}(t_i|t_j)}\frac{g_i^{1-\alpha}(x_i|t_j)}{\bar{G}_i^{1-\alpha}(t_i|t_j)}\,dx_i, \tag{5.18}$$

for $i, j = 1, 2, i \neq j$ and $t_1, t_2 \geq l$.

Note that $I_{X_i,Y_i}(\alpha;t_1,t_2) = I_{(X_i|X_j=t_j),(Y_i|Y_j=t_j)}(\alpha;t_i)$ for $i, j = 1, 2,\ i \neq j$. Hence (5.18) provides dynamic information on the distance between the conditionally specified rvs.

Theorem 5.4.1. *For $i, j = 1, 2, i \neq j$ and $0 < \alpha \neq 1$, the function $I_{X_i,Y_i}(\alpha;t_1,t_2)$ depends only on t_j if and only if $(X_i|X_j = t_j)$ and $(Y_i|Y_j = t_j)$ satisfy CPHR model*

(5.2).

Proof. For $i = 1$, let us suppose that $(X_1|X_2 = t_2)$ and $(Y_1|Y_2 = t_2)$ satisfy (5.2). Then their sfs satisfy $\bar{G}_1(t_1|t_2) = \left[\bar{F}_1(t_1|t_2)\right]^{\theta_1(t_2)}$. Hence from (5.18) if α satisfies $(1-\alpha)\theta_1(t_2) + \alpha > 0$, we get

$$I_{X_1,Y_1}(\alpha; t_1, t_2) = \frac{1}{\alpha - 1} \log \frac{\theta_1^{1-\alpha}(t_2)}{(1-\alpha)\theta_1(t_2) + \alpha},$$

which depends only on t_2. The proof for $i = 2$ is similar.

Conversely, for $i = 1$, let us suppose that $I_{X_1,Y_1}(\alpha; t_1, t_2)$ depends only on t_2 for some $0 < \alpha \neq 1$. Then

$$\frac{1}{\alpha - 1} \log \int_{t_1}^{\infty} \frac{f_1^{\alpha}(x_1|t_2)}{\bar{F}_1^{\alpha}(t_1|t_2)} \frac{g_1^{1-\alpha}(x_1|t_2)}{\bar{G}_1^{1-\alpha}(t_1|t_2)} dx_1 = A_1(t_2) \text{ (say)}.$$

Equivalently,

$$\int_{t_1}^{\infty} f_1^{\alpha}(x_1|t_2) g_1^{1-\alpha}(x_1|t_2) dx_1 = B_1(t_2) \bar{F}_1^{\alpha}(t_1|t_2) \bar{G}_1^{1-\alpha}(t_1|t_2),$$

where $B_1(t_2) = \exp((\alpha - 1)A_1(t_2))$. Differentiating with respect to t_1 and simplifying, we get

$$\frac{1}{B_1(t_2)} = \alpha \phi_1^{\alpha-1}(t_1, t_2) + (1-\alpha)\phi_1^{\alpha}(t_1, t_2),$$

where $\phi_1(t_1, t_2) = h_{Y_1|Y_2}(t_1|t_2)/h_{X_1|X_2}(t_1|t_2)$. Differentiating again with respect to t_1 we get

$$0 = \alpha(\alpha - 1)\phi_1^{\alpha-2}(t_1, t_2)(1 - \phi_1(t_1, t_2)) \frac{\partial}{\partial t_1} \phi_1(t_1, t_2).$$

As $\alpha(\alpha - 1) \neq 0$, we have $\frac{\partial}{\partial t_1}\phi_1(t_1, t_2) = 0$ and hence $\phi_1(t_1, t_2) = \theta_1(t_2)$. Therefore (5.2) holds for $i = 1$. The proof for the case $i = 2$ is similar. □

Now we will state a theorem which is quite similar to Theorem 5.2.3 of Section 5.2 of this chapter.

Theorem 5.4.2. *Let (X_1^w, X_2^w) be a random vector which has the bivariate weighted distribution associated to (X_1, X_2) and to two non-negative and differentiable functions w_1 and w_2. Let us assume that the support of (X_1, X_2) is $S = (l, \infty) \times (l, \infty)$ for $l \geq 0$. Then the following conditions are equivalent:*

(a) (X_1^w, X_2^w) and (X_1, X_2) satisfy the CPHR model (5.2) for $i, j = 1, 2, i \neq j$.

(b) $I_{X_i, X_i{}^w}(\alpha; t_1, t_2)$ depends only on t_j for $i = 1, 2$ and $0 < \alpha \neq 1$.

(c) The conditional reliability functions of (X_1, X_2) satisfy

$$\log \bar{F}_i(t_i|t_j) = \frac{\log(w_i(t_i)/w_i(l))}{\theta_i(t_j) - 1},$$

for $i, j = 1, 2, i \neq j$.

(d) (X_1, X_2) has the following pdf

$$f(x_1, x_2) = ca_1a_2 \frac{w_1'(x_1)w_2'(x_2)}{w_1^{a_1+1}(x_1)w_2^{a_2+1}(x_2)} \exp\left(-\phi a_1 a_2 \big(\log \frac{w_1(x_1)}{w_1(l)}\big)\big(\log \frac{w_2(x_2)}{w_2(l)}\big)\right),$$

for $x_1, x_2 \geq l$, where $c < 0$, $\phi \geq 0$, and $a_i > 1$ or $a_i < 0$ for $i, j = 1, 2$, $i \neq j$.

Proof. The equivalence of (a) and (b) is a direct consequence of Theorem 5.4.1. The equivalences of (a), (c) and (d) were proved in Theorem 5.2.3.

The comments given after Theorem 5.2.3 also hold for the present theorem. In particular note that the model given in (d) is a truncated version of the conditional proportional hazard rate model considered by Arnold and Strauss (1991) and that Arnold and Strauss's model is obtained when $l = 0$. Some particular parametric model can be obtained from this general model. For example if $l = 1$ and $w_i(t) = t$ for $i = 1, 2$, then we get a bivariate Pareto model (see Navarro et al. (2014)). This section ends by obtaining some bounds to CRDI_1 in terms of some well known reliability measures using likelihood ratio ordering. □

Theorem 5.4.3. *For $i, j = 1, 2,\ i \neq j$, if $(X_i|X_j = t_j) \leq_{LR} (Y_i|Y_j = t_j)$ and $\alpha > 1$ ($0 < \alpha < 1$), then*

$$I_{X_i,Y_i}(\alpha; t_1, t_2) \leq (\geq) \frac{\alpha}{\alpha - 1} \log \left[\frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)} \right].$$

Proof. Since $X_i|X_j = t_j \leq_{LR} Y_i|Y_j = t_j$, $\frac{f_i(x_i|t_j)}{g_i(x_i|t_j)}$ is decreasing in $x_i, \forall t_j$. Therefore $\frac{f_i^\alpha(x_i|t_j)}{g_i^\alpha(x_i|t_j)}$ is decreasing in $x_i, \forall t_j$, and for $\alpha > 0$.

$$i.e., \quad \frac{f_i^\alpha(x_i|t_j)}{g_i^\alpha(x_i|t_j)} \leq \frac{f_i^\alpha(t_i|t_j)}{g_i^\alpha(t_i|t_j)} \quad \forall x_i > t_i,\ \forall t_j,\ \alpha > 0,$$

$$\begin{aligned} I_{X_i,Y_i}(\alpha; t_1, t_2) &= \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{f_i^\alpha(x_i|t_j) g_i^{1-\alpha}(x_i|t_j)}{\bar{F}_i^\alpha(t_i|t_j) \bar{G}_i^{1-\alpha}(t_i|t_j)} dx_i, \\ &= \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{f_i^\alpha(x_i|t_j) \bar{G}_i^\alpha(t_i|t_j) g_i(x_i|t_j)}{g_i^\alpha(x_i|t_j) \bar{F}_i^\alpha(t_i|t_j) \bar{G}_i(t_i|t_j)} dx_i. \end{aligned}$$

For $0 < \alpha < 1$,

$$\begin{aligned} I_{X_i,Y_i}(\alpha; t_1, t_2) &\geq \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{h_{X_i|X_j}^\alpha(t_i|t_j) g_i(x_i|t_j)}{h_{Y_i|Y_j}^\alpha(t_i|t_j) \bar{G}_i(t_i|t_j)} dx_i, \\ &= \frac{\alpha}{\alpha - 1} \log \left[\frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)} \right]. \end{aligned}$$

For $\alpha > 1$,

$$I_{X_i,Y_i}(\alpha; t_1, t_2) \leq \frac{\alpha}{\alpha - 1} \log \left[\frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)} \right].$$ □

Corollory 5.4.1. *For $i, j = 1, 2,\ i \neq j$, if $X_i|X_j = t_j \leq_{LR} X_i^w|X_j^w = t_j$ and $\alpha > 1$*

$(0 < \alpha < 1)$, *then*

$$I_{X_i,X_i^w}(\alpha; t_1, t_2) \leq (\geq) \frac{\alpha}{\alpha - 1} \log \left[\frac{h_{X_i|X_j}(t_i|t_j)}{h_{X_i^w|X_j^w}(t_i|t_j)} \right].$$

Example 5.4.1. *Let (X_1, X_2) be a Pareto I rv specified by the pdf*

$$f(t_1, t_2) = Cx_1^{-\alpha_1 - 1} x_2^{-\alpha_2 - 1} x_1^{-\theta \log x_2}; \quad x_1, x_2 > 1,\ C > 0, \alpha_1, \alpha_2, \theta > 1.$$

Take the weight function as $w_i(x_i) = x_i$, so $\frac{f_i(x_i|t_j)}{f_i^w(x_i|t_j)} = \frac{(\alpha_i + \theta logt_j)}{(\alpha_i + \theta logt_j - 1)x_i}$ is decreasing in x_i and the corollary follows. □

Theorem 5.4.4. *For $i, j = 1, 2,\ i \neq j$ and $0 < \alpha \neq 1$, if w_i is increasing, then*

$$I_{X_i,X_i^w}(\alpha; t_1, t_2) \leq \log \left[\frac{E(w_i(X_i)|X_i > t_i, X_j = t_j)}{w_i(t_i)} \right].$$

Proof. Since $w_i(x_i)$ is increasing in x_i, we have $(w_i(x_i))^{1-\alpha}$ is increasing (decreasing) in x_i if $0 < \alpha < 1$ $(\alpha > 1)$. *i.e.*, $(w_i(x_i))^{1-\alpha} \geq (\leq)(w_i(t_i))^{1-\alpha} \quad \forall x_i > t_i$, if $0 < \alpha < 1$ $(\alpha > 1)$. We have

$$I_{X_i,X_i^w}(\alpha; t_1, t_2) = \frac{1}{\alpha - 1} \log \left(\int_{t_i}^{\infty} \frac{f_i(x_i|t_j) w_i(x_i)^{1-\alpha}}{\bar{F}_i(t_i|t_j)(E(w_i(X_i)|X_i > t_i, X_j = t_j))^{1-\alpha}} dx_i \right),$$
$$i \neq j,\ i, j = 1, 2.$$

For $0 < \alpha < 1$,

$$I_{X_i,X_i^w}(\alpha; t_1, t_2) \leq \log \left[\frac{E(w_i(X_i)|X_i > t_i, X_j = t_j)}{w_i(t_i)} \right].$$

The inequality is preserved when $\alpha > 1$ also. □

Remark 5.4.1. *Theorem 5.4.4 can be illustrated using Example 5.4.1.*

Theorem 5.4.5. *For $i, j = 1, 2$, $i \neq j$, if $X_i|X_j = t_j \leq_{LR} Y_i|Y_j = t_j$ and $\alpha > 1$ $(0 < \alpha < 1)$, then*

$$I_{X_i,Z_i}(\alpha; t_1, t_2) \leq (\geq) \frac{\alpha}{\alpha - 1} \log \left[\frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)} \right] + I_{Y_i,Z_i}(\alpha; t_1, t_2).$$

Proof. Since $X_i|X_j = t_j \leq_{LR} Y_i|Y_j = t_j$, $\frac{f_i(x_i|t_j)}{g_i(x_i|t_j)}$ is decreasing in $x_i, \forall t_j$, *i.e.* $\frac{f_i^\alpha(x_i|t_j)}{g_i^\alpha(x_i|t_j)}$ is decreasing in $x_i, \forall t_j$, for $\alpha > 0$.

$$i.e., \ \frac{f_i^\alpha(x_i|t_j)}{g_i^\alpha(x_i|t_j)} \leq \frac{f_i^\alpha(t_i|t_j)}{g_i^\alpha(t_i|t_j)} \quad \forall x_i > t_i, \ \forall t_j, \ \text{for } \alpha > 0.$$

$$\begin{aligned} I_{X_i,Z_i}(\alpha; t_1, t_2) &= \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{f_i^\alpha(x_i|t_j) q_i^{1-\alpha}(x_i|t_j)}{\bar{F}_i^\alpha(t_i|t_j) \bar{Q}_i^{1-\alpha}(t_i|t_j)} dx_i, \\ &= \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{f_i^\alpha(x_i|t_j) g_i^\alpha(x_i|t_j) \bar{G}_i^\alpha(t_i|t_j) q_i^{1-\alpha}(x_i|t_j)}{g_i^\alpha(x_i|t_j) \bar{G}_i^\alpha(t_i|t_j) \bar{F}_i^\alpha(t_i|t_j) \bar{Q}_i^{1-\alpha}(t_i|t_j)} dx_i. \end{aligned}$$

For $0 < \alpha < 1$,

$$\begin{aligned} I_{X_i,Z_i}(\alpha; t_1, t_2) &\geq \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{h_{X_i|X_j}^\alpha(t_i|t_j) g_i^\alpha(x_i|t_j) q_i^{1-\alpha}(x_i|t_j)}{h_{Y_i|Y_j}^\alpha(t_i|t_j) \bar{G}_i^\alpha(t_i|t_j) \bar{Q}_i^{1-\alpha}(t_i|t_j)} dx_i, \\ &= \frac{\alpha}{\alpha - 1} \log \left[\frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)} \right] + I_{Y_i,Z_i}(\alpha; t_1, t_2). \end{aligned}$$

For $\alpha > 1$,

$$\begin{aligned} I_{X_i,Z_i}(\alpha; t_1, t_2) &\leq \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{h_{X_i|X_j}^\alpha(t_i|t_j) g_i^\alpha(x_i|t_j) q_i^{1-\alpha}(x_i|t_j)}{h_{Y_i|Y_j}^\alpha(t_i|t_j) \bar{G}_i^\alpha(t_i|t_j) \bar{Q}_i^{1-\alpha}(t_i|t_j)} dx_i, \\ &= \frac{\alpha}{\alpha - 1} \log \left[\frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)} \right] + I_{Y_i,Z_i}(\alpha; t_1, t_2). \end{aligned}$$

□

Example 5.4.2. *Let (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2) be 3 bivariate independent expo-*

nential rvs with respective pdfs given by $f(x_1, x_2) = \lambda_1\lambda_2 \exp(-\lambda_1 x_1 - \lambda_2 x_2)$*;* $\lambda_1, \lambda_2 > 0$*,* $x_1, x_2 > 0$*,* $g(x_1, x_2) = \mu_1\mu_2 \exp(-\mu_1 x_1 - \mu_2 x_2)$*;* $\mu_1, \mu_2 > 0$*,* $x_1, x_2 > 0$ *and* $q(x_1, x_2) = \gamma_1\gamma_2 \exp(-\gamma_1 x_1 - \gamma_2 x_2)$*;* $\gamma_1, \gamma_2 > 0$*,* $x_1, x_2 > 0$*, such that* $\lambda_i > \mu_i > \gamma_i$*. Clearly,* $\frac{f_i(x_i|t_j)}{g_i(x_i|t_j)} = \frac{\lambda_i}{\mu_i} \exp[-(\lambda_i - \mu_i)x_i]$ *is decreasing in* x_i*,* $\forall t_j$ *and Theorem 5.4.5 can be illustrated.* □

