QUEUES WITH INTERRUPTION IN RANDOM/MARKOVIAN ENVIRONMENT

Thesis submitted to the
Cochin University of Science and Technology
for the award of the degree of

DOCTOR OF PHILOSOPHY

under the Faculty of Science by

JAYA.S

Department of Mathematics
Cochin University of Science and Technology
Cochin-682022

AUGUST 2015

Certificate

This is to certify that the thesis entitled '**QUEUES WITH INTERRUPTION IN RANDOM/MARKOVIAN ENVIRONMENT** ' submitted to the Cochin University of Science and Technology by Ms.Jaya.S for the award of the degree of Doctor of Philosophy under the Faculty of Science is a bonafide record of studies carried out by her under my supervision in the Department of Mathematics, Cochin University of Science and Technology. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.

Dr. B.Lakshmy
(Research Guide)
Associate Professor
Department of Mathematics
Cochin University of Science and Technology
Kochi - 682 022, Kerala

Cochin-22
03-08-2015.

Declaration

I, Jaya.S., hereby declare that this thesis entitled **'Queues with interruption in Random/ Markovian Environment'** contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of the thesis.

Jaya.S
Research Scholar
Registration No. 3367
Department of Mathematics
Cochin University of Science and Technology
Cochin-682 022, Kerala.

Cochin-22
03-08-2015.

To

My Amma, Chachan,

Moorthy Sir

And To

God

Acknowledgement

An eventful journey, filled with anxieties, started eight years ago has come to a fruitful end. At this happy juncture I gratefully remember all those who have helped me directly or indirectly in this journey.

First of all let me bow my head before the font memories of my mother who was with me at the beginning of this journey. She continues to be the greatest inspiration in my life.

It is always an encouraging and enriching experience to interact with great people. Their in-depth knowledge, vision and affection always motivate to achieve high. Let me thank Dr A. Krishnamoorthy, Professor(Emeritus), Department of Mathematics, CUSAT, for the great support, guidance, patience and the valuable time spent for me. Thank you sir, for supporting my attendance at various conferences and engaging me in new ideas.

Now let me thank my Guide and supervisor Dr.B.Lakshmy, Associate Professor, Department of Mathematics, CUSAT, for her unstinting support since the the day I began working on this and for giving me intellectual freedom. I am indebted for her inspiring ideas which contributed a lot in the development of the thesis.

I express my gratitude to UGC for granting me FDP Deputation, without which this project would have a different end.

The teachers in Mathematics Department of CUSAT have been a source of inspiration since the days of my post graduation. I would like to express my respect and thanks to Prof. T. Thrivikraman, who has always been and will continue to be the role model in my pursuit of excellence. I also remember with gratitude the great teachers Dr. M. Jathavedan, Dr. M. K. Ganapathy, Dr. R. S. Chakravarti, Dr. M. N. Narayanan Namboothiri and Dr. A. Vijayakumar.

Dr. Romeo P. G, Head, Dept of Mathematics, CUSAT was a constant support for me. I specially thank him for his valuable suggestions. His approach and consideration helped me a lot. I also thank the office staff and librarian of the Department of Mathematics for their support and help of various kinds. My gratitude also goes to the authorities of Cochin University of Science and Technology for the facilities they provided I also thank Mr. Syam Sunder for his timely help. He supported me academically by sharing his Knowledge in MATLAB and LATEX.

Let me, with great respect recall my teacher, Professor Gopalakrishnan Unnithan, Former Prof. D.B.College Sasthamcottah, who, during my college days, planted in my heart the dream of persuing higher studies at CUSAT.

I also thank all my friends and colleagues in Maharaja's college, Ernaklam, for the care, concern and help extended to me. Special thanks are due to Dr.N.Shaji, Department of Physics, Maharaja's College, who was the coordinator of UGC cell for the timely help and support. I remember with great gratitude the support and valuable counsel of my colleagues Ms. Ashakumari, Ms. Sudha Rani, Dr. George Mathew, Ms. Safiya, Mr. Mathai, Ms.Jaya Augustine, Mr.Murali, Department of Mathematics, Maharaja's college, Ernaklam.

I take this opportunity to place on record my sincere gratitude to the late Mr.Rajendra Prasad Pillai, former HOD, Department of Mathematics, Maharaja's college, Ernakulam, whose untimely demise is deeply mourned.

I would like to thank my fellow research scholars Dr. Sreenivasan, Dr.Varghese Jacob, Mr. Pravas, Mr.Satheesh, Dr.Pramod, Mr.Gopakumar, Dr.Kiran kumar, Mr.Gireesn, Mr.Tijo James, Mr. Balesh, Mr.Didimos, Dr.Chithra, Dr.Raji George, Dr. Pamy Sebastian, Dr. Manikandan, Dr.Jayaprasad, Dr. Resmi Varghese, Ms.Elizabath Reshma, Ms. Anu Varghese, Ms Anusha, Ms.Viji, Ms.Treesa, Miss.Smisha, Ms.Savitha. K.N, Ms.Sreeja, Ms. Sindhu Rani, Ms.Bineetha, Miss.Lijo and others for their love and support.

A very special thanks to my dear friends Ms.Savitha, Ms Resmi, Ms. Seethu, Ms Akhila, Miss Dhanya and Mr. Manjunath for making the life at CUSAT memorable. Their timely help, advice, affection and support helped me a lot to get rid off my tensions and worries.

Let me also express my heart felt gratitude to my friends Ms Lakshmi and Ms Meera Gopal of Maharaja's college, Ernakulam. They have been a constant source of encouragement and motivation from the very beginning.

I specially thank my father K.Sivadasan for his selfless support and understanding. I am very lucky to have a father like him. Childhood is the sweetest period in life and spent time with parents most valuable. My children Hari Krishnan(Kichu), Hari Narayanan(Kunchu), were denied of much of my presence and care during the past two years, but they cooperated with me in their own sweet way.It would be unfair on my part to overlook their support. Finally, and most importantly, I would like to thank my Husband Rajesh. His support, encouragement, quiet patience and unwavering love were undeniably the bedrock upon which the past eight years of my life have been built. His tolerance of my occasional vulgar moods is a testament in itself of his unyielding devotion and love. This accomplishment would not have been possible without him.

QUEUES WITH INTERRUPTION IN RANDOM/MARKOVIAN ENVIRONMENT

Contents

Notations

- $\boldsymbol{e}_a$ denotes column vector of 1's with order a;
- $\mathbf{0}$ is a vector consisting of 0's with appropriate dimension;
- $\boldsymbol{e}_r(j)$ denotes column vector of dimension r with 1 in the j^{th} position and 0 elsewhere;
- $\boldsymbol{e}_r^{'}(j)$ denotes the transpose of the column vector of dimension r with 1 in the j^{th} position and 0 elsewhere;
- I_k denotes identity matrix of order k.
- A^{-1} denotes the inverse of matrix A.
- $A \otimes B$ denotes the Kronecker product of A and B.
- diag(α) denotes diagonal matrix with α as diagonal entry.
- Z^+ denotes the set of all positive integers.