Theorem 5.4.6. *For* $i, j = 1, 2,\ i \neq j$ *and* $0 < \alpha \neq 1$*, if* $Y_i|Y_j = t_j) \leq_{LR} (Z_i|Z_j = t_j)$*, then*

$$I_{X_i,Y_i}(\alpha; t_1, t_2) \geq I_{X_i,Z_i}(\alpha; t_1, t_2) + \log \frac{h_{Z_i|Z_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}.$$

Proof. Since $Y_i|Y_j = t_j \leq_{LR} Z_i|Z_j = t_j$ we have $\frac{g_i(x_i|t_j)}{q_i(x_i|t_j)}$ is decreasing in x_i, $\forall t_j$. Therefore $\frac{g_i^{1-\alpha}(x_i|t_j)}{q_i^{1-\alpha}(x_i|t_j)}$ is decreasing in x_i, $\forall t_j$, if $0 < \alpha < 1$ and $\frac{g_i^{1-\alpha}(x_i|t_j)}{q_i^{1-\alpha}(x_i|t_j)}$ is increasing in x_i, $\forall t_j$ if $\alpha > 1$.

$$i.e., \quad \frac{g_i^{1-\alpha}(x_i|t_j)}{q_i^{1-\alpha}(x_i|t_j)} \leq \frac{g_i^{1-\alpha}(t_i|t_j)}{q_i^{1-\alpha}(t_i|t_j)} \quad \forall x_i > t_i,\ \forall t_j,\ \text{if } 0 < \alpha < 1$$

and

$$\frac{g_i^{1-\alpha}(x_i|t_j)}{q_i^{1-\alpha}(x_i|t_j)} \geq \frac{g_i^{1-\alpha}(t_i|t_j)}{q_i^{1-\alpha}(t_i|t_j)} \quad \forall x_i > t_i,\ \forall t_j,\ \text{if } \alpha > 1.$$

Now for $0 < \alpha < 1$, we have

$$\begin{aligned}
I_{X_i,Y_i}(\alpha; t_1, t_2) &= \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{f_i^{\alpha}(x_i|t_j) g_i^{1-\alpha}(x_i|t_j)}{\bar{F}_i^{\alpha}(t_i|t_j)\bar{G}_i^{1-\alpha}(t_i|t_j)} dx_i, \\
&= \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{f_i^{\alpha}(x_i|t_j) g_i^{1-\alpha}(x_i|t_j) q_i^{1-\alpha}(x_i|t_j)\bar{Q}_i^{1-\alpha}(t_i|t_j)}{\bar{F}_i^{\alpha}(t_i|t_j) q_i^{1-\alpha}(x_i|t_j)\bar{Q}_i^{1-\alpha}(t_i|t_j)\bar{G}_i^{1-\alpha}(t_i|t_j)} dx_i, \\
&\geq \log\left[\frac{h_{Z_i|Z_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}\right] + \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{f_i^{\alpha}(x_i|t_j) q_i^{1-\alpha}(x_i|t_j)}{\bar{F}_i^{\alpha}(t_i|t_j)\bar{Q}_i^{1-\alpha}(t_i|t_j)} dx_i, \\
&= I_{X_i,Z_i}(\alpha; t_1, t_2) + \log\left[\frac{h_{Z_i|Z_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}\right].
\end{aligned}$$

The inequality is preserved when $\alpha > 1$ also. □

Remark 5.4.2. *Theorem 5.4.6 can be illustrated using Example 5.4.2.*

5.5 Conditional Renyi's discrimination information of type 2

In this section, dynamic Renyi's divergence measure given in (2.2) is extended to conditional survival models, the definition to which is as follows:

Definition 5.5.1. *Let (X_1, X_2) and (Y_1, Y_2) be two non-negative bivariate rvs admitting absolutely continuous dfs. If $f_i^*(t_i|t_j)$ and $g_i^*(t_i|t_j)$, $i \neq j$, $i, j = 1, 2$ denote the conditional densities of $X_i|X_j > t_j$ and $Y_i|Y_j > t_j$ respectively, then conditional Renyi's discrimination information of type 2 (CRDI$_2$) for these rvs can be defined as*

$$I^*_{X_i,Y_i}(\alpha; t_1, t_2) = \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{(f_i^*(x_i|t_j))^{\alpha}(g_i^*(x_i|t_j))^{1-\alpha}}{(\bar{F}_i^*(t_i|t_j))^{\alpha}(\bar{G}_i^*(t_i|t_j))^{1-\alpha}} dx_i,$$

where $\bar{F}_i^*(t_i|t_j)$ and $\bar{G}_i^*(t_i|t_j)$ are the sfs of $X_i|X_j > t_j$ and $Y_i|Y_j > t_j$ respectively.

The weighted distribution of X_i given $X_j > t_j$, $i \neq j$, $i, j = 1, 2$, is defined as

$$f_i^{w^*}(t_i|t_j) = \frac{w_i(t_i)}{E(w_i(X_i)|X_j > t_j)} f_i^*(t_i|t_j), \; t_i, t_j > 0. \tag{5.19}$$

The corresponding sf denoted by $\bar{F}_i^{w^*}(t_i|t_j) = P(X_i^w > t_i|X_j^w > t_j)$ is of the form

$$\bar{F}_i^{w^*}(t_i|t_j) = \frac{E(w_i(X_i)|X_i > t_i, X_j > t_j)}{E(w_i(X_i)|X_j > t_j)} \bar{F}_i^*(t_i|t_j), i \neq j, i, j = 1, 2, \tag{5.20}$$

CRDI$_2$ of $X_i|X_j > t_j$ and $X_i^w|X_j^w > t_j$ denoted by $I^*_{X_i,X_i^w}(\alpha; t_1, t_2)$ is defined as

$$I^*_{X_i,X_i^w}(\alpha; t_1, t_2) = \frac{1}{\alpha - 1} \log \int_{t_i}^{\infty} \frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)} \frac{(w_i(x_i))^{1-\alpha}}{(E(w_i(X_i)|X_i > t_i, X_j > t_j)^{1-\alpha}} dx_i,$$
$$i \neq j, i, j = 1, 2.$$

Theorem 5.5.1. *For $i, j = 1, 2$, $i \neq j$, $I^*_{X_i,Y_i}(\alpha; t_1, t_2)$ is locally constant if and only if $X_i|X_j > t_j$ and $Y_i|Y_j > t_j$ satisfy CPHR model (5.5).*

Proof. Assume that $X_i|X_j > t_j$ and $Y_i|Y_j > t_j$ satisfy CPHR model (5.5). So we have

$$
\begin{aligned}
&\bar{G}^*_i(t_i|t_j) = [\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)}.\\
&I^*_{X_i,Y_i}(\alpha; t_1, t_2) = \frac{1}{\alpha-1}\log\int_{t_i}^{\infty}\frac{f_i^{*\alpha}(x_i|t_j)g_i^{*1-\alpha}(x_i|t_j)}{\bar{F}_i^{*\alpha}(t_i|t_j)\bar{G}_i^{*1-\alpha}(t_i|t_j)}dx_i,\\
&= \frac{1}{\alpha-1}\log\int_{t_i}^{\infty}\frac{f_i^{*\alpha}(x_i|t_j)[\theta_i(t_j)[\bar{F}^*_i(x_i|t_j)]^{\theta_i(t_j)-1}f^*_i(x_i|t_j)]^{1-\alpha}}{\bar{F}_i^{*\alpha}(t_i|t_j)[\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)(1-\alpha)}}dx_i,\\
&= \frac{1}{\alpha-1}\log\int_{t_i}^{\infty}\frac{[\theta_i(t_j)]^{1-\alpha}[\bar{F}^*_i(x_i|t_j)]^{(\theta_i(t_j)-1)(1-\alpha)}f^*_i(x_i|t_j)}{[\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)(1-\alpha)+\alpha}}dx_i,\\
&= \frac{1}{\alpha-1}\log\int_{t_i}^{\infty}\frac{[\theta_i(t_j)]^{1-\alpha}[\bar{F}^*_i(x_i|t_j)]^{\theta_i(t_j)(1-\alpha)+\alpha-1}f^*_i(x_i|t_j)}{[\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)(1-\alpha)+\alpha}}dx_i,\\
&= \frac{1}{\alpha-1}\log\int_{t_i}^{\infty}\frac{[\theta_i(t_j)]^{1-\alpha}[\theta_i(t_j)(1-\alpha)+\alpha][\bar{F}^*_i(x_i|t_j)]^{\theta_i(t_j)(1-\alpha)+\alpha-1}f^*_i(x_i|t_j)}{[\theta_i(t_j)(1-\alpha)+\alpha][\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)(1-\alpha)+\alpha}}dx_i,\\
&= \frac{1}{\alpha-1}\log\left[\frac{[\theta_i(t_j)]^{1-\alpha}[\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)(1-\alpha)+\alpha}}{[\theta_i(t_j)(1-\alpha)+\alpha][\bar{F}^*_i(t_i|t_j)]^{\theta_i(t_j)(1-\alpha)+\alpha}}\right],\\
&= \frac{1}{\alpha-1}\log\left[\frac{[\theta_i(t_j)]^{1-\alpha}}{[\theta_i(t_j)(1-\alpha)+\alpha]}\right],\ \text{locally constant.}
\end{aligned}
$$

Conversely suppose that $I^*_{X_i,Y_i}(\alpha; t_1, t_2)$ is locally constant, *i.e.* $I^*_{X_i,Y_i}(\alpha; t_1, t_2) = B_i(t_j)$ (say).

$$
\begin{aligned}
&\frac{1}{\alpha-1}\log\int_{t_i}^{\infty}\frac{f_i^{*\alpha}(x_i|t_j)g_i^{*1-\alpha}(x_i|t_j)}{\bar{F}_i^{*\alpha}(t_i|t_j)\bar{G}_i^{*1-\alpha}(t_i|t_j)}dx_i = B_i(t_j),\\
&i.e.\quad \int_{t_i}^{\infty}\frac{f_i^{*\alpha}(x_i|t_j)g_i^{*1-\alpha}(x_i|t_j)}{\bar{F}_i^{*\alpha}(t_i|t_j)\bar{G}_i^{*1-\alpha}(t_i|t_j)}dx_i = C_i(t_j)\ \text{where}\ C_i(t_j) = e^{(\alpha-1)B_i(t_j)},
\end{aligned}
$$

$$
\int_{t_i}^{\infty} f_i^{*\alpha}(x_i|t_j)g_i^{*1-\alpha}(x_i|t_j)dx_i = C_i(t_j)\bar{F}_i^{*\alpha}(t_i|t_j)\bar{G}_i^{*1-\alpha}(t_i|t_j).
$$

Differentiating with respect to t_i, we get

$$\begin{aligned}
-f_i^{*\alpha}(t_i|t_j)g_i^{*1-\alpha}(t_i|t_j) &= -\alpha C_i(t_j)\bar{F}_i^{*1-\alpha}(t_i|t_j)f_i^*(t_i|t_j)\bar{G}_i^{*1-\alpha}(t_i|t_j) \\
&\quad - (1-\alpha)C_i(t_j)\bar{F}_i^{*\alpha}(t_i|t_j)\bar{G}_i^{*-\alpha}(t_i|t_j)g_i^*(t_i|t_j), \\
f_i^{*\alpha}(t_i|t_j)g_i^{*1-\alpha}(t_i|t_j) &= \alpha C_i(t_j)\bar{F}_i^{*1-\alpha}(t_i|t_j)f_i^*(t_i|t_j)\bar{G}_i^{*1-\alpha}(t_i|t_j) \\
&\quad + (1-\alpha)C_i(t_j)\bar{F}_i^{*\alpha}(t_i|t_j)\bar{G}_i^{*-\alpha}(t_i|t_j)g_i^*(t_i|t_j). \qquad (5.21)
\end{aligned}$$

Dividing with $f_i^{*\alpha}(t_i|t_j)g_i^{*1-\alpha}(t_i|t_j)$, equation (5.21) reduces to

$$1 = \alpha C_i(t_j)h_{X_i|X_j}^{*1-\alpha}(t_i|t_j)h_{Y_i|Y_j}^{*\alpha-1}(t_i|t_j) + (1-\alpha)C_i(t_j)h_{X_i|X_j}^{*-\alpha}(t_i|t_j)h_{Y_i|Y_j}^{*\alpha}(t_i|t_j),$$

or

$$\frac{1}{C_i(t_j)} = \alpha\left[\frac{h_{Y_i|Y_j}^*(t_i|t_j)}{h_{X_i|X_j}^*(t_i|t_j)}\right]^{\alpha-1} + (1-\alpha)\left[\frac{h_{Y_i|Y_j}^*(t_i|t_j)}{h_{X_i|X_j}^*(t_i|t_j)}\right]^{\alpha}. \qquad (5.22)$$

Putting $\frac{h_{Y_i|Y_j}^*(t_i|t_j)}{h_{X_i|X_j}^*(t_i|t_j)} = \psi_i(t_1,t_2)$, equation (5.22) becomes

$$\frac{1}{C_i(t_j)} = \alpha\left[\psi_i(t_1,t_2)\right]^{\alpha-1} + (1-\alpha)\left[\psi_i(t_1,t_2)\right]^{\alpha}. \qquad (5.23)$$

Differentiating (5.23) with respect to t_i, we get

$$\begin{aligned}
0 &= \alpha(\alpha-1)\left[\psi_i(t_1,t_2)\right]^{\alpha-2}\frac{\partial}{\partial t_i}\psi_i(t_1,t_2) + (1-\alpha)\alpha\left[\psi_i(t_1,t_2)\right]^{\alpha-1}\frac{\partial}{\partial t_i}\psi_i(t_1,t_2), \\
0 &= \alpha(\alpha-1)\left[\psi_i(t_1,t_2)\right]^{\alpha-2}\big(\frac{\partial}{\partial t_i}\psi_i(t_1,t_2)\big)\left[1-\psi_i(t_1,t_2)\right], \\
0 &= \big(\frac{\partial}{\partial t_i}\psi_i(t_1,t_2)\big)\left[1-\psi_i(t_1,t_2)\right],
\end{aligned}$$

which implies $\psi_i(t_1,t_2)=1$ or $\frac{\partial}{\partial t_i}\psi_i(t_1,t_2)=0$

$$\frac{\partial}{\partial t_i}\psi_i(t_1,t_2)=0 \Rightarrow \psi_i(t_1,t_2)=\theta_i(t_j),$$

$$i.e.,\ \frac{h^*_{Y_i|Y_j}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)}=\theta_i(t_j).$$

Thus $X_i|X_j>t_j$ and $Y_i|Y_j>t_j$ satisfy CPHR model (5.5). □

Corollory 5.5.1. *For $i,j=1,2,\ i\neq j$, $I^*_{X_i,X_i^w}(\alpha;t_1,t_2)$ is locally constant if and only if $X_i|X_j>t_j$ and $X_i^w|X_j^w>t_j$ satisfy CPHR model (5.5).*

Corollory 5.5.2. *For $i,j=1,2,\ i\neq j$, $I^*_{X_i,X_i^w}(\alpha;t_1,t_2)$ is locally constant if and only if $w_i(t_i)=[\bar{F}_i^*(t_i|t_j)]^{\theta_i(t_j)-1}$, where $\theta_i(t_j)$ is a function of t_j only.*

Theorem 5.5.2. *Let (X_1,X_2) be a non-negative bivariate random vector and (X_1^s,X_2^s) be its size-biased version. Then $I^*_{X_i,X_i^s}(\alpha;t_1,t_2)$ is locally constant if and only if (X_1,X_2) follows bivariate Pareto I distribution specified by the sf*

$$\bar{F}(x_1,x_2)=x_1^{-\alpha_1}x_2^{-\alpha_2}x_1^{-\theta\log x_2};\quad x_1,x_2>1,\ \alpha_1,\alpha_2,\theta\geq 0.$$

Proof. Assume that (X_1,X_2) follows bivariate Pareto I model, then for $i=1$ we have

$$\begin{aligned}
I^*_{X_1,X_1^s}(\alpha;t_1,t_2) &= \frac{1}{\alpha-1}\log\int_{t_1}^{\infty}\frac{x_1^{\beta(1-\alpha)}f_1^*(x_1|t_2)}{\left[E(X_1^\beta)|X_1>t_1,X_2>t_2\right]^{1-\alpha}\bar{F}_1^*(t_1|t_2)}dx_1,\\
&= \frac{1}{\alpha-1}\log\int_{t_1}^{\infty}\frac{x_1^{\beta(1-\alpha)}x_1^{-\alpha_1-1}t_2^{-\alpha_2}x_1^{-\theta\log t_2}(\alpha_1+\theta\log t_2)}{\left[\frac{(\alpha_1+\theta\log t_2)t_1^\beta}{(\alpha_1+\theta\log t_2-\beta)}\right]^{1-\alpha}t_1^{-\alpha_1}t_2^{-\alpha_2}t_1^{-\theta\log t_2}}dx_1,\\
&= \frac{1}{\alpha-1}\log\int_{t_1}^{\infty}\frac{x_1^{-\alpha_1-\theta\log t_2-\beta(\alpha-1)-1}(\alpha_1+\theta\log t_2)^\alpha}{(\alpha_1+\theta\log t_2-\beta)^{\alpha-1}t_1^{-\alpha_1-\theta\log t_2-\beta(\alpha-1)}}dx_1,\\
&= \frac{1}{\alpha-1}\log\left[\frac{(\alpha_1+\theta\log t_2)^\alpha}{(\alpha_1+\theta\log t_2-\beta)^{\alpha-1}(\alpha_1+\theta\log t_2+\beta(\alpha-1))}\right],\\
&= \log\left[\frac{\alpha_1+\theta\log t_2}{\alpha_1+\theta\log t_2-\beta}\right]+\frac{1}{\alpha-1}\log\left[\frac{\alpha_1+\theta\log t_2}{\alpha_1+\theta\log t_2+\beta(\alpha-1)}\right].
\end{aligned}$$

Similarly for $i = 2$ we have

$$I^*_{X_2,X_2^s}(\alpha; t_1, t_2) = \log\left[\frac{\alpha_2 + \theta\log t_1}{\alpha_2 + \theta\log t_1 - \beta}\right] + \frac{1}{\alpha - 1}\log\left[\frac{\alpha_2 + \theta\log t_1}{\alpha_2 + \theta\log t_1 + \beta(\alpha - 1)}\right].$$

$$i.e., \; I^*_{X_i,X_i^s}(\alpha; t_1, t_2) = \log\left[\frac{\alpha_i + \theta\log t_j}{\alpha_i + \theta\log t_j - \beta}\right] + \frac{1}{\alpha - 1}\log\left[\frac{\alpha_i + \theta\log t_j}{\alpha_i + \theta\log t_j + \beta(\alpha - 1)}\right],$$

locally constant.

Conversely suppose that $I^*_{X_i,X_i^s}(\alpha; t_1, t_2)$ is locally constant. Now using Corollary 5.5.1 we have

$$h^*_{X_i^s|X_j^s}(t_i|t_j) = K_i(t_j)h^*_{X_i|X_j}(t_i|t_j).$$

The rest of the proof follows similar to that for Theorem 5.3.3. □

In the following theorems we use the likelihood ratio ordering of conditional survival rvs to obtain some bounds for CRDI_2.

Theorem 5.5.3. *For $i, j = 1, 2$, $i \neq j$, if $X_i|X_j > t_j \leq_{LR} Y_i|Y_j > t_j$ and $\alpha > 1$ ($0 < \alpha < 1$), then*

$$I^*_{X_i,Y_i}(\alpha; t_1, t_2) \leq (\geq) \frac{\alpha}{\alpha - 1}\log\left[\frac{h^*_{X_i|X_j}(t_i|t_j)}{h^*_{Y_i|Y_j}(t_i|t_j)}\right].$$

Proof. The proof is similar to that of Theorem 5.4.3. □

Corollory 5.5.3. *For $i, j = 1, 2$, $i \neq j$, if $X_i|X_j > t_j \leq_{LR} X_i^w|X_j^w > t_j$ and $\alpha > 1$ ($0 < \alpha < 1$), then*

$$I^*_{X_i,X_i^w}(\alpha; t_1, t_2) \leq (\geq) \frac{\alpha}{\alpha - 1}\log\left[\frac{h^*_{X_i|X_j}(t_i|t_j)}{h^*_{X_i^w|X_j^w}(t_i|t_j)}\right].$$

Example 5.5.1. *Let (X_1, X_2) be a bivariate Pareto I rv specified by the sf $\bar{F}(x_1, x_2) = x_1^{-\alpha_1}x_2^{-\alpha_2}x_1^{-\theta\log t_2}$; $x_1, x_2 > 1$, α_1, α_2, $\theta > 1$. Take the weight function as $w_i(x_i) = x_i$ so* *that*

$$\frac{f_i^*(x_i|t_j)}{f_i^{w*}(x_i|t_j)} = \frac{(\alpha_i + \theta \log t_j)}{(\alpha_i + \theta \log t_j - 1)x_i} \text{ is decreasing in } x_i.$$

Therefore,

$$\begin{aligned} I^*_{X_i,X_i^w}(\alpha; t_1, t_2) &= \frac{\alpha}{\alpha-1} \log\left[\frac{\alpha_i + \theta \log t_j}{\alpha_i + \theta \log t_j - 1}\right] + \frac{1}{\alpha-1} \log\left[\frac{\alpha_i + \theta \log t_j - 1}{\alpha_i + \theta \log t_j + \alpha - 1}\right], \\ &\leq (\geq) \frac{\alpha}{\alpha-1} \log\left[\frac{\alpha_i + \theta \log t_j}{\alpha_i + \theta \log t_j - 1}\right] = \frac{\alpha}{\alpha-1} \log\left[\frac{h^*_{X_i|X_j}(t_i|t_j)}{h^*_{X_i^w|X_j^w}(t_i|t_j)}\right], \end{aligned}$$

according as $\alpha > 1$ *(*$0 < \alpha < 1$*).* □

Theorem 5.5.4. *If* w_i *is increasing, then*

$$I^*_{X_i,X_i^w}(\alpha; t_1, t_2) \leq \log\left[\frac{E(w_i(X_i)|X_i > t_i, X_j > t_j)}{w_i(x_i)}\right].$$

Proof. The proof is similar to that of Theorem 5.4.4. □

Example 5.5.2. *Consider the bivariate rv taken in Example 5.5.1. Using the weight function* $w_i(x_i) = x_i$*, which is increasing in* x_i*, we have*

$$\begin{aligned} I^*_{X_i,X_i^w}(\alpha; t_1, t_2) &= \log\left[\frac{\alpha_i + \theta \log t_j}{\alpha_i + \theta \log t_j - 1}\right] + \frac{1}{\alpha-1} \log\left[\frac{\alpha_i + \theta \log t_j}{\alpha_i + \theta \log t_j - 1}\right], \\ &\leq \log\left[\frac{\alpha_i + \theta \log t_j}{\alpha_i + \theta \log t_j - 1}\right] = \log\left[\frac{E(w_i(X_i)|X_i > t_i, X_j > t_j)}{w_i(t_i)}\right]. \end{aligned}$$ □

Theorem 5.5.5. *For* $i, j = 1, 2$*,* $i \neq j$*, if* $X_i|X_j > t_j \leq_{LR} Y_i|Y_j > t_j$ *and* $\alpha > 1$ *(*$0 < \alpha < 1$*), then*

$$I^*_{X_i,Z_i}(\alpha; t_1, t_2) \leq (\geq) \frac{\alpha}{\alpha-1} \log\left[\frac{h^*_{X_i|X_j}(t_i|t_j)}{h^*_{Y_i|Y_j}(t_i|t_j)}\right] + I^*_{Y_i,Z_i}(\alpha; t_1, t_2).$$

Proof. The proof is similar to that of Theorem 5.4.5. □

Example 5.5.3. *Let $(X_1, X_2), (Y_1, Y_2)$ and (Z_1, Z_2) be 3 bivariate independent exponential rvs with respective sfs given by $\bar{F}(x_1, x_2) = \exp(-\lambda_1 x_1 - \lambda_2 x_2);\ \lambda_1, \lambda_2 > 0, x_1, x_2 > 0$, $\bar{G}(x_1, x_2) = \exp(-\mu_1 x_1 - \mu_2 x_2);\ \mu_1, \mu_2 > 0, x_1, x_2 > 0$ and $\bar{Q}(x_1, x_2) = \exp(-\gamma_1 x_1 - \gamma_2 x_2);\ \gamma_1, \gamma_2 > 0, x_1, x_2 > 0$ such that $\lambda_i > \mu_i > \gamma_i$. Clearly,*

$$\frac{f_i^*(x_i|t_j)}{g_i^*(x_i|t_j)} = \frac{\lambda_i}{\mu_i} \exp[-(\lambda_i - \mu_i)x_i]$$

is decreasing in $x_i, \forall t_j$. Therefore,

$$\begin{aligned} I^*_{X_i,Z_i}(\alpha; t_1, t_2) &= \frac{\alpha}{\alpha-1} \log\left[\frac{\lambda_i^{\alpha}\gamma_i^{1-\alpha}}{(\lambda_i - \gamma_i)\alpha + \gamma_i}\right], \\ &= \frac{\alpha}{\alpha-1} \log\left[\frac{\lambda_i}{\mu_i}\right] + \frac{1}{\alpha-1} \log\left[\frac{\mu_i^{\alpha}\gamma_i^{1-\alpha}}{(\mu_i - \gamma_i)\alpha + \gamma_i}\right] \\ &\quad + \frac{1}{\alpha-1} \log\left[\frac{(\mu_i - \gamma_i)\alpha + \gamma_i}{(\lambda_i - \gamma_i)\alpha + \gamma_i}\right], \\ &\leq (\geq) \frac{\alpha}{\alpha-1} \log\left[\frac{\lambda_i}{\mu_i}\right] + \frac{1}{\alpha-1} \log\left[\frac{\mu_i^{\alpha}\gamma_i^{1-\alpha}}{(\mu_i - \gamma_i)\alpha + \gamma_i}\right], \\ &= \frac{\alpha}{\alpha-1} \log\left[\frac{h^*_{X_i|X_j}(t_i|t_j)}{h^*_{Y_i|Y_j}(t_i|t_j)}\right] + I^*_{Y_i,Z_i}(\alpha; t_1, t_2), \end{aligned}$$

provided $\alpha > 1$ ($0 < \alpha < 1$). □

Theorem 5.5.6. *For $i, j = 1, 2,\ i \neq j$, if $Y_i|Y_j > t_j \leq_{LR} Z_i|Z_j > t_j$ and $\alpha > 1$ ($0 < \alpha < 1$), then*

$$I^*_{X_i,Y_i}(\alpha; t_1, t_2) \leq (\geq) I^*_{X_i,Z_i}(\alpha; t_1, t_2) + \log\left[\frac{h^*_{Z_i|Z_j}(t_i|t_j)}{h^*_{Y_i|Y_j}(t_i|t_j)}\right].$$

Proof. The proof is similar to that of Theorem 5.4.6 □

Example 5.5.4. *Consider the rvs discussed in Example 5.5.3. so that*

$$\frac{g_i^*(x_i|t_j)}{q_i^*(x_i|t_j)} = \frac{\mu_i}{\gamma_i} e^{-(\mu_i - \gamma_i)x_i}$$

is decreasing in x_i. *Therefore,*

$$\begin{aligned}
I^*_{X_i,Y_i}(\alpha;t_1,t_2) &= \frac{1}{\alpha-1}\log\left[\frac{\lambda_i^{\alpha}\mu_i^{1-\alpha}}{(\lambda_i-\mu_i)\alpha+\mu_i}\right], \\
&= \frac{1}{\alpha-1}\log\left[\frac{\lambda_i^{\alpha}\gamma_i^{1-\alpha}}{(\lambda_i-\gamma_i)\alpha+\gamma_i}\right] \\
&\qquad + \log(\frac{\gamma_i}{\mu_i}) + \frac{1}{\alpha-1}\log\left[\frac{(\lambda_i-\gamma_i)\alpha+\gamma_i}{(\lambda_i-\mu_i)\alpha+\mu_i}\right], \\
&\leq (\geq)\frac{1}{\alpha-1}\log\left[\frac{\lambda_i^{\alpha}\gamma_i^{1-\alpha}}{(\lambda_i-\gamma_i)\alpha+\gamma_i}\right] + \log(\frac{\gamma_i}{\mu_i}), \\
&= \log\left[\frac{h^*_{Z_i|Z_j}(t_i|t_j)}{h^*_{Y_i|Y_j}(t_i|t_j)}\right] + I^*_{X_i,Z_i}(\alpha;t_1,t_2),
\end{aligned}$$

provided $\alpha > 1$ *(*$0 < \alpha < 1$*).* □

5.6 Conditional Kerridge's inaccuracy measure of type 1

In this section we extend the Kerridge's inaccuracy measure, given in (1.12) to the conditionally specified rvs $(X_i|X_j = t_j)$ and $(Y_i|Y_j = t_j)$.

Definition 5.6.1. *Let* (X_1, X_2) *and* (Y_1, Y_2) *be two non-negative bivariate random vectors admitting absolutely continuous dfs. If* $f_i(t_i|t_j)$ *and* $g_i(t_i|t_j)$, $i \neq j$, $i, j = 1, 2$ *denote the conditional densities of* $X_i|X_j = t_j$ *and* $Y_i|Y_j = t_j$ *respectively, then conditional Kerridge's inaccuracy measure of type 1 (CKIM*$_1$*) is defined by*

$$K_{X_i,Y_i}(t_1,t_2) = -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log \frac{g_i(x_i|t_j)}{\bar{G}_i(t_i|t_j)} dx_i,$$

for $i, j = 1, 2$, $i \neq j$ *and* $t_1, t_2 \geq l$, *where* $\bar{F}_i(t_i|t_j)$ *and* $\bar{G}_i(t_i|t_j)$ *are the sfs of* $X_i|X_j = t_j$ *and* $Y_i|Y_j = t_j$ *respectively. Note that* $K_{X_i,Y_i}(t_1,t_2) = K_{(X_i|X_j=t_j),(Y_i|Y_j=t_j)}(t_i)$ *for* $i, j = 1, 2, i \neq j$.

Then we have the following characterization result for the Arnold and Strauss's bivariate exponential distribution obtained in Arnold and Strauss (1991).

Theorem 5.6.1. *Let* (X_1, X_2) *and* (Y_1, Y_2) *be two random vectors with common support* $(0, \infty) \times (0, \infty)$ *and that satisfy the CPHR model given in (5.2) for* $i, j = 1, 2,\ i \neq j$ *then the following conditions are equivalent:*

(a) $K_{X_i,Y_i}(t_1, t_2)$ *depends only on* t_j *for* $i, j = 1, 2, i \neq j$.