Abbreviation used

PH	:	Phase type;
MAP	:	Markovian Arrival Process;
$CTMC$	:	Continuous-time Markov Chain;
QBD	:	Quasi-birth-death;
LST	:	Laplace-Stieltjes Transform;
$LIQBD$	:	Level Independent Quasi-Birth-Death;
$LDQBD$	:	Level Dependent Quasi-Birth-Death;

Chapter 1

Preliminaries

1.1 Introduction

The history of queues starts from the pre-historic time. Recall the queue for the Noha's Ark. Even in nature the seasons are in queue for its turn. In all the cases where the demand for service is more than the facility available the result is a queue. A well organized queueing system is the requirement of any society to deliver the service in an efficient and effective manner. Both the service providing facility(server) and the people coming for service(customer) don't like queues. To provide delightful service to the customer using the limited resources at an optimal cost a scientific study became inevitable. This leads to the emergence and development of queueing theory.

The mathematical study of the queueing system is generally called as Queueing Theory. The history of queuing theory goes back to more than a century. The mathematical analysis of the queuing systems starts with the works by A.K. Erlang and T.O.Engest in the beginning of 1900s. Though lot of works were going on after Erlang published his first paper in 1909, with the introduction of Metric Analytic Method by Neuts, the study of queueing system gained a new momentum.

With the staggering growth in the fields of networking and communication technology, study of queueing system become very important. The queueing theoretic analysis is very important for the effective and economic use of the resources for rendering service. The scientific analysis of queueing system helps us to study the characteristics of the queuing system such as waiting time, service cost, optimum service rate, etc.
Some important elementary aspects of queueing theory which are required for the understanding of the thesis are discussed here.

1.2 Foundation

Stochastic processes

In many situations probability models are more realistic than deterministic models. Several phenomena occurring in physics are studied as random phenomena changing with time and space. Stochastic processes originated from the needs of physicists.
Let $X(t)$ be a random variable where t is a parameter assuming values from the set T. Then the collection of random variables $\{X(t), t \in T\}$ is called a stochastic process. We denote the state of the process at time t by $X(t)$ and the collection of all possible values $X(t)$ can assume, is called

state space.

Example: Consider the case of throwing a unbiased die. Let $X(n)$ be the outcome of n^{th} throw, $n \geq 1$. Then $\{X(n), n \geq 1\}$ is a stochastic process with state space$\{1, 2, \ldots, 6\}$.

Markov chain

Consider a Stochastic process $\{X_n, n \in T\}$, then $X_n = i$ implies that the process is in state i at time t. A Stochastic process $\{X_n, n \in T\}$ is called a Markov chain if

$Pr\{X_n = i_n | X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = Pr\{X_n = i_n | X_{n-1} = i_{n-1}\}$.

Transition probability matrix

$p_{ij} = Pr\{X_n = j | X_{n-1} = i\}$ is called the transition probability from state i to state j. The matrix $P = (p_{ij})$, where i, j are elements of the state space, is called the one-step transition probability matrix of the Markov chain.

Transient and recurrent states

A subset of the state space of a Markov chain is said to be closed if no state outside that subset can be reached from any state within it. If the chain has no proper closed subset other than the state space itself, it is called an irreducible chain. A state i is recurrent if and only if, starting from state i, the probability of returning to state i after some finite time

is certain. A non-recurrent state is said to be transient. For a recurrent state if the mean recurrence time is finite, it is called positive recurrent. The greatest common divisor of the recurrence times of a state is called its period. If the period is one, the state is said to be aperiodic. A positive recurrent aperiodic state of a Markov chain is said to be Ergodic. A Markov chain is ergodic if all its states are ergodic.
Theorem:If a Markov chain is irreducible and positive recurrent, there exists a unique solution to the linear system $\boldsymbol{\pi} P = \boldsymbol{\pi}, \boldsymbol{\pi} e = 1$ where $\boldsymbol{\pi}$ is the stationary probability vector. If the chain is aperiodic, the probabilities $Pr(x_n = i)$ will converge to π_i as $n \to \infty$.

Counting process

A Stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of events that have occurred upto time t.

Poisson process

A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda \geq 0$, if

1. $N(0) = 0$.
2. The process has independent increments.
3. For $s, t \geq 0$, $P\left\{N(t+s) - N(s) = n\right\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0, 1, \ldots$.

λ is called the rate of the process and $E[N(t)] = \lambda t$.

Phase Type distribution

Consider a Markov chain on the states $\{1, 2, 3 \ldots, m, m+1\}$ with the infinitesimal generator $Q = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$
where the $m \times m$ matrix T satisfies $T_{ii} < 0$ for $1 \leq i \leq m$, and $T_{ij} \geq 0$; for $i \neq j$. Also $Te + T^0 = 0$. Let initial probability vector of this process be $(\boldsymbol{\alpha}, \alpha_{m+1})$ with $\boldsymbol{\alpha}e + \alpha_{m+1} = 1$. Also assume that the states $1, 2, \ldots, m$ are transient so that absorption into the state $m+1$ is certain. A probability distribution $F(.)$ on $[0; \infty)$ is said be a phase type distribution (PH-distribution) of order m with representation $(\boldsymbol{\alpha}, T)$ if and only if it is the distribution of the time until absorption of a finite Markov process.

- If $F(.)$ is a phase type distribution, then $F(x) = 1 - e^{(Tx)}e$, for $x \geq 0$.

- For a PH distribution $F(.)$ with representation $(\boldsymbol{\alpha}, T)$, The distribution $F(.)$ has a jump at $x = 0$ of magnitude α_{m+1}.

- The corresponding probability density function $f(.)$ is given by $f(x) = \boldsymbol{\alpha}e^{(Tx)}T^0, x \geq 0$.

- The Laplace-Stieltjes transform $\widehat{f}(s)$ of $F(.)$ is given by
$\widehat{f}(s) = \alpha_{m+1} + \boldsymbol{\alpha}(sI - T)^{-1}T^0$, for $Re(s) \geq 0$.

- The i^{th} raw moment $\mu_i' \geq 0$ is given by $\mu_i' = (-1)^i i! \boldsymbol{\alpha} T^{-1} e$, $i = 1, 2, 3, \ldots$

Erlang distribution

In PH Distribution if the transition starts only from first phase, absorption is possible only from the m^{th} phase, one step forward transition is the only possible transition, the transition rate is μ and $\boldsymbol{\alpha} = (1, 0, \ldots, 0)$,

$$T = \begin{bmatrix} -\mu & \mu & & \\ & -\mu & \mu & \\ & & & \\ & & & \\ & & & -\mu \end{bmatrix}$$

then the corresponding distribution is called an Erlang distribution.

Exponential distribution

In PH Distribution if
$m = 1,\ \ T = [-\mu],\ \ T^0 = [\mu]$, and $\boldsymbol{\alpha} = 1$
then the distribution is called exponential distribution. The density function of Exponential distribution is given by $f(x) = \mu e^{-\mu x}, x \geq 0$.