(b) (X_1, X_2) *has the following joint pdf*

$$f(x_1, x_2) = c\exp(-\lambda_1 x_1 - \lambda_2 x_2 - \theta\lambda_1\lambda_2),$$

for $x_1, x_2 \geq 0$, *where* $c > 0$, $\theta \geq 0$ *and* $\lambda_i > 0$ *for* $i = 1, 2$.

Proof. If (X_1, X_2) and (Y_1, Y_2) satisfy the CPHR model given in (5.2), then

$$\bar{G}_i(t_i|t_j) = \left[\bar{F}_i(t_i|t_j)\right]^{\theta_i(t_j)}.$$

To prove that (a) implies (b), let us assume that $K_{X_i,Y_i}(t_1, t_2)$ depends only on t_j for $i, j = 1, 2,\ i \neq j$.

Then for $i = 1$, we have

$$-\int_{t_1}^{\infty} f_1(x_1|t_2)\log(\theta_1(t_2)\left[\bar{F}_1(x_1|t_2)\right]^{\theta_1(t_2)-1} f_1(x_1|t_2))dx_1$$
$$= C_1(t_2)\bar{F}_1(t_1|t_2) - \theta_1(t_2)\bar{F}_1(t_1|t_2)\log\bar{F}_1(t_1|t_2).$$

Differentiating with respect to t_1, we get

$$f_1(t_1|t_2)\log(\theta_1(t_2)\left[\bar{F}_1(t_1|t_2)\right]^{\theta_1(t_2)-1} f_1(t_1|t_2))$$
$$= -C_1(t_2)f_1(t_1|t_2) + \theta_1(t_2)f_1(t_1|t_2)(1+\log\bar{F}_1(t_1|t_2)).$$

Hence

$$\log\left(\theta_1(t_2)\left[\bar{F}_1(t_1|t_2)\right]^{\theta_1(t_2)-1} f_1(t_1|t_2)\right) = -C_1(t_2) + \theta_1(t_2) + \log\left[\bar{F}_1(t_1|t_2)\right]^{\theta_1(t_2)}$$

and

$$\log(\theta_1(t_2)h_{X_1|X_2}(t_1|t_2)) = \theta_1(t_2) - C_1(t_2).$$

Therefore $h_{X_1|X_2}(t_1|t_2)$ depends only on t_2. Analogously it can be proved that $h_{X_2|X_1}(t_2|t_1)$ depends only on t_1. Hence, both the conditional distributions are Exponential and from Arnold and Strauss (1991) the pdf is as given in (b).

The converse part is straight forward. □

Finally in this section, we obtain bounds for CKIM$_1$ function by using the LR ordering.

Theorem 5.6.2. *For $i, j = 1, 2$, $i \neq j$, if $g_i(t_i|t_j)$ is decreasing, then*

$$K_{X_i,Y_i}(t_1,t_2) \geq -\log h_{Y_i|Y_j}(t_i|t_j).$$

Proof. Since $g_i(x_i|t_j)$ is decreasing in x_i, $g_i(x_i|t_j) \leq g_i(t_i|t_j)$, $\forall x_i > t_i$ and $\forall t_j$

$$\begin{aligned} K_{X_i,Y_i}(t_1,t_2) &= -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log\frac{g_i(x_i|t_j)}{\bar{G}_i(t_i|t_j)} dx_i, \\ &\geq -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log h_{Y_i|Y_j}(t_i|t_j) dx_i, \\ &= -\log(h_{Y_i|Y_j}(t_i|t_j)). \end{aligned}$$

□

Corollory 5.6.1. *For $i, j = 1, 2$, $i \neq j$, if $f_i^w(t_i|t_j)$ is decreasing, then*

$$K_{X_i, X_i^w}(t_1, t_2) \geq -\log(h_{X_i|X_i^w}(t_i|t_j)).$$

Theorem 5.6.3. *For $i, j = 1, 2$, $i \neq j$, if w_i is increasing, then*

$$K_{X_i, X_i^w}(t_1, t_2) \leq I_{X_i}(t_1, t_2) + \log\left[\frac{E(w_i(X_i)|X_i > t_i, X_j = t_j)}{w_i(t_i)}\right],$$

where $I_{X_i}(t_1, t_2) = -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} dx_i$ is the residual entropy of $(X_i|X_j = t_j)$.

Proof. Since $w_i(x_i)$ is increasing x_i, $w_i(x_i) \geq w_i(t_i)$, $\forall x_i > t_i$

$$\begin{aligned}
K_{X_i, X_i^w}(t_1, t_2) &= -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log\left[\frac{w_i(x_i) f_i(x_i|t_j)}{E(w_i(X_i)|X_i > t_i, X_j = t_j)\bar{F}_i(t_i|t_j)}\right] dx_i, \\
&\leq -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log\left[\frac{w_i(t_i) f_i(x_i|t_j)}{E(w_i(X_i)|X_i > t_i, X_j = t_j)\bar{F}_i(t_i|t_j)}\right] dx_i, \\
&= -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log\left(\frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)}\right) dx_i + \log\left[\frac{E(w_i(X_i)|X_i > t_i, X_j = t_j)}{w_i(t_i)}\right], \\
&= I_{X_i}(t_1, t_2) + \log\left[\frac{E(w_i(X_i)|X_i > t_i, X_j = t_j)}{w_i(t_i)}\right].
\end{aligned}$$

□

Remark 5.6.1. *Using joint pdf considered in Example 5.4.1. and using weight function as $w_i(x_i) = x_i$, Corollary 5.6.1. and Theorem 5.6.3. can be illustrated.*

Theorem 5.6.4. *For $i, j = 1, 2$, $i \neq j$, if $(X_i|X_j = t_j) \leq_{LR} (Y_i|Y_j = t_j)$, then*

$$K_{X_i, Z_i}(t_1, t_2) \geq \frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)} K_{Y_i, Z_i}(t_1, t_2).$$

Proof. Since $X_i|X_j = t_j \leq_{\text{LR}} Y_i|Y_j = t_j$, $\frac{f_i(x_i|t_j)}{g_i(x_i|t_j)}$ is decreasing in x_i, $\forall t_j$.

$$i.e., \quad \frac{f_i(x_i|t_j)}{g_i(x_i|t_j)} \leq \frac{f_i(t_i|t_j)}{g_i(t_i|t_j)}, \ \forall x_i > t_i \text{ and } \forall t_j.$$

$$
\begin{aligned}
K_{X_i,Z_i}(t_1,t_2) &= -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log \frac{q_i(x_i|t_j)}{\bar{Q}_i(t_i|t_j)} dx_i, \\
&= -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j) g_i(x_i|t_j) \bar{G}_i(t_i|t_j)}{\bar{F}_i(t_i|t_j) \bar{G}_i(t_i|t_j) g_i(x_i|t_j)} \log \frac{q_i(x_i|t_j)}{\bar{Q}_i(t_i|t_j)} dx_i, \\
&\geq -\int_{t_i}^{\infty} \frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)} \frac{g_i(x_i|t_j)}{\bar{G}_i(t_i|t_j)} \log \frac{q_i(x_i|t_j)}{\bar{Q}_i(t_i|t_j)} dx_i, \\
&= \frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)} K_{Y_i,Z_i}(t_1,t_2).
\end{aligned}
$$

□

Theorem 5.6.5. *For $i,j = 1,2,\ i \neq j$, if $(Y_i|Y_j = t_j) \leq_{LR} (Z_i|Z_j = t_j)$, then*

$$
K_{X_i,Y_i}(t_1,t_2) \geq K_{X_i,Z_i}(t_1,t_2) + \log \frac{h_{Z_i|Z_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}.
$$

Proof. Since $Y_i|Y_j = t_j \leq_{\text{LR}} Z_i|Z_j = t_j$, $\frac{g_i(x_i|t_j)}{q_i(x_i|t_j)}$ is decreasing in x_i, $\forall t_j$,

$$
\begin{aligned}
&i.e., \quad \frac{g_i(x_i|t_j)}{q_i(x_i|t_j)} \leq \frac{g_i(t_i|t_j)}{q_i(t_i|t_j)}\ \forall x_i > t_i \text{ and } \forall t_j. \\
K_{X_i,Y_i}(t_1,t_2) &= -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log \frac{g_i(x_i|t_j)}{\bar{G}_i(t_i|t_j)} dx_i, \\
&= -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log \frac{g_i(x_i|t_j)}{q_i(x_i|t_j)} \frac{q_i(x_i|t_j)}{\bar{Q}_i(t_i|t_j)} \frac{\bar{Q}_i(t_i|t_j)}{\bar{G}_i(t_i|t_j)} dx_i, \\
&\geq -\int_{t_i}^{\infty} \frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \log \frac{q_i(x_i|t_j)}{\bar{Q}_i(t_i|t_j)} \frac{h_{Y_i|Y_j}(t_i|t_j)}{h_{Z_i|Z_j}(t_i|t_j)} dx_i, \\
&= K_{X_i,Z_i}(t_1,t_2) + \log \frac{h_{Z_i|Z_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}.
\end{aligned}
$$

□

Example 5.6.1. *Let (X_1,X_2), (Y_1,Y_2) and (Z_1,Z_2) be 3 bivariate Pareto rvs with respective pdfs given by $f(x_1,x_2) = (c_1-1)(c_1-2)(1+a_1x_1+a_2x_2)^{-c_1}$; $x_1,x_2 > 0,\ a_1,a_2 > 0, c_1 > 1$; $g(x_1,x_2) = (c_2-1)(c_2-2)(1+a_1x_1+a_2x_2)^{-c_2}$; $x_1,x_2 > 0,\ a_1,a_2 > 0, c_2 > 1$ and $q(x_1,x_2) = (c_3-1)(c_3-2)(1+a_1x_1+a_2x_2)^{-c_3}$; $x_1,x_2 > 0,\ a_1,a_2 > 0, c_3 > 1$ such that $c_1 > c_2 > c_3$. So $\frac{g_i(x_i|t_j)}{q_i(x_i|t_j)} = \frac{(c_2-1)}{(c_3-1)} \frac{(1+a_ix_i+a_jt_j)^{-(c_2-c_3)}}{(1+a_jt_j)^{-(c_2-c_3)}}$ is decreasing in x_i, $\forall t_j$ and can be seen that Theorem 5.6.5. follows.* □

5.7 Conditional Kerridge's inaccuracy measure of type 2

In this section we study conditional Kerridge's inaccuracy measure of type 2 ($CKIM_2$), an extension of residual Kerridge's inaccuracy measure (1.12) to conditional survival rvs. The definition is as follows:

Definition 5.7.1. *Let (X_1, X_2) and (Y_1, Y_2) be two non-negative bivariate random vectors admitting absolutely continuous dfs. If $f_i^*(t_i|t_j)$ and $g_i^*(t_i|t_j)$, $i \neq j$, $i, j = 1, 2$ denote the conditional densities of $X_i|X_j > t_j$ and $Y_i|Y_j > t_j$ respectively, then dynamic inaccuracy measure for these conditionally specified distributions, called $CKIM_2$, can be defined as*

$$K^*_{X_i,Y_i}(t_1, t_2) = -\int_{t_i}^{\infty} \frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)} \log \frac{g_i^*(x_i|t_j)}{\bar{G}_i^*(t_i|t_j)} dx_i,$$

where $\bar{F}_i^*(t_i|t_j)$ and $\bar{G}_i^*(t_i|t_j)$ are the sfs of $X_i|X_j > t_j$ and $Y_i|Y_j > t_j$ respectively.

The $CKIM_2$ of $X_i|X_j > t_j$ and the corresponding weighted rv $X_i^w|X_j^w > t_j$ denoted by $K_{X_i,X_i^w}(t_1, t_2)$ is defined as

$$K^*_{X_i,X_i^w}(t_1, t_2) = -\int_{t_i}^{\infty} \frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)} \log \frac{f_i^{w^*}(x_i|t_j)}{\bar{F}_i^{w^*}(t_i|t_j)}, \quad i \neq j,\ i, j = 1, 2. \tag{5.24}$$

Using (5.19) and (5.20), (5.24) can be written as

$$K^*_{X_i,X_i^w}(t_1, t_2) = -\int_{t_i}^{\infty} \frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)} \log \left(\frac{w_i(x_i) f_i^*(x_i|t_j)}{E(w_i(X_i)|X_i > t_i, X_j > t_j) \bar{F}_i^*(t_i|t_j)} \right) dx_i,$$
$$i \neq j, i, j = 1, 2.$$

Theorem 5.7.1. *Let (X_1, X_2) be a non-negative bivariate random vector and let (X_1^w, X_2^w) be its weighted version. Assume that $X_i|X_j > t_j$ and $X_i^w|X_j^w > t_j$ satisfy conditional* *proportional hazards model (5.5). Then $K^*_{X_i,X_i^w}(t_1,t_2)$ is locally constant if and only if (X_1,X_2) follows Gumbel's bivariate exponential distribution.*

Proof. Assume that (X_1,X_2) follows Gumbel's bivariate exponential distribution. Then

$$K^*_{X_i,X_i^w}(t_1,t_2) = -\int_{t_i}^{\infty} \frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)} \log \frac{f_i^{w^*}(x_i|t_j)}{\bar{F}_i^{w^*}(t_i|t_j)},$$

$$= -\int_{t_i}^{\infty} \frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)} \log \left[\frac{\theta_i(t_j)\left[\bar{F}_i^*(x_i|t_j)\right]^{\theta_i(t_j)-1} f_i^*(x_i|t_j)}{\left[\bar{F}_i^*(t_i|t_j)\right]^{\theta_i(t_j)}} \right] dx_i.$$

For $i=1$, we have

$$\begin{aligned}
K^*_{X_1,X_1^w}(t_1,t_2) &= -\int_{t_1}^{\infty} \frac{f_1^*(x_1|t_2)}{\bar{F}_1^*(t_1|t_2)} \log \left[\frac{\theta_1(t_2)[\bar{F}_1^*(x_1|t_2)]^{\theta_1(t_2)-1} f_1^*(x_1|t_2)}{[\bar{F}_1^*(t_1|t_2)]^{\theta_1(t_2)}} \right] dx_1,\\
&= -\int_{t_1}^{\infty} \frac{-\frac{\partial}{\partial x_1}\bar{F}(x_1,t_2)}{\bar{F}(t_1,t_2)} \log \left[\frac{\theta_1(t_2)[\bar{F}(x_1,t_2)]^{\theta_1(t_2)-1}\left(-\frac{\partial}{\partial x_1}\bar{F}(x_1,t_2)\right)}{[\bar{F}(t_1,t_2)]^{\theta_1(t_2)}} \right] dx_1,\\
&= -\int_{t_1}^{\infty} \frac{(\lambda_1+\theta t_2)e^{-\lambda_1 x_1-\lambda_2 t_2-\theta x_1 t_2}}{e^{-\lambda_1 t_1-\lambda_2 t_2-\theta t_1 t_2}}\\
&\quad \log \left[\frac{\theta_1(t_2)[e^{-\lambda_1 x_1-\lambda_2 t_2-\theta x_1 t_2}]^{\theta_1(t_2)-1}\left((\lambda_1+\theta t_2)e^{-\lambda_1 x_1-\lambda_2 t_2-\theta x_1 t_2}\right)}{\left[e^{-\lambda_1 t_1-\lambda_2 t_2-\theta t_1 t_2}\right]^{\theta_1(t_2)}} \right] dx_1,\\
&= -\log\left(\frac{\theta_1(t_2)(\lambda_1+\theta t_2)}{e^{-(\lambda_1+\theta t_2)\theta_1(t_2)t_1}}\right) + \frac{(\lambda_1+\theta t_2)^2\theta_1(t_2)}{e^{-(\lambda_1+\theta t_2)t_1}} \int_{t_1}^{\infty} x_1 e^{-(\lambda_1+\theta t_2)x_1} dx_1,\\
&= -\log\left(\frac{\theta_1(t_2)(\lambda_1+\theta t_2)}{e^{-(\lambda_1+\theta t_2)\theta_1(t_2)t_1}}\right) + \frac{(\lambda_1+\theta t_2)^2\theta_1(t_2)}{e^{-(\lambda_1+\theta t_2)t_1}} \left[\frac{t_1 e^{-(\lambda_1+\theta t_2)t_1}}{(\lambda_1+\theta t_2)} + \frac{e^{-(\lambda_1+\theta t_2)t_1}}{(\lambda_1+\theta t_2)^2} \right],\\
&= \theta_1(t_2) - \log[\theta_1(t_2)(\lambda_1+\theta t_2)].
\end{aligned}$$

Similarly for $i=2$, we have

$$K^*_{X_2,X_2^w}(t_1,t_2) = \theta_2(t_1) - \log[\theta_2(t_1)(\lambda_2+\theta t_1)],$$

$$\textit{i.e. } K^*_{X_i,X_i^w}(t_1,t_2) = \theta_i(t_j) - \log\left[\theta_i(t_j)(\lambda_i+\theta t_j)\right], \text{ locally constant.}$$

Conversely, assume that $K^*_{X_i,X_i^w}(t_1,t_2)$ is locally constant. Then we have

$$-\int_{t_i}^{\infty}\frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)}\log\left[\frac{\theta_i(t_j)\left[\bar{F}_i^*(x_i|t_j)\right]^{\theta_i(t_j)-1}f_i^*(x_i|t_j)}{\left[\bar{F}_i^*(t_i|t_j)\right]^{\theta_i(t_j)}}\right]dx_i = C_i(t_j),$$

$$-\int_{t_i}^{\infty}\frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)}\log\left[\theta_i(t_j)\left[\bar{F}_i^*(x_i|t_j)\right]^{\theta_i(t_j)-1}f_i^*(x_i|t_j)\right]dx_i + \theta_i(t_j)\log\bar{F}_i^*(t_i|t_j) = C_i(t_j). \tag{5.25}$$

Differentiating (5.25) with respect to t_i, we get

$$-h^*_{X_i|X_j}(t_i|t_j)\int_{t_i}^{\infty}\frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)}\log\left[\theta_i(t_j)\left[\bar{F}_i^*(x_i|t_j)\right]^{\theta_i(t_j)-1}f_i^*(x_i|t_j)\right]dx_i + h^*_{X_i|X_j}(t_i|t_j)\log\left[\theta_i(t_j)\left[\bar{F}_i^*(x_i|t_j)\right]^{\theta_i(t_j)}h^*_{X_i|X_j}(t_i|t_j)\right] - \theta_i(t_j)h^*_{X_i|X_j}(t_i|t_j) = 0. \tag{5.26}$$

Using (5.25) and then dividing by $h^*_{X_i|X_j}(t_i|t_j)$, (5.26) reduces to

$$\log\left[\theta_i(t_j)h^*_{X_i|X_j}(t_i|t_j)\right] = \theta_i(t_j) - C_i(t_j)$$

or $h^*_{X_i|X_j}(t_i|t_j) = K_i(t_j)$ where $K_i(t_j) = \frac{e^{\theta_i(t_j)-C_i(t_j)}}{\theta_i(t_j)}$ which implies

$$\frac{f_i^*(t_i|t_j)}{\bar{F}_i^*(t_i|t_j)} = K_i(t_j),$$

$$\frac{-\frac{\partial}{\partial t_i}\bar{F}(t_1,t_2)}{\bar{F}(t_1,t_2)} = K_i(t_j),$$

$$\frac{\partial}{\partial t_i}\log\bar{F}(t_1,t_2) = -K_i(t_j),$$

$$\log\bar{F}(t_1,t_2) = -K_i(t_j)t_i + A_i(t_j),$$

$$\bar{F}(t_1, t_2) = e^{-K_i(t_j)t_i + A_i(t_j)}.$$

i.e.,

$$\bar{F}(t_1, t_2) = e^{-K_1(t_2)t_1 + A_1(t_2)}, \tag{5.27}$$

$$\bar{F}(t_1, t_2) = e^{-K_2(t_1)t_2 + A_2(t_1)}. \tag{5.28}$$

As $t_1 \to 0^+$ in (5.27) we have

$$\bar{F}_2(t_2) = e^{A_1(t_2)}. \tag{5.29}$$

Using (5.29), (5.27) becomes

$$\bar{F}(t_1, t_2) = e^{-K_1(t_2)t_1} \bar{F}_2(t_2). \tag{5.30}$$

Similarly, we get

$$\bar{F}(t_1, t_2) = e^{-K_2(t_1)t_2} \bar{F}_1(t_1). \tag{5.31}$$

As $t_2 \to 0^+$ in (5.30), we get

$$\bar{F}_1(t_1) = e^{-K_1(0)t_1} = e^{-\lambda_1 t_1}, \text{ where } \lambda_1 = K_1(0).$$

So (5.31) becomes

$$\bar{F}(t_1, t_2) = e^{-K_2(t_1)t_2} e^{-\lambda_1 t_1}. \tag{5.32}$$

Similarly, we get

$$\bar{F}(t_1, t_2) = e^{-K_1(t_2)t_1} e^{-\lambda_2 t_2}. \tag{5.33}$$

Equating (5.32) and (5.33), we get

$$e^{-K_2(t_1)t_2}e^{-\lambda_1 t_1} = e^{-K_1(t_2)t_1}e^{-\lambda_2 t_2}.$$

$$e^{(K_1(t_2)-\lambda_1)t_1} = e^{(K_2(t_1)-\lambda_2)t_2}.$$

Taking logarithms, we have

$$(K_1(t_2) - \lambda_1)t_1 = (K_2(t_1) - \lambda_2)t_2,$$

$$\frac{(K_1(t_2) - \lambda_1)}{t_2} = \frac{(K_2(t_1) - \lambda_2)}{t_1} = \theta,$$

which implies

$$K_i(t_j) = \lambda_i + \theta t_j, \quad i \neq j,\ i, j = 1, 2. \tag{5.34}$$

Substituting (5.34) either in (5.32) or (5.33) we get

$$\bar{F}(t_1, t_2) = e^{-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2}, \quad t_1, t_2 > 0,\ \lambda_1, \lambda_2, \theta \geq 0.$$

Hence the theorem follows. □

Theorem 5.7.2. *For $i, j = 1, 2, i \neq j$, if $g_i^*(t_i|t_j)$ is decreasing, then*

$$K^*_{X_i,Y_i}(t_1, t_2) \geq -\log\left(h^*_{Y_i|Y_j}(t_i|t_j)\right).$$

Proof. The proof is similar to that of Theorem 5.6.2. □

Corollary 5.7.1. *For $i, j = 1, 2, i \neq j$, if $f_i^{w^*}(x_i|t_j)$ is decreasing, then*

$$K^*_{X_i,X_i^w}(t_1, t_2) \geq -\log\left(h^*_{X_i^w|X_j^w}(t_i|t_j)\right).$$

Example 5.7.1. *Let (X_1, X_2) be a bivariate Pareto I rv specified by the sf*

$$\bar{F}(x_1, x_2) = x_1^{-\alpha_1} x_2^{-\alpha_2} x_1^{-\theta \log t_2}; \quad x_1, x_2 > 1,\ \alpha_1, \alpha_2, \theta > 1.$$

Using the weight function $w_i(x_i) = x_i$, we have that $f_i^{*w}(x_i|t_j)$ is decreasing in x_i. Therefore,

$$\begin{aligned} K^*_{X_i, X_i^w}(t_1, t_2) &= \log\big(\frac{t_i}{\alpha_i + \theta \log t_j - 1}\big) + 1 \geq \log\big(\frac{t_i}{\alpha_i + \theta \log t_j - 1}\big), \\ &= \log\left[\frac{1}{h^*_{X_i^w | X_j^w}(t_i|t_j)}\right]. \end{aligned}$$

□

Theorem 5.7.3. *For $i, j = 1, 2, i \neq j$, if w_i is increasing, then*

$$K^*_{X_i, X_i^w}(t_1, t_2) \leq I^*_{X_i}(t_1, t_2) + \log\left[\frac{E(w_i(X_i)|X_i > t_i, X_j > t_j)}{w_i(t_i)}\right],$$

where $I^*_{X_i}(t_1, t_2) = -\int_{t_i}^{\infty} \frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)} \log \frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)} dx_i$ is the residual entropy function of $X_i|X_j > t_j$.

Proof. The proof is similar to that of Theorem 5.6.3 □

Example 5.7.2. *Consider the sf given in Example 5.7.1. Take the weight function as $w_i(x_i) = x_i$ which is increasing in x_i. Then,*

$$\begin{aligned} K^*_{X_i, X_i^w}(t_1, t_2) &= \log\big(\frac{t_i}{\alpha_i + \theta \log t_j - 1}\big) + 1, \\ &= \log\big(\frac{t_i}{\alpha_i + \theta \log t_j}\big) + \big(\frac{\alpha_i + \theta \log t_j + 1}{\alpha_i + \theta \log t_j}\big) + \log\big(\frac{\alpha_i + \theta \log t_j}{\alpha_i + \theta \log t_j - 1}\big) \\ &\qquad - \big(\frac{1}{\alpha_i + \theta \log t_j}\big), \\ &\leq \log\big(\frac{t_i}{\alpha_i + \theta \log t_j}\big) + \big(\frac{\alpha_i + \theta \log t_j + 1}{\alpha_i + \theta \log t_j}\big) + \log\big(\frac{\alpha_i + \theta \log t_j}{\alpha_i + \theta \log t_j - 1}\big), \\ &= I^*_{X_i}(t_1, t_2) + \log\left[\frac{E(w_i(X_i)|X_i > t_i, X_j > t_j)}{w_i(t_i)}\right]. \end{aligned}$$

□

Theorem 5.7.4. *For $i, j = 1, 2, i \neq j$, if $(X_i|X_j > t_j) \leq_{LR} (Y_i|Y_j > t_j)$, then*

$$K^*_{X_i,Z_i}(t_1, t_2) \geq \frac{h^*_{X_i|X_j}(t_i|t_j)}{h^*_{Y_i|Y_j}(t_i|t_j)} K^*_{Y_i,Z_i}(t_1, t_2).$$

Proof. The proof is similar to that of Theorem 5.6.4. □

Theorem 5.7.5. *For $i, j = 1, 2, i \neq j$, if $(Y_i|Y_j > t_j) \leq_{LR} (Z_i|Z_j > t_j)$, then*

$$K^*_{X_i,Y_i}(t_1, t_2) \geq K^*_{X_i,Z_i}(t_1, t_2) + \log \left[\frac{h^*_{Z_i|Z_j}(t_1|t_2)}{h^*_{Y_i|Y_j}(t_1|t_2)} \right].$$

Proof. The proof is similar to that of Theorem 5.6.5. □

Example 5.7.3. *Let $(X_1, X_2), (Y_1, Y_2)$ and (Z_1, Z_2) be 3 bivariate Pareto rvs with respective sfs given by $\bar{F}(x_1, x_2) = (1 + a_1x_1 + a_2x_2)^{-c_1}$; $x_1, x_2 > 0$, $a_1, a_2, c_1 > 0$; $\bar{G}(x_1, x_2) = (1 + a_1x_1 + a_2x_2)^{-c_2}$; $x_1, x_2 > 0$, $a_1, a_2, c_2 > 0$; and $\bar{Q}(x_1, x_2) = (1 + a_1x_1 + a_2x_2)^{-c_3}$; $x_1, x_2 > 0$, $a_1, a_2, c_3 > 0$ such that $c_1 > c_2 > c_3$. So*

$$\frac{g^*_i(x_i|t_j)}{q^*_i(x_i|t_j)} = \frac{c_2}{c_3} \frac{(1 + a_ix_i + a_jt_j)^{-(c_2-c_3)}}{(1 + a_jt_j)^{-(c_2-c_3)}}$$

is decreasing in x_i, $\forall t_j$. Therefore,

$$\begin{aligned}
K^*_{X_i,Y_i}(t_1, t_2) &= \log \Big(\frac{1 + a_1t_1 + a_2t_2}{c_2a_i}\Big) + \Big(\frac{c_2 + 1}{c_1}\Big),\\
&= \log \Big(\frac{1 + a_1t_1 + a_2t_2}{c_3a_i}\Big) + \Big(\frac{c_3 + 1}{c_1}\Big) + \log \Big(\frac{c_3}{c_2}\Big) + \frac{c_2 - c_3}{c_1},\\
&\geq \log \Big(\frac{1 + a_1t_1 + a_2t_2}{c_3a_i}\Big) + \Big(\frac{c_3 + 1}{c_1}\Big) + \log \Big(\frac{c_3}{c_2}\Big),\\
&= \log \frac{h^*_{Z_i|Z_j}(t_i|t_j)}{h^*_{Y_i|Y_j}(t_i|t_j)} + K^*_{X_i,Z_i}(t_1, t_2).
\end{aligned}$$

□

Chapter 6

Some properties of residual R-norm entropy and divergence measures

6.1 Introduction

Several generalizations of the classical Shannon's entropy are available in literature, by introducing some additional parameters which make these entropies sensitive to different shapes of probability distributions (see, for example, Renyi (1961), Kapur (1967) and Tsallis (1988)). All these entropies, when the additional parameters tend to one, reduce to the classical Shannon's entropy (1.4). Another important generalization of Shannon's information is due to Boekee and Lubbe (1980), called as R-norm entropy, which is initially defined in the discrete case. One can also refer to Kumar and Choudhary (2011). A continuous version of R-norm entropy is available in Nanda and Das (2006), given by

$$H_X(R) = \frac{R}{R-1}\left[1 - \left(\int_0^\infty f^R(x)dx\right)^{1/R}\right], \tag{6.1}$$

which is a real-valued function, for $R \neq 1$, $R \in (0, \infty)$. When $R \to 1$ (6.1) reduces to (1.4). Using similar argument in (1.5), the residual R-norm entropy for an item survived for t units of time is given by (Nanda and Das (2006))

$$H_X(R;t) = \frac{R}{R-1}\left[1 - \left(\int_t^\infty \left(\frac{f(x)}{\bar{F}(t)}\right)^R dx\right)^{1/R}\right], \tag{6.2}$$

for $R \neq 1$, $R \in (0, \infty)$. Note that, as $R \to 1$, (6.2) approaches (1.5). Different generalizations of R-norm entropy could be found in Hooda and Ram (1998), Hooda (2001) Kumar (2009), Choudhary and Kumar (2011).

There are different generalizations on Kullback-Leibler divergence. In preceding chapters we came across Renyi's divergence measure, a well known generalization of Kullback-Leibler divergence measure. Another generalization of Kullback-Leibler divergence is due to Nanda and Das (2006), based on the R-norm entropy, and its residual form. This residual R-norm divergence measure for two absolutely continuous rvs X and Y with common support $S = (l, \infty)$ for $l \geq 0$ is given by

$$H_{X,Y}(R;t) = \frac{R}{R-1}\left[\left(\int_t^\infty \frac{f(x)}{\bar{F}(t)}\left(\frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)}\right)^{R-1} dx\right)^{\frac{1}{R}} - 1\right]. \tag{6.3}$$

For properties of the residual R-norm entropy (6.2) and its divergence measure (6.3), we refer to Nanda and Das (2006).

In this chapter, we further study R-norm entropy and divergence in the context of weighted models and study its properties using some stochastic orderings. We also extend these measures to the doubly truncated rvs, conditionally specified models and conditional survival models and prove results that characterize some well known bivariate lifetime models.