Little's Formula

One of the most powerful formulae in queueing theory is developed by John D.C.Little. If L is the expected number of customers in the system, W is the mean waiting time in the system and λ is the arrival rate then

$$L = \lambda W, L_q = \lambda W_q,$$

where W_q is the mean waiting time in the queue and L_q is the expected number of customers in the queue.

Quasi birth-death process

Consider a Markov Chain with state space $S = \bigcup_{n\geq 0} = \{(n,i) : 1 \leq i \leq m\}$. Here the first component n is called level of the Chain and the second component i is called a phase of the n^{th} level. The Markov Chain is called a Quasi-birth-death (QBD) process if the one step transitions from a state is restricted within the same level or to the two adjacent levels. If the transition rates are level independent, the resulting QBD process is called level independent quasi-birth-death process (LIQBD), else it is called level dependent quasi-birth-death process (LDQBD). Arranging the elements of S in lexicographic order, the infinitesimal generator of a LIQBD process has the block tridiagonal matrix form in which three diagonal blocks repeat after some initial levels. We write such a matrix, with modification depending on boundary states, as $Q = \begin{bmatrix} B_1 & A_0 & & \\ B_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & & \ddots & \ddots & \ddots \end{bmatrix}$ where the sub matrices A_0, A_1, A_2 are square and have the same dimension; matrix B_1 is also square and need not have the same size as A_1. Also, $B_1\boldsymbol{e} + A_0\boldsymbol{e} = B_2\boldsymbol{e} + A_1\boldsymbol{e} + A_0\boldsymbol{e} = (A_0 + A_1 + A_2)\boldsymbol{e} = 0$.

Matrix Analytic method

The introduction of Matrix analytic method is a land mark in the history of queueing theory. Matrix analytic method was introduced by M.F. Neuts in late 1970's. It is a tool to construct and analyze a wide class of stochastic models using a matrix formalism to develop algorithmically tractable solution. When queueing theory found its applications in several new areas, the usual methods like method of generating functions, methods using transforms etc. failed to provide much tractability in the analysis of many models especially when the distribution of inter-arrival time or service time is not exponential. The introduction of Matrix analytic methods provided a smooth way to analyze much complicated Stochastic models in an algorithmic way and to numerically explore the problems more deeply. For further details regarding matrix analytic method one may refer books by Marcel. F Neuts[39], Latouche and Ramaswami [29] and Breuer and Baum [4].

Theorem:The matrix Q defined above is positive recurrent if and only if the minimal non-negative solution R to the matrix-quadratic equation

$$R^2A_2 + RA_1 + A_0 = 0 \tag{1.1}$$

has all its eigenvalues inside the unit disk and the finite system of equations
$x_0(B_1 + RB_2) = 0$
$x_0(I - R)^{-1}\boldsymbol{e} = 1$
has a unique positive solution x_0. If the matrix $A = A_0 + A_1 + A_2$ is irreducible, then $sp(R) < 1$ if and only if $\Pi A_2\boldsymbol{e} > \Pi A0\boldsymbol{e}$
where Π is the stationary probability vector of A.

Computation of R matrix

The stationary probability vector $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ of Q is given by $\boldsymbol{x}_i = x_0 R^i, i > 1$.
Once R, the rate matrix, is obtained, the vector $\boldsymbol{x}$ can be computed. Using logarithmic reduction algorithm we can compute R. In some cases we can have analytic form for the elements of R

Part of this thesis is developed based on the above discussions. For basic reference the following books are used. Karlin and Taylor [17, 18], Medhi [37, 38], Gross and Harris [13], Neuts [?], Ross [43], Breuer and Baum [4], Bellman [2], Latouche and Ramaswami [29], Pakes [40] and Takagi [46].

1.3 Motivation

All of us are familiar with queues. A queue is formed when there are more people in demand of service than the number of servers. At times interruption occurs to service process. Interruption means a break in the service process. Interruption can be due to server breakdown, complications created by the customer to own service, arrival of high priority customers, server taking prescheduled vacation, etc. In all walks of life we face interruption. In a journey through road we may come across interruption due to traffic block, break down of vehicle etc. At a billing counter we may find interruption due to some problems of the billing machine. In some cases interruption due to more than one factor may occur to the same service process. We label these factors as environmental factors. Different environmental factors may be the cause of interruption at various occasions. As an example, consider the case of Radio-communication. In this, signals are sent in the form of electromagnetic waves produced in the space. As the electromagnetic waves travel through space the following factors affects the radio communication.

Atmospheric condition, the nature of the objects on earth surface between the transmitting and receiving centers, distance of transmission, density of signals from other stations, power of transmitter and capacity of receiver, method of transmission, nature of antenna, frequency of carrier wave and methods of detection. In this case there are mainly nine environmental factors causing interruption. Interruption due to some factors are temporary. So we ignore such interruptions. Interruption due to some factors are identified only at a later stage. The method of rectification of interruption depends on the nature of environmental factor causing interruption.

As another example, consider the case of a patient admitted to hospital for emergency operation. Interruption can occur due to unavailability of operation theatre, lack of fitness of the patient, rare blood group of the patient, in the case of organ transplantation unavailability of matching organ, frequently changing physical condition of the patient, patient's response to medicines, etc. In the case of organ transplantation, the response of the body to the organ cannot be predetermined. Sometimes the body will accept the organ or reject it. We will get the response only at a later stage. Then only the surgeon can take necessary steps to save the life of the patient. In the case of intake of medicine the response of the body can be judged only at a later stage. If the particular medicine is not receptive for the particular patient, correction has to be done in the treatment. There are different factors interrupting the treatment and correction of interruption caused by each factor is different. The time duration for correction may vary from case to case.

As another example we consider the case of usage of internet. While browsing the net the possible interruptions are power failure, congestion in network, connectivity problem, software or hardware issues of the PC, etc. The interruption due to some of these factors are detectible only at a later stage. In some cases correction can be done. But interruption due to the same factors may occur again. Sometimes the impatient customer give up the effort due to the repeated interruption.

The above mentioned examples are related to interruption in service process. Another important aspect is the interruption in vacation. When ever the queue becomes empty the server goes for vacation. During vacation the server can go for maintenance, can provide services in some other queueing system. This is for the effective utilization of free time of server and to reduce the waiting time of the customers in other queues. Depending on the environment the server can opt for either normal vacation or working vacation. During normal vacation, based on the environment the server can pick up different options.

Consider a super market. If there is no customer in the billing counter, the sales girl can utilize the time for tallying the account, can record the price list of new arrivals in the computer, can arrange the new stock in the shelf, can clean the shop, can go to take food, can take the list of items finished, etc. Depending on the requirement she selects the vacation job. When customers arrive to the queue she returns from vacation interrupting it.