6.2 Weighted residual R-norm entropy and divergence

Using (6.2), the residual R-norm entropy for weighted rv X^w is given by

$$H_{X^w}(R,t) = \frac{R}{R-1}\left[1 - \left(\int_t^\infty \left(\frac{f^w(x)}{\bar{F}^w(t)}\right)^R dx\right)^{\frac{1}{R}}\right]. \tag{6.4}$$

The corresponding weighted residual R-norm divergence measure based on (6.2) is of the form

$$H_{X,X^w}(R,t) = \frac{R}{R-1}\left[\left(\int_t^\infty \frac{f(x)}{\bar{F}(t)}\left(\frac{f(x)/\bar{F}(t)}{f^w(x)/\bar{F}^w(t)}\right)^{R-1} dx\right)^{\frac{1}{R}} - 1\right]. \tag{6.5}$$

In the following theorem we obtain a bound that connects $H_{X^w}(R;t)$ and $H_X(R;t)$

Theorem 6.2.1. *If $w(x)$ is increasing in x, then for $R \neq 1$*

$$\left[1 - \left(\frac{R-1}{R}\right) H_{X^w}(R;t)\right] \geq \frac{h_{X^w}(t)}{h_X(t)}\left(1 - \left(\frac{R-1}{R}\right) H_X(R;t)\right),$$

where $h_X(t)$ and $h_{X^w}(t)$ denote the failure (hazard) rate functions of X and X^w respectively.

Proof. Since $w(x)$ is increasing in x, we have $w(x) \geq w(t)\ \forall x > t$. Therefore

$$\begin{aligned}
\left[1 - \left(\frac{R-1}{R}\right) H_{X^w}(R;t)\right] &= \left(\int_t^\infty \left(\frac{f^w(x)}{\bar{F}^w(t)}\right)^R dx\right)^{\frac{1}{R}}, \\
&= \left(\int_t^\infty \left(\frac{w(x)f(x)}{[E(w(X)|X>t)]\bar{F}(t)}\right)^R dx\right)^{\frac{1}{R}}, \\
&\geq \frac{w(t)}{[E(w(X)|X>t)]}\left(\int_t^\infty \left(\frac{f(x)}{\bar{F}(t)}\right)^R dx\right)^{\frac{1}{R}},
\end{aligned}$$

$$= \frac{h_{X^w}(t)}{h_X(t)} \left[1 - \left(\frac{R-1}{R}\right) H_X(R;t)\right]. \qquad \square$$

Example 6.2.1. *Consider Pareto I distribution with pdf given by $f(x) = ck^c x^{-c-1}$, $x > k$, $k > 0$, $c > 1$. Using the weight function $w(x) = x$ which is increasing we get*

$$\begin{aligned}
\left[1 - \left(\frac{R-1}{R}\right) H_{X^w}(R;t)\right] &= \frac{(c-1)}{t^{R-1}(cR-1)}, \\
&= \left(\frac{c-1}{c}\right)\left(\frac{cR+R-1}{cR-1}\right)\left(\frac{c}{t^{R-1}(cR+R-1)}\right), \\
&= \left(\frac{cR+R-1}{cR-1}\right)\left(\frac{h_{X^w}(t)}{h_X(t)}\right)\left[1 - \left(\frac{R-1}{R}\right) H_X(R;t)\right], \\
&\geq \frac{h_{X^w}(t)}{h_X(t)} \left[1 - \left(\frac{R-1}{R}\right) H_X(R;t)\right]. \qquad \square
\end{aligned}$$

We now obtain a bound for the residual R-norm divergence using likelihood ratio (LR) ordering.

Theorem 6.2.2. *If $X \leq_{LR} Y$, then for $R > 1$ $(0 < R < 1)$*

$$1 + \left(\frac{R-1}{R}\right) H_{X,Y}(R;t) \leq (\geq) \left(\frac{h_X(t)}{h_Y(t)}\right)^{\frac{R-1}{R}}.$$

Proof. Since $X \leq_{LR} Y$, $\frac{f(x)}{g(x)}$ is decreasing in x. Therefore $\frac{f(x)}{g(x)} \leq \frac{f(t)}{g(t)}$ for every $x > t$. For $R > 1$,

$$\begin{aligned}
1 + \left(\frac{R-1}{R}\right) H_{X,Y}(R;t) &= \left(\int_t^\infty \frac{f(x)}{\bar{F}(t)} \left(\frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)}\right)^{R-1} dx\right)^{\frac{1}{R}}, \\
&\leq \left(\int_t^\infty \frac{f(x)}{\bar{F}(t)} \left(\frac{h_X(t)}{h_Y(t)}\right)^{R-1} dx\right)^{\frac{1}{R}},
\end{aligned}$$

proves the result. The case for $0 < R < 1$ is similar. $\square$

Corollary 6.2.1. *If $X \leq_{LR} X^w$, then for $R > 1$ $(0 < R < 1)$*

$$\left(1 + \left(\frac{R-1}{R}\right) H_{X,X^w}(R;t)\right) \leq (\geq) \left(\frac{h_X(t)}{h_{X^w}(t)}\right)^{\frac{R-1}{R}}.$$

Remark 6.2.1. *Corollary 6.2.1 can be easily illustrated using Example 6.2.1.*

The following theorem provides bounds for $H_{X,Y}(R;t)$ in terms of the hazard function of Y and $H_X(R;t)$.

Theorem 6.2.3. *If $g(x)$ is decreasing in x, then for $R > 1$ $(0 < R < 1)$*

$$\left[1 + \left(\frac{R-1}{R}\right) H_{X,Y}(R;t)\right] \geq (\leq) \left[\frac{1}{h_Y(t)}\right]^{\frac{R-1}{R}} \left[1 - \left(\frac{R-1}{R}\right) H_X(R;t)\right].$$

Proof. Since $g(x)$ is decreasing in x, $g(x) \leq g(t)\ \forall x > t$. For $R > 1$, we have

$$\begin{aligned}
\left[1 + \left(\frac{R-1}{R}\right) H_{X,Y}(R;t)\right] &= \left(\int_t^\infty \frac{f(x)}{\bar{F}(t)} \left(\frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)}\right)^{R-1} dx\right)^{\frac{1}{R}}, \\
&\geq \left[\frac{1}{h_Y(t)}\right]^{\frac{R-1}{R}} \left(\int_t^\infty \frac{f^R(x)}{\bar{F}^R(t)} dx\right)^{\frac{1}{R}}, \\
&= \left[\frac{1}{h_Y(t)}\right]^{\frac{R-1}{R}} \left[1 - \left(\frac{R-1}{R}\right) H_X(R;t)\right].
\end{aligned}$$

For $0 < R < 1$, the inequality is reversed. □

Corollary 6.2.2. *If $f^w(x)$ is decreasing in x, then for $R > 1$ $(0 < R < 1)$*

$$\left[1 + \left(\frac{R-1}{R}\right) H_{X,X^w}(R;t)\right] \geq (\leq) \left[\frac{1}{h_{X^w}(t)}\right]^{\frac{R-1}{R}} \left[1 - \left(\frac{R-1}{R}\right) H_X(R;t)\right].$$

Example 6.2.2. *Consider an exponential rv with pdf $f(x) = \lambda e^{-\lambda x}$, $x > 0$, $\lambda > 0$. Take the weight function as $w(x) = e^{-ax}$, $a > 0$. Then $f^w(x) = (\lambda + a)e^{-(\lambda+a)x}$, $\lambda, a > 0$,*

$x > 0$ *is decreasing in* x*. Then* $R > 1$*, we get*

$$\begin{aligned}
\left[1+\left(\frac{R-1}{R}\right)H_{X,X^w}(R;t)\right] &= \frac{\lambda}{(\lambda-a(R-1))^{\frac{1}{R}}(\lambda+a)^{\frac{R-1}{R}}},\\
&= \frac{\lambda^{\frac{R-1}{R}}}{R^{\frac{1}{R}}(\lambda+a)^{\frac{R-1}{R}}}\left(\frac{\lambda R}{\lambda-a(R-1)}\right)^{\frac{1}{R}},\\
&= \left(\frac{\lambda R}{\lambda-a(R-1)}\right)^{\frac{1}{R}}\left(\frac{1}{h_{X^w}(t)}\right)^{\frac{R-1}{R}}\left[1-\left(\frac{R-1}{R}\right)H_X(R;t)\right],\\
&\geq \left(\frac{1}{h_{X^w}(t)}\right)^{\frac{R-1}{R}}\left[1-\left(\frac{R-1}{R}\right)H_X(R;t)\right]. \qquad \square
\end{aligned}$$

In certain reliability studies, often one has information about the lifetime only between two time points. That is, individuals whose event time lies within a certain time interval are only observed. Based on this idea, Kotlarski (1972) studied the conditional expectation for the doubly (interval) truncated rvs. Later, Navarro and Ruiz (1996) generalized the failure rate and the conditional expectation to the doubly truncated rvs. It is shown that generalized failure rate (GFR) and the conditional expectation for doubly truncated rvs determine the distribution uniquely. More properties of GFR and conditional expectation in the context of characterization problems are available in Ruiz and Navarro (1996), Navarro and Ruiz (2004) and Sunoj et al. (2009) and the references therein. Motivated by this idea, we extend the definition of R-norm entropy to the double truncated rvs and study certain properties. A straightforward extension of R-norm entropy to the doubly truncated rvs is given by

$$H_X(R;t_1,t_2) = \frac{R}{R-1}\left[1-\left(\int_{t_1}^{t_2}\frac{f^R(x)}{[\bar{F}(t_1)-\bar{F}(t_2)]^R}dx\right)^{\frac{1}{R}}\right],$$

$H_X(R;t_1,t_2)$ obtained for different distributions is given in Table 6.1.

The corresponding residual R-norm divergence measure for doubly truncated rvs is of

Sl. No.	Distributions	$f(t)$	$H_X(R;t_1,t_2)$
1	Uniform	$\frac{1}{a}$; $0<t<a,\ a>0$	$\frac{R}{R-1}\left[1-\frac{(t_2-t_1)^{1/R}}{(t_2-t_1)}\right]$
2	Exponential	$\lambda e^{-\lambda t}$; $\lambda>0,\ t>0$	$\frac{R}{R-1}\left[1-\frac{\lambda}{(e^{-\lambda t_1}-e^{-\lambda t_2})}\left[\frac{(e^{-\lambda R t_1}-e^{-\lambda R t_2})}{\lambda R}\right]^{\frac{1}{R}}\right]$
3	Pareto I	$\frac{c}{k}\left(\frac{k}{t}\right)^{c+1}$ $t>k,\ c,k>0$	$\frac{R}{R-1}\left[1-\frac{ck^c}{k^c\left(t_1^{-c}-t_2^{-c}\right)}\left[\frac{t_1^{-cR-R+1}-t_2^{-cR-R+1}}{cR+R-1}\right]^{\frac{1}{R}}\right]$
4	Pareto II	$pq(1+pt)^{-q-1}$; $p>0,\ q>0,\ t>0$	$\frac{R}{R-1}\left[1-\frac{pq}{(1+pt_1)^{-q}-(1+pt_2)^{-q}}\left[\frac{(1+pt_1)^{-Rq-R+1}-(1+pt_2)^{-Rq-R+1}}{p(Rq+R-1)}\right]^{\frac{1}{R}}\right]$
5	Beta	$ab(1-at)^{b-1}$; $a>0,\ b>0,\ 0<t<\frac{1}{a}$	$\frac{R}{R-1}\left[1-\frac{ab}{(1-at_1)^b-(1-at_2)^b}\left[\frac{(1-at_1)^{bR-R+1}-(1-at_2)^{bR-R+1}}{a(bR-R+1)}\right]^{\frac{1}{R}}\right]$

Table 6.1:

the form

$$H_{X,Y}(R;t_1,t_2)=\frac{R}{R-1}\left[\left\{\int_{t_1}^{t_2}\frac{f(x)}{[\bar{F}(t_1)-\bar{F}(t_2)]}\left(\frac{f(x)/\bar{F}(t_1)-\bar{F}(t_2)}{g(x)/\bar{G}(t_1)-\bar{G}(t_2)}\right)^{R-1}dx\right\}^{\frac{1}{R}}-1\right].$$

If X^w denotes the weighted rv corresponding to X, then the residual R-norm divergence measure for doubly truncated rvs between X and X^w is given by

$$H_{X,X^w}(R;t_1,t_2)=\frac{R}{R-1}\left[\left\{\int_{t_1}^{t_2}\frac{f(x)}{[\bar{F}(t_1)-\bar{F}(t_2)]}\left(\frac{f(x)/\bar{F}(t_1)-\bar{F}(t_2)}{f^w(x)/\bar{F}^w(t_1)-\bar{F}^w(t_2)}\right)^{R-1}dx\right\}^{\frac{1}{R}}-1\right].$$

In the following theorem we have obtained bounds for $\left[1+\left(\frac{R-1}{R}\right)H_{X,Y}(R;t_1,t_2)\right]$ in terms of the generalised failure rates of X and Y.

Theorem 6.2.4. *If $X\leq_{LR}Y$, then for $R>1$ $(0<R<1)$*

$$\left[\frac{h_2^X(t_1,t_2)}{h_2^Y(t_1,t_2)}\right]^{\frac{R-1}{R}}\leq(\geq)\left[1+\left(\frac{R-1}{R}\right)H_{X,Y}(R;t_1,t_2)\right]\leq(\geq)\left[\frac{h_1^X(t_1,t_2)}{h_1^Y(t_1,t_2)}\right]^{\frac{R-1}{R}},\quad(6.6)$$

where $h_i^X(t_1,t_2) = \frac{f(t_i)}{\bar{F}(t_1)-\bar{F}(t_2)}$*,* $i = 1,2$ *and* $h_i^Y(t_1,t_2) = \frac{g(t_i)}{\bar{G}(t_1)-\bar{G}(t_2)}$*,* $i = 1,2$ *are the generalized failure rates (GFRs) of the rvs* X *and* Y *respectively.*

Proof. Since $X \le_{LR} Y$, we have $\frac{f(x)}{g(x)}$ is decreasing in x. That is,

$$\frac{f(t_2)}{g(t_2)} \le \frac{f(x)}{g(x)} \le \frac{f(t_1)}{g(t_1)}, \quad \forall t_1 < x < t_2.$$

Then for $R > 1$ we have

$$\begin{aligned}\left[1+\left(\frac{R-1}{R}\right)H_{X,Y}(R;t_1,t_2)\right] &= \left[\int_{t_1}^{t_2}\frac{f(x)}{[\bar{F}(t_1)-\bar{F}(t_2)]}\left(\frac{f(x)/\bar{F}(t_1)-\bar{F}(t_2)}{g(x)/\bar{G}(t_1)-\bar{G}(t_2)}\right)^{R-1}dx\right]^{\frac{1}{R}},\\ &\le \left[\int_{t_1}^{t_2}\frac{f(x)}{[\bar{F}(t_1)-\bar{F}(t_2)]}\left(\frac{h_1^X(t_1,t_2)}{h_1^Y(t_1,t_2)}\right)^{R-1}dx\right]^{\frac{1}{R}},\end{aligned}$$

which proves the upper bound of (6.6). In a similar way, we can obtain the lower bound. The case for $0 < R < 1$ is similar. □

Corollory 6.2.3. *If* $w(x)$ *is increasing in* x *(or* $X \le_{LR} X^w$*) then for* $R > 1$ *(*$0 < R < 1$*)*

$$\left[\frac{h_2^X(t_1,t_2)}{h_2^{X^w}(t_1,t_2)}\right]^{\frac{R-1}{R}} \le (\ge) \left[1+\left(\frac{R-1}{R}\right)H_{X,X^w}(R;t_1,t_2)\right] \le (\ge) \left[\frac{h_1^X(t_1,t_2)}{h_1^{X^w}(t_1,t_2)}\right]^{\frac{R-1}{R}}.$$