In a net cafe if there is no customer waiting the operator can use the system for data entry, software updation, rearrangement, etc. If more customers arrive the operator stops his job and provides the PC for browsing depending on the cost effectiveness.These are some of the real life situations

which motivated us to focus on queues with interruption in random environment.
The works reported in the literature discuss about interruptions, either server induced or customer induced, rather than the cause of interruption. In this thesis we introduce the concept of environment dependent service interruption and vacation interruption.
The models in this thesis are the results of inspiration drawn from the following works.

- *Queues with environment dependent interruption*: White and Christie. [48], Awi Federgruen and Linda Green. [1], Bhaskar Sengupta. [3], Dudin. A.N., Varghese Jacob, and Krishnamoorthy.[7], A Krishnamoorthy. A., Pramod. P.K, and Deepak. T.G.[22], Krishnamoorthy. A., Pramod. P.K, and Chakravarthy.[24], Krishnamoorthy. A, Pramod.P.K, Chakravarthy.S.R. [27]

- *Queues with environment dependent vacation*: O.C. Ibe, Olubukola A. Isijola.[14], O C. Ibe, Olubukola A. Isijola. [15], Y. Levy, U. Yechiali. [30], Fuhrmann.S, R.Cooper [10], Doshi.B.T. [5], [6], Shanthikumar.J.G. [42], Takagi.H. [46], Servi.L.D, S.G. Finn. [41], D.A. Wu, H. Takagi. [50], N. Tian, G. Zhang. [46], D.A. Wu, H. Takagi. [50], Li.J, N.Tian. [36], Li.W, X.Xu, N.Tian. [32], Kim.J.D, D.W.Choi, K.C. Chae[20], Li.J, N.Tian, Z.Ma[33], Li.J, N.Tian[34], Li.J, N. Tian, Z. Zhang, H. Luh. [35], Ke.Jau-Chuan, Chia-Huang Wu, Z.G.Zhang. [19], Zhang, Z. Hou. [51], Sreenivasan.C, A. Krishnamoorthy. [44], Sreenivasan.C, S. R. Chakravarthy, A. Krishnamoorthy. [45], Li.J, N.Tian. [31].

1.4 Summary of the thesis

This thesis entitled "Queues with interruption in random/Markovian environment" contains eight chapters including the present introductory chapter. The main tools used for the development of the thesis are Matrix analytic method, method of induction, method of generating function and supplementary variable technique. Distributions like Exponential distribution, Erlang distribution, PH distribution are considered. A brief discussion about these preliminaries are included in the first chapter. Chapters 2-5 deal with queues with environment dependent interruption and chapters 6-8 deal with queues with environment dependent vacation.
Chapter 2 is devoted to a queueing system with service interruption in which service gets interrupted due to different environmental factors. Even though any number of interruptions can occur during the service of a customer, the maximum number of interruptions is restricted to a finite number K and if the number of interruptions exceeds the maximum, the customer leaves the system without completing service. The difference between the model under discussion and those considered earlier in literature is that the customer / server is unaware of the interruption until a random amount of time elapses from the moment interruption strikes. At the moment the interruption occurs, a random clock and a superclock start ticking. The interruption is identified only when the random clock is realized. The superclock measures the total interruption time during the service of a customer. On realization of superclock the customer goes out of the system without completing service. The kind of service to be started after the interruption depends on the environmental factor that caused the interruption. Here we first analyze the service process to find the response time and to compute the stability condition. The optimal

values of K for a suitable cost function is investigated. Numerical investigation indicates the cost function as convex/increasing/decreasing in K

In chapter 3 a queueing system similar to one discussed in chapter 2 is analyzed. The main difference is that in this model the interruption causing environmental factors forms a Markov chain with initial probability vector $p_i, i = 1, 2, \ldots, n$ and transition probability matrix $P = (p_{ij}), i, j = 1, 2, \ldots, n$. The condition for stability of the system is obtained. Steady state probability vector and important performance measures are calculated with the help of Matrix analytic method. A comparison between the two models, Queue with interruption in random environment and Queue with interruption in Markovian environment, is carried out.

In chapter 4 we consider a single server queueing system in which arrival occurs according to a Poisson Process. On arrival if the customer finds the server busy, he joins the tail of the queue otherwise he gets service immediately. The service is Erlang distributed. During service there is a possibility for interruption in service due to different factors. Here we assume that there are $n+1$ environmental factors causing interruption to the service. These factors are numbered 1 to $n+1$ depending on the ascending order of severity of interruption caused by them. The interruption occurs according to a Poisson Process. When the interruption due to i^{th} factor occurs the rate of service changes. On the onset of interruption an exponentially distributed random clock and a PH distributed interruption clock are started. Only forward phase change is allowed for the interruption clock. When the interruption occurs due to any one of the first n factors it is ignored in the beginning and service is continued with interruption. When the interruption clock realizes the service is stopped and the server is repaired immediately. After repair the service to the customer in service is resumed if the interruption clock is realized before the random clock otherwise the service has to be restarted. There is a possibility for customer completing service with interruption. In that case the server goes for repair after the service completion. If the interruption is due to $(n+1)^{th}$ factor the customer goes out of the system and the server is replaced immediately. Once the interruption starts getting attended both the clocks are reset to zero position.

As the duration of ignored interruption increases the severity of interruption also increases. After some duration, the cause of interruption changes from i^{th} factor to j^{th} factor, where $j \geq i$ and i^{th} factor is the one causing initial interruption. Then the service rate also changes. Again if the interruption remains unattended for sometime, the cause of interruption changes from j^{th} factor to k^{th} factor, where $k \geq j$. The server is replaced on being interrupted by the $(n+1)^{th}$ factor. The customer in service is also lost when the interruption is due to $(n+1)^{th}$ factor. The $n+1$ environmental factors are the states of a Markov chain with initial probability vector $p_i, i = 1, 2, \ldots, n+1$ and transition probability matrix $P = (p_{ij}), i, j = 1, 2, \ldots, n+1$. Stability of the system is verified. Using Matrix analytic method Steady state probability vector and important performance measures are obtained. The important performance measures are numerically explored.

In chapter 5 we analyze two queueing models. In the first model we consider a single server queueing system with arrival following Poisson process. The service time is Erlang distributed. At times there is a possibility for interruption in service process. It occurs according to a Poisson process. The duration of interruption is exponentially distributed. The service continues ignoring the interruption. During interrupted service there is a scope for self correction of interruption. Self correction occurs according to a Poisson process. On the onset of interruption an interruption clock is started which is Erlang distributed. If the interruption clock is realized before service completion the server goes for repair and after repair the service is resumed. Repair time is exponentially distributed. If service is completed before the realization of interruption clock the next customer in the head of the queue enters for service.