Theorem 6.2.5. *If* $g(x)$ *is decreasing in* x*, then for* $R > 1$ *(*$0 < R < 1$*)*

$$\begin{aligned}\left[\frac{1}{h_1^Y(t_1,t_2)}\right]^{\frac{R-1}{R}}\left[1-\left(\frac{R-1}{R}\right)H_X(R;t_1,t_2)\right] &\le (\ge) \left[1+\left(\frac{R-1}{R}\right)H_{X,Y}(R;t_1,t_2)\right],\\ &\le (\ge) \left[\frac{1}{h_2^Y(t_1,t_2)}\right]^{\frac{R-1}{R}}\left[1-\left(\frac{R-1}{R}\right)H_X(R;t_1,t_2)\right]. \qquad (6.7)\end{aligned}$$

Proof. If $g(x)$ is decreasing in x, we have $g(t_1) \ge g(x) \ge g(t_2)$, $\forall t_1 < x < t_2$. For $R > 1$,

$$\left[1+\left(\frac{R-1}{R}\right)H_{X,Y}(R;t_1,t_2)\right] = \left[\int_{t_1}^{t_2}\frac{f(x)}{[\bar{F}(t_1)-\bar{F}(t_2)]}\left(\frac{f(x)/\bar{F}(t_1)-\bar{F}(t_2)}{g(x)/\bar{G}(t_1)-\bar{G}(t_2)}\right)^{R-1}dx\right]^{\frac{1}{R}},$$

$$\geq \left[\int_{t_1}^{t_2} \frac{f(x)}{[\bar{F}(t_1)-\bar{F}(t_2)]} \left(\frac{f(x)/\bar{F}(t_1)-\bar{F}(t_2)}{g(t_1)/\bar{G}(t_1)-\bar{G}(t_2)}\right)^{R-1} dx\right]^{\frac{1}{R}},$$

$$= \left[\int_{t_1}^{t_2} \left(\frac{f(x)}{[\bar{F}(t_1)-\bar{F}(t_2)]}\right)^{R} \left(\frac{1}{h_1^Y(t_1,t_2)}\right)^{R-1} dx\right]^{\frac{1}{R}}.$$

proves the lower bound for (6.7). The upper bound for the same is similarly obtained. The case for $0 < R < 1$ is similar. □

6.3 Residual R-norm entropy and divergence for conditionally specified models

In this section we study the residual R-norm entropy measure (6.2) and divergence measure (6.3) based on it for conditionally specified models. Let (X_1, X_2) and (Y_1, Y_2) be two bivariate random vectors with respect to Lesbegue measure in the positive quadrant $\mathbb{R}_2^+ = \{(t_1, t_2)|t_i > 0, i = 1, 2\}$ of the two dimensional Euclidean space $\mathbb{R}_2$. The joint pdf and sf of (X_1, X_2) are denoted by f and $\bar{F}$ and that of (Y_1, Y_2) by g and $\bar{G}$, respectively. Consider the conditionally specified rvs $(X_i|X_j = t_j)$ and $(Y_i|Y_j = t_j)$ for $i, j = 1, 2$, $i \neq j$. Their pdfs, sfs and hazard rates are denoted by $f_i(t_i|t_j)$, $g_i(t_i|t_j)$, $\bar{F}_i(t_i|t_j)$, $\bar{G}_i(t_i|t_j)$, $h_{X_i|X_j}(t_i|t_j)$, $h_{Y_i|Y_j}(t_i|t_j)$ respectively for $i, j = 1, 2$, $i \neq j$. Using (6.2), the residual R-norm entropy for conditionally specified rv $(X_i|X_j = t_j)$ is defined by

$$H_{X_i}(R; t_1, t_2) = \frac{R}{R-1}\left[1 - \left(\int_{t_i}^{\infty} \left(\frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)}\right)^{R} dx_i\right)^{\frac{1}{R}}\right]. \tag{6.8}$$

Note that $H_{X_i}(R; t_1, t_2) = H_{(X_i|X_j=t_j)}(R; t_i)$. Now we have a characterization theorem that establishes a relationship between $H_{X_i}(R; t_1, t_2)$ and $h_{X_i|X_j}(t_i|t_j)$.

Theorem 6.3.1. *For the random vector (X_1, X_2), the relationship*

$$H_{X_i}(R;t_1,t_2) = \frac{R}{R-1}\left[1-\left(C(h_{X_i|X_j}(t_i|t_j))^{R-1}\right)^{\frac{1}{R}}\right], \tag{6.9}$$

where C is a constant independent of t_1 and t_2 holds if and only if it is distributed as

(a) Bivariate distribution with Pareto conditional given in Arnold (1987) with pdf

$$f(x_1,x_2) = c_1(1+a_1x_1+a_2x_2+bx_1x_2)^{-c}, \ \ a_1,a_2>0, b\geq 0, \ c>2, \ c_1>0, \ \textit{the normalizing constant}; \ x_1,x_2>0,$$

or

(b) Bivariate distribution with exponential conditionals of Arnold and Strauss (1988) with pdf

$$f(x_1,x_2) = c_2 e^{-\alpha_1x_1-\alpha_2x_2-\beta x_1x_2}, \ \ \alpha_1,\alpha_2>0, \beta\geq 0, \ c_2>0, \ \textit{the normalizing constant}; \ x_1,x_2>0,$$

or

(c) Bivariate distribution with beta conditionals with pdf

$$f(x_1,x_2) = c_3(1-p_1x_1-p_2x_2+qx_1x_2)^{d}, \ \ p_1,p_2,d>0, q\geq 0, c_3>0, \ \textit{the normalizing constant}; \ 0<x_1<\frac{1}{p_1}, 0<x_2<\frac{1-p_1x_1}{p_2-qx_1},$$

according as $C \lesseqqgtr \frac{1}{R}$ for $R>1$ and $C \gtreqqless \frac{1}{R}$ for $0<R<1$.

Proof. The first part is straightforward. To prove the converse, we assume that (6.9) holds and assume that $R > 1$.

$$\text{Then,} \quad \int_{t_i}^{\infty} \left(\frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)} \right)^R dx_i = C(h_{X_i|X_j}(t_i|t_j))^{R-1},$$

$$\int_{t_i}^{\infty} (f_i(x_i|t_j))^R dx_i = C(h_{X_i|X_j}(t_i|t_j))^{R-1}(\bar{F}_i(t_i|t_j))^R. \tag{6.10}$$

Differentiating (6.10) with respect to t_i, we get

$$\begin{aligned} -(f_i(t_i|t_j))^R = C(R-1)(h_{X_i|X_j}(t_i|t_j))^{R-2}\frac{\partial}{\partial t_i}h_{X_i|X_j}(t_i|t_j)(\bar{F}_i(t_i|t_j))^R \\ - CR(h_{X_i|X_j}(t_i|t_j))^{R-1}(\bar{F}_i(t_i|t_j))^{R-1}f_i(t_i|t_j). \end{aligned} \tag{6.11}$$

Dividing (6.11) by $(\bar{F}_i(t_i|t_j))^R(h_{X_i|X_j}(t_i|t_j))^R$, we get

$$-1 = \frac{C(R-1)}{(h_{X_i|X_j}(t_i|t_j))^2}\frac{\partial}{\partial t_i}h_{X_i|X_j}(t_i|t_j) - CR$$

or

$$\frac{CR-1}{C(R-1)} = -\frac{\partial}{\partial t_i}\left(\frac{1}{h_{X_i|X_j}(t_i|t_j)}\right).$$

Equivalently,

$$\frac{CR-1}{C(1-R)} = \frac{\partial}{\partial t_i}\left(\frac{1}{h_{X_i|X_j}(t_i|t_j)}\right). \tag{6.12}$$

Integrating (6.12) with respect to t_i, we obtain

$$\left(\frac{1}{h_{X_i|X_j}(t_i|t_j)}\right) = \frac{CR-1}{C(1-R)}t_i + B_i(t_j) = At_i + B_i(t_j). \tag{6.13}$$

where $A = \frac{CR-1}{C(1-R)}$. Equation (6.13) is equivalent to

$$h_{X_i|X_j}(t_i|t_j) = \frac{1}{At_i + B_i(t_j)}.$$

The remaining part of the proof follows directly from Theorem 4.4.2 of Chapter 4. Similar steps holds for $0 < R < 1$. $\square$

Now we consider some results on bivariate weighted distribution when the status of one component is known in advance, more discussion on which is available in Section 5.2 of Chapter 5.

Theorem 6.3.2. *If $w_i(x_i)$ is decreasing in x_i for $i = 1, 2$, then for $R \neq 1$*

$$\left[1 - \left(\frac{R-1}{R}\right) H_{X_i^w}(R; t_1, t_2)\right] \leq \frac{h_{X_i^w|X_j^w}(t_i|t_j)}{h_{X_i|X_j}(t_i|t_j)} \left[1 - \left(\frac{R-1}{R}\right) H_{X_i}(R; t_1, t_2)\right].$$

Proof. By definition, we have

$$\begin{aligned}
\left[1 - \left(\frac{R-1}{R}\right) H_{X_i^w}(R; t_1, t_2)\right] &= \left(\int_{t_i}^{\infty} \frac{(f_i^w(x_i|t_j))^R}{(\bar{F}_i^w(t_i|t_j))^R} dx_i\right)^{\frac{1}{R}}, \\
&= \left(\int_{t_i}^{\infty} \frac{(w_i(x_i) f_i(x_i|t_j))^R}{(E(w_i(X_i)|X_i > t_i, X_j = t_j)\bar{F}_i(t_i|t_j))^R} dx_i\right)^{\frac{1}{R}}, \\
&\leq \frac{w_i(t_i)}{E(w_i(X_i)|X_i > t_i, X_j = t_j)} \left(\int_{t_i}^{\infty} \frac{(f_i(x_i|t_j))^R}{(\bar{F}_i(t_i|t_j))^R} dx_i\right)^{\frac{1}{R}}, \\
&= \frac{h_{X_i^w|X_j^w}(t_i|t_j)}{h_{X_i|X_j}(t_i|t_j)} \left[1 - \left(\frac{R-1}{R}\right) H_{X_i}(R; t_1, t_2)\right].
\end{aligned}$$

$\square$

Example 6.3.1. *Suppose (X_1, X_2) follows Arnold and Strauss bivariate exponential distribution with joint pdf $f(t_1, t_2) = Ke^{-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2}$, $t_1, t_2 > 0$, $K, \lambda_1, \lambda_2 > 0, \theta \geq 0$, Taking the weight $w_i(x_i) = \frac{1}{x_i}$, a decreasing function, we obtain*

$$\begin{aligned}
\left[1 - \left(\frac{R-1}{R}\right) H_{X_i^w}(R; t_1, t_2)\right] &= \frac{(\lambda_i + \theta t_j + 1)}{[R(\lambda_i + \theta t_j + 1)]^{\frac{1}{R}}}, \\
&= \left(\frac{\lambda_i + \theta t_j + 1}{\lambda_i + \theta t_j}\right) \frac{(\lambda_i + \theta t_j)}{[R(\lambda_i + \theta t_j)]^{\frac{1}{R}}} \left(\frac{\lambda_i + \theta t_j}{\lambda_i + \theta t_j + 1}\right)^{\frac{1}{R}}, \\
&= \frac{h_{X_i^w|X_j^w}(t_i|t_j)}{h_{X_i|X_j}(t_i|t_j)} \left[1 - \left(\frac{R-1}{R}\right) H_{X_i}(R; t_1, t_2)\right] \left(\frac{\lambda_i + \theta t_j}{\lambda_i + \theta t_j + 1}\right)^{\frac{1}{R}},
\end{aligned}$$

$$\leq \frac{h_{X_i^w|X_j^w}(t_i|t_j)}{h_{X_i|X_j}(t_i|t_j)}\left[1-\left(\frac{R-1}{R}\right)H_{X_i}(R;t_1,t_2)\right]. \qquad \square$$

We now define the conditional residual R-norm divergence between the rvs $(X_i|X_j = t_j)$ and $(Y_i|Y_j = t_j)$ as

$$H_{X_i,Y_i}(R;t_1,t_2) = \frac{R}{R-1}\left[\left\{\int_{t_i}^{\infty}\frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)}\left(\frac{f_i(x_i|t_j)/\bar{F}_i(t_i|t_j)}{g_i(x_i|t_j)/\bar{G}_i(t_i|t_j)}\right)^{R-1}dx_i\right\}^{\frac{1}{R}}-1\right], \tag{6.14}$$

where $H_{X_i,Y_i}(R;t_1,t_2) = H_{(X_i|X_j=t_j),(Y_i|Y_j=t_j)}(R;t_i)$. Hence (6.14) provides dynamic information on the distance between the conditionally specified rvs $(X_i|X_j = t_j)$ and $(Y_i|Y_j = t_j)$.

Now we study what is the impact on $H_{X_i,Y_i}(R;t_1,t_2)$ when $(X_i|X_j = t_j)$ and $(Y_i|Y_j = t_j)$ satisfy PHR models. Nanda and Das (2006) obtained the following result for univariate rvs.

Theorem 6.3.3. *(Nanda and Das(2006)) $H_R(X,Y;t)$is independent of t if and only if F and G satisfy the PHR model.*

Then we have the following result.

Theorem 6.3.4. *For $i,j = 1,2$, $i \neq j$, the function $H_{X_i,Y_i}(R;t_1,t_2)$ depends only on t_j if and only if $(Y_i|Y_j = t_j)$ and $(X_i|X_j = t_j)$ satisfy the CPHR model* (5.2).

Proof. The proof is obtained from Theorem 6.3.3 using (5.2) and the fact that

$$H_{X_i,Y_i}(R;t_1,t_2) = H_{(X_i|X_j=t_j),(Y_i|Y_j=t_j)}(R;t_i). \qquad \square$$

Theorem 6.3.5. *Let (X_1^w, X_2^w) be a random vector having bivariate weighted distribution associated to (X_1, X_2) and to non-negative differentiable functions w_1 and w_2.*

Assume that the support of (X_1, X_2) is $S = (l,\infty)\times(l,\infty)$ for $l \geq 0$. Then the following conditions are equivalent.

(a) (X_1^w, X_2^w) and (X_1, X_2) satisfy the CPHR model (5.2)

(b) $H_{X_i,X_i^w}(R; t_1, t_2)$ is independent of t_i for $i = 1, 2$ and $(\theta_i(t_j) - 1)(1 - R) + 1 > 0$

(c) The conditional reliability functions of (X_1, X_2) satisfy

$$\log \bar{F}_i(t_i|t_j) = \frac{\log[w_i(t_i)/w_i(l)]}{\theta_i(t_j) - 1}.$$

d) (X_1, X_2) has the following joint pdf

$$f(x_1, x_2) = ca_1a_2\frac{w_1'(x_1)w_2'(x_2)}{w_1^{a_1+1}(x_1)w_2^{a_2+1}(x_2)}\exp\left(-\phi a_1a_2\log\left[\frac{w_1(x_1)}{w_1(l)}\right]\log\left[\frac{w_2(x_2)}{w_2(l)}\right]\right),$$

for $x_1, x_2 \geq l$ where $c > 0$, $\phi \geq 0$ and $a_i > 1$ or $a_i < 0$ for $i = 1, 2$.

Proof. The equivalence between (a) and (b) is a consequence of Theorem 6.3.2. The rest of the proof is similar to Theorem 5.2.3 of Chapter 5. □

In the following theorem, a bound for conditional residual R-norm divergence is obtained using the LR ordering.

Theorem 6.3.6. *If $(X_i|X_j = t_j) \leq_{LR} (Y_i|Y_j = t_j)$ for $i, j = 1, 2$, $i \neq j$, then for $R > 1$ ($0 < R < 1$)*

$$\left[1 + \left(\frac{R-1}{R}\right)H_{X_i,Y_i}(R; t_1, t_2)\right] \leq (\geq) \left(\frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}\right)^{\frac{R-1}{R}}.$$

Proof. $(X_i|X_j = t_j) \leq_{\mathrm{LR}} (Y_i|Y_j = t_j)$ implies that $\frac{f_i(x_i|t_j)}{g_i(x_i|t_j)}$ is decreasing in x_i.

$$i.e., \frac{f_i(x_i|t_j)}{g_i(x_i|t_j)} \leq \frac{f_i(t_i|t_j)}{g_i(t_i|t_j)}, \quad \forall x_i > t_j.$$

Now using (6.18) and for $R > 1$ we obtain

$$\begin{aligned}
\left[1+\left(\frac{R-1}{R}\right)H_{X_i,Y_i}(R;t_1,t_2)\right] &= \left(\int_{t_i}^{\infty}\frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)}\left(\frac{f_i(x_i|t_j)/\bar{F}_i(t_i|t_j)}{g_i(x_i|t_j)/\bar{G}_i(t_i|t_j)}\right)^{R-1}dx_i\right)^{\frac{1}{R}},\\
&\leq \left(\frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}\right)^{\frac{R-1}{R}}\left(\int_{t_i}^{\infty}\frac{f_i(x_i|t_j)}{\bar{F}_i(t_i|t_j)}dx_i\right)^{\frac{1}{R}},\\
&= \left(\frac{h_{X_i|X_j}(t_i|t_j)}{h_{Y_i|Y_j}(t_i|t_j)}\right)^{\frac{R-1}{R}}.
\end{aligned}$$

The case for $0 < R < 1$ is similar. □

Corollory 6.3.1. *If $(X_i|X_j = t_j) \leq_{LR} (X_i^w|X_j^w = t_j)$ for $i, j = 1, 2$, $i \neq j$, then for $R > 1$ ($0 < R < 1$)*

$$\left[1+\left(\frac{R-1}{R}\right)H_{X_i,X_i^w}(R;t_1,t_2)\right] \leq (\geq) \left(\frac{h_{X_i|X_j}(t_i|t_j)}{h_{X_i^w|X_j^w}(t_i|t_j)}\right)^{\frac{R-1}{R}}.$$

6.4 Residual R-norm entropy and divergence for conditional survival models

In this section, we consider the conditional survival rvs $(X_i|X_j > t_j)$ and $(Y_i|Y_j > t_j)$; $i, j = 1, 2$, $i \neq j$. Their pdf, sf and hazard rates are denoted by $f_i^*(t_i|t_j)$, $g_i^*(t_i|t_j)$, $\bar{F}_i^*(t_i|t_j)$, $\bar{G}_i^*(t_i|t_j)$, $h^*_{X_i|X_j}(t_i|t_j)$, $h^*_{Y_i|Y_j}(t_i|t_j)$ respectively for $i, j = 1, 2$, $i \neq j$. Using (6.2), the conditional survival residual R-norm entropy for $(X_i|X_j > t_j)$ is defined as

$$H^*_{X_i}(R;t_1,t_2) = \frac{R}{R-1}\left[1-\left(\int_{t_i}^{\infty}\frac{(f_i^*(x_i|t_j))^R}{(\bar{F}_i^*(t_i|t_j))^R}dx_i\right)^{\frac{1}{R}}\right]. \tag{6.15}$$

Now we have the following characterization theorem.

Theorem 6.4.1. *For the random vector (X_1, X_2) the relationship*

$$H^*_{X_i}(R;t_1,t_2) = \frac{R}{R-1}\left(1-\left(K(h^*_{X_i|X_j}(t_i|t_j))^{R-1}\right)^{\frac{1}{R}}\right), \tag{6.16}$$

where K is a constant independent of t_1 and t_2 holds if and if it is distributed as

(a) Bivariate Pareto with sf

$$\bar{F}(x_1,x_2) = (1+a_1x_1+a_2x_2+bx_1x_2)^{-c}, \quad a_1,a_2,c>0, b\geq 0\,; x_1,x_2>0, \tag{6.17}$$

or

(b) Gumbel's bivariate exponential with sf

$$\bar{F}(x_1,x_2) = \exp(-\alpha_1x_1-\alpha_2x_2-\beta x_1x_2), \quad \alpha_1,\alpha_2>0, \beta\geq 0; x_1,x_2>0, \tag{6.18}$$

or

(c) Bivariate beta with sf

$$\bar{F}(x_1,x_2) = (1-p_1x_1-p_2x_2+qx_1x_2)^{d}, \quad p_1,p_2,d>0, q\geq 0$$
$$0<x_1<\frac{1}{p_1},\ 0<x_2<\frac{1-p_1x_1}{p_2-qx_1}, \tag{6.19}$$

according as $K \underset{>}{\overset{\leq}{=}} \frac{1}{R}$ for $R>1$ and $K \underset{<}{\overset{\geq}{=}} \frac{1}{R}$ for $0<R<1$.

Proof. The proof for the first part of the theorem is direct. To prove the converse part, assume that (6.16) holds. For $R>1$, equation (6.16) is equivalent to

$$\int_{t_i}^{\infty}\frac{(f^*_i(x_i|t_j))^R}{(\bar{F}^*_i(t_i|t_j))^R}dx_i = K(h^*_{X_i|X_j}(t_i|t_j))^{R-1},$$
$$\int_{t_i}^{\infty}(f^*_i(x_i|t_j))^R dx_i = K(h^*_{X_i|X_j}(t_i|t_j))^{R-1}(\bar{F}^*_i(t_i|t_j))^R. \tag{6.20}$$

Differentiating (6.20) with respect to t_i, we get

$$-(f_i^*(t_i|t_j))^R = K(R-1)(h^*_{X_i|X_j}(t_i|t_j))^{R-2}\frac{\partial}{\partial t_i}h^*_{X_i|X_j}(t_i|t_j)(\bar{F}_i^*(t_i|t_j))^R \\ - KR(h^*_{X_i|X_j}(t_i|t_j))^{R-1}(\bar{F}_i^*(t_i|t_j))^{R-1}f_i^*(t_i|t_j),$$

$$-(f_i^*(t_i|t_j))^R = K(R-1)(h^*_{X_i|X_j}(t_i|t_j))^{R-2}\frac{\partial}{\partial t_i}h^*_{X_i|X_j}(t_i|t_j)(\bar{F}_i^*(t_i|t_j))^R \\ - KR(h^*_{X_i|X_j}(t_i|t_j))^{R}(\bar{F}_i^*(t_i|t_j))^{R}. \quad (6.21)$$

Dividing (6.21) by $(\bar{F}_i^*(t_i|t_j))^R(h^*_{X_i|X_j}(t_i|t_j))^R$, yield

$$K(R-1)\frac{\frac{\partial}{\partial t_i}(h^*_{X_i|X_j}(t_i|t_j))}{(h^*_{X_i|X_j}(t_i|t_j))^2} = KR-1,$$

$$K(1-R)\frac{\partial}{\partial t_i}\left(\frac{1}{h^*_{X_i|X_j}(t_i|t_j)}\right) = KR-1. \quad (6.22)$$

Integrating (6.22) with respect to t_i, we obtain

$$\frac{1}{(h^*_{X_i|X_j}(t_i|t_j))} = \frac{KR-1}{K(1-R)}t_i + B_i(t_j) = At_i + B_i(t_j),$$

where $A = \frac{KR-1}{K(1-R)}$. Thus, $h^*_{X_i|X_j}(t_i|t_j) = \frac{1}{At_i+B_i(t_j)}$. Now using the result of Roy (1989), the models (6.17), (6.18) and (6.19) follow. The case for $0 < R < 1$ can be similarly obtained. □

The following theorems give the bounds for conditional residual R-norm entropy.

Theorem 6.4.2. *If $w_i(x_i)$ is decreasing in x_i for $i = 1, 2$, then for $R \neq 1$*

$$\left[1 - \left(\frac{R-1}{R}\right) H^*_{X_i^w}(R; t_1, t_2)\right] \leq \frac{h^*_{X_i^w|X_j^w}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)} \left[1 - \left(\frac{R-1}{R}\right) H^*_{X_i}(R; t_1, t_2)\right].$$

Proof. The proof is similar to that of Theorem 6.3.2. □

Example 6.4.1. *Suppose (X_1, X_2) follows bivariate Pareto I distribution with joint sf*

$$\bar{F}(x_1, x_2) = x_1^{-\alpha_1} x_2^{-\alpha_2} x_1^{-\theta \log x_2};\ x_1, x_2 > 1, \alpha_1, \alpha_2, \theta > 0.$$

Taking the weight $w_i(x_i) = \frac{1}{x_i}$, a decreasing function, we obtain

$$\begin{aligned}
\left[1 - \left(\frac{R-1}{R}\right) H^*_{X_i^w}(R; t_1, t_2)\right] &= (\alpha_i + \theta \log t_j + 1) \left[\frac{t_i^{-R+1}}{\alpha_i R + \theta R \log t_j + 2R - 1}\right]^{\frac{1}{R}}, \\
&= \left(\frac{\alpha_i + \theta \log t_j + 1}{\alpha_i + \theta \log t_j}\right) \left[(\alpha_i + \theta \log t_j) \left(\frac{t_i^{-R+1}}{\alpha_i R + \theta R \log t_j + R - 1}\right)^{\frac{1}{R}}\right] \\
&\quad \left(\frac{\alpha_i R + \theta R \log t_j + R - 1}{\alpha_i R + \theta R \log t_j + 2R - 1}\right)^{\frac{1}{R}}, \\
&= \frac{h^*_{X_i^w|X_j^w}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)} \left[1 - \left(\frac{R-1}{R}\right) H^*_{X_i}(R; t_1, t_2)\right] \left(\frac{\alpha_i R + \theta R \log t_j + R - 1}{\alpha_i R + \theta R \log t_j + 2R - 1}\right)^{\frac{1}{R}}, \\
&\leq \frac{h^*_{X_i^w|X_j^w}(t_i|t_j)}{h^*_{X_i|X_j}(t_i|t_j)} \left[1 - \left(\frac{R-1}{R}\right) H^*_{X_i}(R; t_1, t_2)\right].
\end{aligned}$$

□

We now define the conditional survival residual R-norm divergence between the rvs $(X_i|X_j > t_j)$ and $(Y_i|Y_j > t_j)$ as

$$H^*_{X_i,Y_i}(R; t_1, t_2) = \frac{R}{R-1} \left[\left\{\int_{t_i}^{\infty} \frac{f_i^*(x_i|t_j)}{\bar{F}_i^*(t_i|t_j)} \left(\frac{f_i^*(x_i|t_j)/\bar{F}_i^*(t_i|t_j)}{g_i^*(x_i|t_j)/\bar{G}_i^*(t_i|t_j)}\right)^{R-1} dx_i\right\}^{\frac{1}{R}} - 1\right].$$

It is to be noted that $H^*_{X_i,Y_i}(R; t_1, t_2) = H_{(X_i|X_j>t_j),(Y_i|Y_j>t_j)}(R; t_i)$. Hence $H^*_{X_i,Y_i}(R; t_1, t_2)$ provides dynamic information on the distance between the conditional survival rvs

$(X_i|X_j > t_j)$ and $(Y_i|Y_j > t_j)$. Now we have the following characterization theorem using conditional proportional hazards model (5.5).

Theorem 6.4.3. *For $i, j = 1, 2$, $i \neq j$, $H^*_{X_i,Y_i}(R; t_1, t_2)$ depends only on t_j if and only if $(X_i|X_j > t_j)$ and $(Y_i|Y_j > t_j)$ satisfy conditional proportional hazards model (5.5).*

Proof. The proof is obtained from Theorem 6.3.3 using (5.5) and the fact that

$$H^*_{X_i,Y_i}(R; t_1, t_2) = H_{(X_i|X_j>t_j),(Y_i|Y_j>t_j)}(R; t_i). \qquad \square$$

Corollory 6.4.1. *For $i, j = 1, 2$, $i \neq j$, $H^*_{X_i,X_i^w}(R; t_1, t_2)$ depends only on t_j if and only if $X_i|X_j > t_j$ and $X_i^w|X_j^w > t_j$ satisfy conditional proportional hazards model (5.5).*

Corollory 6.4.2. *For $i, j = 1, 2$, $i \neq j$, $H^*_{X_i,X_i^w}(R; t_1, t_2)$ depends only on t_j if and only if $w_i(t_i) = [\bar{F}_i^*(t_i|t_j)]^{\theta_i(t_j)-1}$ where $\theta_i(t_j)$ is a function of t_j only.*

Proof. Using Theorem 5.3.1 and Corollary 6.4.1, Corollary 6.4.2 can be proved. $\square$

The following theorems provide bounds for conditional survival residual R-norm divergence.

Theorem 6.4.4. *If $(X_i|X_j > t_j) \leq_{LR} (Y_i|Y_j > t_j)$ for $i, j = 1, 2$, $i \neq j$, then for $R > 1$ $(0 < R < 1)$*

$$\left[1 + \left(\frac{R-1}{R}\right) H^*_{X_i,Y_i}(R; t_1, t_2)\right] \leq (\geq) \left(\frac{h^*_{X_i|X_j}(t_i|t_j)}{h^*_{Y_i|Y_j}(t_i|t_j)}\right)^{\frac{R-1}{R}}.$$

Proof. The proof is similar to that of Theorem 6.3.6. $\square$

Corollory 6.4.3. *If $(X_i|X_j > t_j) \leq_{LR} (X_i^w|X_j^w > t_j)$ for $i, j = 1, 2$, $i \neq j$, then for $R > 1$ $(0 < R < 1)$*

$$\left[1 + \left(\frac{R-1}{R}\right) H^*_{X_i,X_i^w}(R; t_1, t_2)\right] \leq (\geq) \left(\frac{h^*_{X_i|X_j}(t_i|t_j)}{h^*_{X_i^w|X_j^w}(t_i|t_j)}\right)^{\frac{R-1}{R}}.$$

6.5 Future Study

In the present study we have considered three types of uncertainty measures *viz.* Shannon's entropy, Renyi's entropy and R-norm entropy. These measures came under a family of uncertainty measures namely ϕ entropy introduced by Burbea and Rao (1982) which is given by

$$H_\phi(X) = H_\phi(P_\theta) = \int_X \phi(f_\theta(x))d\mu(x),$$

where P_θ is the probability distribution of the rv X, $\phi : (0, \infty) \to \mathbb{R}$ is a continuous concave function and $\phi(0) = \lim_{t \downarrow 0} \phi(t) \in (-\infty, \infty)$. Shannon entropy and Renyi's entropy can be viewed as special cases of ϕ entropy with $\phi(x) = -x \log x$ and $\phi(x) = x^\alpha$ respectively. Studies similar to that considered in the present work can be taken up for ϕ entropies also. Similarly one can come across ϕ divergence measures defined by

$$D_\phi(P_{\theta_1}, P_{\theta_2}) = \int_X f_{\theta_2}(x)\phi\left(\frac{f_{\theta_1}(x)}{f_{\theta_2}(x)}\right) d\mu(x), \ \phi \in \Phi_*$$

where Φ_* is the class of all convex functions $\phi(x)$, $x \geq 0$. If $\phi(x) = x \log x - x + 1$ the above measure reduces to Kullback-Leibler divergence. Similar works in the present study can be carried out for ϕ divergence measures also.

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