In the second model the arrival process and the service process are same as in the first model. During service interruption occurs according to a Poisson process. There are n environmental factors causing interruption. Interruption due to i^{th} environmental factor occurs with probability p_i. If the interruption is due to first m factors it is ignored and service continues. But the service will be at lower rate. The duration up to which the server works without breakdown is assessed with the help of an interruption clock. This clock starts ticking with the initiation of the first interruption to the service of a customer. The duration of the clock is exponentially distributed. During that period there is a possibility for self correction of interruption. This self correction duration is exponentially distributed. If self correction occurs the service rate changes. On realization of the interruption clock the server goes for repair. The repair time is exponentially distributed. After repair the interrupted service is resumed. If the service of a customer is completed while server in interruption the next customer in the head of the queue enters for service at the interrupted server. If the interruption is due to the remaining $n-m$ factors the server directly goes for repair. Taking into account the severity of interruption caused by these $n-m$ factors, protection for remaining service is provided at the epoch of resumption of service after repair. The stability of both systems are analyzed. Steady state probability vector is calculated using matrix analytic method. Important performance measures are numerically substantiated.

Chapter 6 analyzes a single server multiple vacation queueing system. There are mainly two types of vacation: the server goes for type I vacation after a non-empty busy period of serving at least one customer. On returning from type I vacation if the server finds the system empty, it goes for a type II vacation. In type I vacation, depending on the environment, there are n distinct kinds of vacations. Interruption can occur to all types of vacations. The interruption to vacation is controlled by the length of the queue. We calculate the long run system probabilities, mean and variance of the number of customers in the system. Using Little's formula waiting time is also calculated and numerically illustrated. An optimization problem is discussed with numerical illustration.

Chapter 7 is devoted to a single server queueing system with working vacation in which arrival occurs according to a Poisson process. The service time is exponentially distributed. On completion of a service if the server finds the system empty he goes for a working vacation. There are n types of working vacations. Depending on the environment, after a busy period, the server goes for i^{th} type of vacation with probability $p_i, 1 \leq i \leq n$. The duration of vacations are exponentially distributed with different parameters. During vacation if customers arrive, the server provides service at a lower rate . On completion of service during vacation, if there is no customer left in the system the server continues its vacation.

Otherwise the vacation is interrupted, i.e. the server returns to normal service without completing the vacation and starts service in the normal rate. On completion of vacation if the server finds the system empty, he remains in the corresponding vacations. We demonstrate stochastic decomposition of the queue length and waiting time processes using method of induction and Little's formula.

In chapter 8 we carry out the study an $M/G/1$ queue with multiple vacation and vacation interruption. Both normal vacation (type I) and working vacation (type II) are considered. The exhaustive service discipline is assumed in this. At the end of a busy period, depending on the environment, the server either opts for normal vacation or working vacation. On completion of type I vacation if the server finds the system empty he goes for type II vacation. On completion of type II vacation if the server finds the system empty he goes for another type II vacation and so on. On completion of service in type II vacation, if the server finds one or more customers in queue he returns to normal service, interrupting the vacation. An arriving customer, during type I vacation, joins the queue with probability q or leaves the system with probability $1 - q$ and during type II vacation all the arriving customers join the queue. Using supplementary variable technique we derive the distributions for the queue length and service status under steady state condition. Laplace-Stieltjes transform of the stationary waiting time is also developed. Some numerical illustrations are also given.

Chapter 2

Queues with interruption in Random Environment

Introduction

Queues with interruption was first studied by White and Christie [48] in the context of a two priority system with preemption. In some queueing systems, the service process is subject to interruptions due to (i) (unscheduled) breakdowns of the server(s), (ii) scheduled off-periods, or (iii) arrival of customers of a higher priority class. In such systems, the distribution of service time in interruption free system is replaced by distribution of completion time which is the time a customer spends in the system after leaving the queue.

Some results of this chapter are included in the following paper.
A.Krishnamoorthy, Jaya.S, B.Lakshmy. : Queue with interruption in random environment, Annals of Operations Research, Springer (Accepted for publication).

Several researchers have discussed queues with service interruption due to server taking vacation and (or) due to arrival of priority customers. A recent survey paper by Krishnamoorthy et al [24] gives a detailed description of research on queueing models with service interruptions induced by server breakdown and also those with customer induced service break; in the latter case the server starts service for the next customer if any available. In server induced interruption such as break down, no service takes place when such failures occur.

In another paper Krishnamoorthy et.al [27] discuss phase type distributed interruption, Markovian arrival process and phase type service time distribution. Here the maximum number of interruptions for a customer is fixed. An interrupted service is either resumed from where it got interrupted or repeated from the beginning, based on the realization of a threshold clock that starts ticking at the epoch of onset of an interruption. A super clock is started at the epoch of the first interruption to a customer's service. If the super clock expires before fixing an interruption no further interruption is allowed to occur. The interruption occurs according to a Poisson process and the interruption duration, threshold and super clocks follow mutually independent phase type distributions.

The important features of the model discussed in this chapter, which are not discussed in all the above cited papers, are (i) there are a finite number of n factors causing interruption and the service after interruption is not necessarily the continuation of the previous service, but a new service depending on the environmental factor causing interruption, (ii)the customer/server is not aware of the interruption for some time, and during that period, service provided is not effective, (iii) interruption can occur due to any of the n environmental factors, but when the service is interrupted no further interruption is allowed until the rectification of current interruption, (iv) only a finite number of interruptions are allowed, and (v) a customer in service is going out of the system without completing service, based on the realization of the super clock or the number of interruptions that strikes the customer in service exceeding the permitted maximum number.

As a real life example for customer induced interruption we consider the case of a patient undergoing medical treatment. During the course of treatment the patient will have certain restrictions on the diet. The food items that are taboos can be considered as the environmental factors. Consumption of any of such food item causes interruption to the medication. If the patient consumes any of the restricted food item the result of the treatment may not be as desired. The realization of the violation of restriction is identified only at a later stage (in earlier reported research works (see paper [22]) it is instantaneous identification). Then corrective actions are taken and the treatment can be continued or sometimes may be modified. If the total time from the moment of consumption of restricted food item till the starting of a new treatment after the interruption, exceeds a certain amount of time the treatment may become ineffective and the patient has to stop taking the treatment. At the same time, repeated use of restricted food item also affects the result of the treatment. So the number of possible interruptions is restricted to a fixed quantity. If the number of interruptions exceeds that upper bound then also the treatment is ineffective. In the model under discussion the server is assumed to be affected by interruption of the kind described in the example.

As another example we consider the online blend headers used in refineries for making different grades of petrol and diesel. Various streams arrive at the blend header from different processing units. The streams are expected to have a set of properties. Based on the expected properties of different streams and the flow rates we expect a set of properties for the blend header output. There is an online analyser in the blend header which will continuously monitor the properties of the blend header output. Sometimes the quality of the streams may vary due to variation in the conditions of the processing units. Here the factors causing variation to the conditions of the processing units are considered as environmental factors. Due to different reasons the analyser may not identify the variations. Then the values shown by the analyser may not be correct, but depending on that values the blend header go on working. Only while validating the blend header readings by cross checking in the laboratory at preset intervals, the interruption is identified. If there is a mismatch in the value shown by the analyzer and lab result, the next step is the identification of the factor which caused the error and depending on that corrective actions are taken to get the output with desired properties. This can happen any number of times. If the total time for blending and correction exceeds a particular time limit the properties of the blended batch will be totally different from the target and the final product will be offspec. There are other real life examples occurring in medical, engineering, communication, and educational fields.

2.1 Model description

The queueing system that we consider is such that arrivals occur according to a Poisson process with parameter λ. The service time is exponentially distributed with parameter μ. During service, interruption occurs according to a Poisson process with parameter β. There are n environmental factors causing interruption. The interruption due to the i^{th} factor occurs with probability δ_i. For a certain duration of time the server is unaware of the interruption. At the start of the interruption, a random clock which is exponentially distributed with parameter η_i, starts ticking when the interruption is caused by the i^{th} environmental factor. The random clock measures the time elapsed from the epoch of occurrence of interruption until the identification of interruption. The realization of the random clock indicates the identification of interruption. The fixing time is exponentially distributed with parameter α_i, if the i^{th} factor is the cause of interruption. On fixing, a new service starts which is exponentially distributed with parameter μ_i, provided the i^{th} factor caused the interruption.

The super clock, which is exponentially distributed with parameter γ_i, started at the beginning of the first interruption that strikes the customer in service, will freeze when the new service starts after interruption and it will again start functioning from the position where it stopped, when another interruption occurs to the same customer. Sometimes the super clock may be realized before the random clock. The number of interruptions during the service of a customer is limited to K. On realization of the super clock or when the number of interruptions exceeds K, whichever occurs first to the customer in service, he goes out of the system without completing the service. Finally when the customer leaves the system the superclock is reset to the zero position. A diagrammatic representation of the model is given in Figure 2.1.

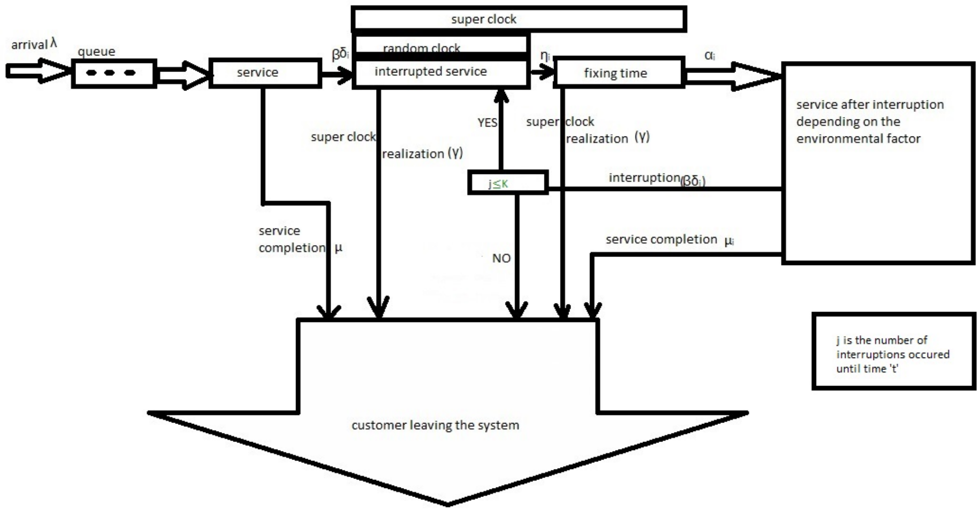

Figure 2.1: Model description

2.2 Analysis of service process with interruptions (Response time)

The service process $\{X(t), t \geq 0\}$ where X(t)= $(S(t), I_1(t), I_2(t), I_3(t), I_4(t))$ is a Markov Chain with $3nK+1$ transient states given by $\{0\} \cup \{(1, m, i, j, l)/m = 1, 2, \ldots, K; i = 0, 1; j = 0, 1; l = 1, \ldots, n\} \cup \{(2, m, l)/m = 1, 2, \ldots, K; l = 1, 2.....n\}$ and one absorbing state denoted by $\{*\}$. The absorbing state represents the customer moving out of the system either after service completion or without completing service. Here $S(t)$ denotes the status of the server at time t:

$$S(t) = \begin{cases} 0, & \text{if the service going on and has not} \\ & \text{undergone any interruption so far} \\ 1, & \text{if the service is interrupted} \\ 2, & \text{if service is continuing after interruption ;} \end{cases}$$

$I_1(t)$ denotes the number of interruptions occurred until time t to the current customer in service. $I_1(t)$ varies from 0 to K;

$I_2(t)$ represents the status of the random clock:

$$I_2(t) = \begin{cases} 0, & \text{if the random clock is realized} \\ 1, & \text{otherwise;} \end{cases}$$

$I_3(t)$ corresponds to the status of the super clock:

$$I_3(t) = \begin{cases} 0, & \text{if the superclock is freezed} \\ 1, & \text{otherwise;} \end{cases}$$

If the server is interrupted at time t, $I_4(t)$ corresponds to the environmental factors that caused the current interruption to the service. In this model we consider n environmental factors. Thus $I_4(t)$ has values varying from 1 to n. The state space of the process is $X(t)=\{0\} \cup \{(1, m, i, j, l)/m = 1, 2, \ldots, K; i = 0, 1; j = 0, 1; l = 1, \ldots, n\} \cup \{(2, m, l)/m = 1, 2, \ldots, K; l = 1, 2.....n\} \cup \{*\}$, where $\{*\}$ is the absorbing state. The infinitesimal generator of the process is given by

$$\bar{Q} = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$$

$$\text{where } T = \begin{bmatrix} B'_0 & B'_1 & 0 \\ 0 & B'_2 & B'_3 \\ 0 & B'_4 & B'_5 \end{bmatrix}_{(3nK+1)\times(3nK+1)} .$$

Let $\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}$, $\bar{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$, $\delta = \begin{bmatrix} \delta_1, & \delta_2, & \dots & \delta_n \end{bmatrix}$.

Then $B_0' = [-(\mu + \beta)]$ where B_0' is a matrix of order one.

$B_1' = \begin{bmatrix} \beta\delta & 0 \end{bmatrix}_{(1\times 2nK)}$, $B_2' = I_K \otimes U_0$.

$$U_0(i,j) = \begin{cases} -(\eta_i + \gamma_i) & \text{for } i = j = 1, 2, ...n \\ -(\alpha_{i-n} + \gamma_{i-n}) & \text{for } i = j = n+1, ...2n \\ \eta_i & \text{for } j = i+n, i = 1, 2, ...n \\ 0 & \text{otherwise.} \end{cases} \quad i, j = 1, 2, \dots 2n$$

$B_3' = I_K \otimes U_2$, where $U_2 = \begin{bmatrix} 0 \\ U_1 \end{bmatrix}_{(2n\times n)}$, and $U_1 = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$;

$B_4' = \begin{bmatrix} 0 & I_{K-1} \otimes U_3 \\ 0 & \mathbf{0} \end{bmatrix}_{(nK\times 2nK)}$ with $U_3 = \begin{bmatrix} (e \otimes \beta\delta) & 0 \end{bmatrix}_{(n\times 2n)}$.

$B_5' = I_K \otimes U_4$ and $U_4(i,j) = \begin{cases} -(\mu_i + \beta) & \text{for } i = j \\ 0 & \text{otherwise.} \end{cases} \quad i, j = 1, 2, \dots, n$

$T^0 = \begin{bmatrix} B_{03} \\ B_{13} \\ B_{23} \end{bmatrix}_{(3nK+1)\times 1}$ with

$$B_{03} = \begin{bmatrix} \mu \end{bmatrix}, \; B_{13} = \begin{bmatrix} \gamma \\ \vdots \\ \gamma \end{bmatrix}_{(2nK\times 1)}, \; B_{23} = \begin{bmatrix} \bar{\mu} \\ \vdots \\ \bar{\mu} \\ \bar{\mu} + e \otimes \beta \end{bmatrix}_{(nK\times 1)}.$$

The initial probability vector is $\boldsymbol{\zeta} = (1, 0, \dots)$ which means that at first the interruption starts. Here the service process with interruption follows PH Distribution. So using the property that residual service time in a phase type distributed service process is also phase type we see that

- The expected time for service completion/customer leaving the system without completing service due to realization of super clock or attaining maximum K is $E(ST) = \boldsymbol{\zeta}(-T)^{-1}\boldsymbol{e}$, and hence the expected service rate is $\mu_s = 1/E(ST)$.

- Probability for service completion without any interruption is $\boldsymbol{\zeta}(-B_0')^{-1}B_{03}$.

2.2.1 Expected number of interruptions

To calculate the expected number of interruptions during the service of a customer, we consider the Markov Chain $\{Y(t), t \geq 0\}$ where $Y(t) = (I_1(t), S(t), I_2(t), I_3(t), I_4(t))$ with state space $\{0\} \cup \{(m, 1, i, j, l)/m = 0, 1, 2, \ldots, K; i = 0, 1; j = 0, 1; l = 1, \ldots, n\} \cup \{(m, 2, l)/m = 0, 1, 2, \ldots, K; l = 1, 2.....n\} \cup \{\nabla\}$ where $\{\nabla\}$ is the absorbing state which represents the customer moving out from the system either after service completion or without completing service. Here $S(t)$ and $I_j(t), j = 1, 2, 3, 4$ are as defined in section 2.2. The infinitesimal generator of $\{Y(t), t \geq 0\}$ is given by

$$\hat{Q} = \begin{bmatrix} \Delta & \Delta^0 \\ 0 & 0 \end{bmatrix} \text{ where } \Delta = \begin{bmatrix} B_{00} & B_{01} & & & & \\ & A_{10} & A_{11} & & & \\ & & A_{10} & A_{11} & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \\ & & & & & A_{10} \end{bmatrix},$$

$$B_{00} = [-(\mu + \beta)], \; B_{01} = \begin{bmatrix} \beta\delta & 0 \end{bmatrix}_{(1\times 3n)},$$

$$A_{10} = \begin{bmatrix} L_1 & L_2 \\ 0 & L_3 \end{bmatrix}_{3n\times 3n}, \; A_{11} = \begin{bmatrix} 0 & 0 \\ L_4 & 0 \end{bmatrix}_{3n\times 3n},$$

with

$$L_1 = \begin{cases} -(\eta_i + \gamma_i), & \text{for } i = j = 1, 2, ...n \\ \eta_i, & \text{for } j = i + n, i = 1, 2, ...n \\ -(\alpha_{(i-n)} + \gamma_{(i-n)}), & \text{for } i = j = n + 1, ...2n \\ 0, & \text{otherwise;} \end{cases} \quad i, j = 1, 2, \ldots, 2n$$

$$L_2 = \begin{cases} \alpha_j, & \text{for } i = j + n; \\ 0, & \text{otherwise;} \end{cases} \quad i, j = 1, 2, \ldots, n$$

$$L_3 = \begin{cases} -(\beta + \mu_i), & \text{for } i = j \\ 0, & \text{otherwise;} \end{cases} \quad i, j = 1, 2, \ldots, n$$

$$L_4 = \begin{cases} \beta\delta_j, & \text{for } i, j = 1, 2, ...n \\ 0, & \text{otherwise;} \end{cases} \quad i, j = 1, 2, \ldots, 2n$$

and

$$\Delta^0 = \begin{bmatrix} C_0' \\ A_2' \\ A_2' \\ . \\ . \\ B_2' \end{bmatrix} \text{ where } C_0' = [\mu] \; A_2' = \begin{bmatrix} L_5 \\ L_6 \end{bmatrix}_{3n\times 1}, \; B_2' = \begin{bmatrix} L_5 \\ L_7 \end{bmatrix}_{3n\times 1},$$

where $L_5 = \begin{bmatrix} \gamma \\ \gamma \end{bmatrix}_{2n\times 1}$ $L_6 = \begin{bmatrix} \overline{\mu} \end{bmatrix}_{n\times 1}$, $L_7 = \begin{bmatrix} \overline{\mu} + \beta \end{bmatrix}_{n\times 1}$.

The initial probability vector $\zeta = \begin{pmatrix} 1 & 0 & 0 & . & . \end{pmatrix}$.

From the above matrices we get the following system characteristics:

- Probability for absorption after $r(r \leq K)$ interruption is given by

$$a_r = \begin{cases} (B_{00})^{-1}C_0', & \text{for } r = 0, \\ (B_{00}^{-1}B_{01})(A_{10}^{-1}A_{11})^{(r-1)}(A_{10}^{-1}B_2'), & \text{for } r = K, \\ (B_{00}^{-1}B_{01})(A_{10}^{-1}A_{11})^{(r-1)}(A_{10}^{-1}A_2'), & \text{otherwise.} \end{cases}$$

- Expected number of interruptions E(I) for a customer $= \sum_{r=1}^{K} r a_r$
- Probability for service completion after exactly $r(1 \leq r \leq K)$ interruptions

  $c_r = \zeta(-B_{00}^{-1}B_{01})(-A_{10}^{-1}A_{11})^{(r-1)}(-A_{10}^{-1}A_3')$ where $A_3' = \begin{bmatrix} 0 \\ 0 \\ \bar{\mu} \end{bmatrix}$.
- Expected number of interruptions E(IS) before service completion for a single service$= \sum_{r=1}^{K} r c_r$.
- Probability for service completion without any interruption $P(s) = \zeta(-B_{00})^{-1}C_0'$.
- Probability for the customer leaving the system due to the realization of super clock during the r^{th} interruption,

$$y_r = \zeta(-B_{00}^{-1}B_{01})(-A_{10}^{-1}A_{11})^{(r-1)}(-A_{10}^{-1}A_4^{'}), r = 1, 2, \ldots, K,$$

  and $A_4^{'} = \begin{bmatrix} L_5 \\ 0 \end{bmatrix}$.
- Expected number of interruptions E(I) before the realization of super clock $=\sum_{r=1}^{K} r y_r$.

Having computed the measures indicated above, we describe the queueing model and the condition for it to be stable.

2.3 The queueing model

Consider the queueing model $Z = \{Z(t), t \geq 0\}$, where $Z(t) = (N(t), S(t), I_1(t), I_2(t), I_3(t), I_4(t))$ where $N(t)$ is the number of customers in the system. Here $S(t)$ and $I_j(t), j = 1, 2, 3, 4$ are as defined in section 3. Z is a continuous time Markov chain with state space $\{0\} \cup \{(h, m, 1, i, j, l)/h = 1, 2, \ldots, \infty; m = 0, 1, 2, \ldots, K; i = 0, 1; j = 0, 1; l = 1, \ldots, n\} \cup \{(h, m, 2, l)/h = 1, 2, \ldots, \infty; m = 0, 1, 2, \ldots, K; l = 1, 2.....n\}$. Its infinitesimal generator Q is given by

$$\mathrm{Q} = \begin{bmatrix} B_0 & B_1 & & & & \\ B_2 & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \text{ where } B_0 = [-\lambda],\ B_1 = \begin{bmatrix} \lambda & 0 & \cdots \end{bmatrix}_{1\times 3nK},$$

$B_2 = T^0$; $A_0 = \lambda I_{3nK+1}$; $A_1 = T - \lambda I_{3nK+1}$; $A_2 = T^0\zeta$.

Lemma: The system Z is stable when $\lambda < \mu_s$, where μ_s is defined in sec.2.2.

Proof:When the right drift rate (arrival of a customer) is less than the rate of drift to the left (departure of the customer from the system), the system is stable.

Stationary distribution

The stationary distribution, under the condition of stability, $\lambda < \mu_s$, has Matrix Geometric solution. Let χ=$(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ be the steady state probability vector of the Markov chain $\{Z(t), t \geq 0\}$. Each $\mathbf{x}_i, i > 0$ are vectors with $3nK+1$ elements. We assume that $\mathbf{x}_2 = \mathbf{x}_1 R$, and $\mathbf{x}_i = \mathbf{x}_1 R^{i-1}, i \geq 2$,

where R is the minimal non- negative solution to the matrix quadratic equation
$R^2A_2 + RA_1 + A_0$=0.
From χQ=0 we get

$$\mathbf{x}_0 B_0 + \mathbf{x}_1 B_2 = 0.$$

$$\mathbf{x}_0 B_1 + \mathbf{x}_1 (A_1 + RA_2) = 0.$$

Solving the above two equations we get $\mathbf{x}_0$ and $\mathbf{x}_1$ subject to the normalizing condition $\mathbf{x}_0 e + \mathbf{x}_1 (I - R)^{-1} e$=1.

2.4 Performance measures

After calculating the steady state probability vector we now concentrate on some important performance measures of the system.

2.4.1 Expected waiting time

Computation of expected waiting time of a particular customer who joins the queue as the r^{th} customer, is done by considering the Markov Chain X'= $\{(M(t), S(t), I_1(t), I_2(t), I_3(t), I_4(t)), t \geq 0\}$ where $M(t)$ is the rank of the tagged customer.The tagged customers rank will decrease to 1 as the customers ahead of him leave the system. The rank of the customer is not affected by the arrival of customers following the tagged customer. Here $S(t)$ and $I_j(t), j = 1, 2, 3, 4$ are as defined in section 2.2. X' is a Markov chain with state space $\{(h, 0)/h = m, \ldots, 1\} \cup \{(h, 1, s, i, j, l)/h = m, \ldots, 1; s = 1, 2, \ldots, K; i = 0, 1; j = 0, 1; l = 1, \ldots, n\} \cup \{(h, 2, s, l)/h =$

$m, \ldots, 1; s = 1, 2, \ldots, K; l = 1, 2.....n\} \cup \{\Phi\}$ where $\{\Phi\}$ is the absorbing state. The infinitesimal generator matrix $\tilde{Q}$ is given by

$\tilde{Q} = \begin{bmatrix} W & W^0 \\ 0 & 0 \end{bmatrix}$ where

$$W = \begin{bmatrix} T & T^0\boldsymbol{\zeta} & & & \\ & T & T^0\boldsymbol{\zeta} & & \\ & & \ddots & \ddots & \\ & & & T & T^0\boldsymbol{\zeta} \\ & & & & T \end{bmatrix} \text{ and } W^0 = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ T^0 \end{bmatrix}.$$

The expected waiting time of the tagged customer, according to the position of the customer being served at the time of arrival of the tagged customer, is a column vector which is obtained from the formula

$E_W^r = -T^{-1}(I - (T^0\boldsymbol{\zeta}T^{-1})^{(r-1)})(I - T^0\boldsymbol{\zeta}T^{-1})^{-1}\boldsymbol{e}$.

Hence the expected waiting time of a customer in the queue is

$$E(W) = \sum_{r=1}^{\infty} \mathbf{x}_r E_W^r.$$

2.4.2 Other important performance characteristics

- Expected number of customers completing service without interruption $E(NI) = \sum_{i=1}^{\infty} i x_{i,0}$.
- Probability that there are i ($i \geq 0$) customers in the system, $P_i = x_i\boldsymbol{e}$.
- Expected number of customers in the system, $E(S) = \sum_{i=1}^{\infty} iP_i$.
- Fraction of time the server in the interrupted state,

$$FT(I) = \sum_{i=1}^{\infty} x_{i,1}\boldsymbol{e}.$$

- Fraction of time the server is busy,

$$FT(B) = \sum_{i=1}^{\infty} x_{i,0} + \sum_{i=1}^{\infty} x_{i,2}e + \sum_{i=1}^{\infty}\sum_{r=1}^{K} x_{i,1,r,1}\boldsymbol{e}.$$

- Fraction of time the server in the unidentified interrupted state,

$$FT(NI) = \sum_{i=1}^{\infty}\sum_{i_1=1}^{K} x_{i,1,i_1,1}\boldsymbol{e}.$$

- Fraction of time the server in fixing state,

$$FT(FS) = \sum_{i=1}^{\infty}\sum_{i_1=1}^{K} x_{i,1,i_1,0}\boldsymbol{e}.$$

- Fraction of time the super clock is freezed,

$$FT(SF) = \sum_{i_1=1}^{\infty} x_{i,2}\boldsymbol{e}.$$

- Fraction of time the super clock is on,

$$FT(ON) = \sum_{i=1}^{\infty}\sum_{i_1=1}^{K} x_{i,1,i_1}\boldsymbol{e}.$$

- Rate of customer leaving the system, due to realization of superclock, after r^{th} interruption due to j^{th} environmental factor $=\gamma_j \sum_{i=1}^{\infty}\sum_{l=0}^{1} x_{i,1,r,l,0,j}$

2.5 Cost function and Numerical illustrations

The purpose of this section is to determine the optimal value of K through numerical experiments. To this end we construct a cost function as a function involving K. The following costs are considered.

- C1- service cost.
- C2 - Holding cost of the service interrupted customer.
- C3 - Cost of lost service i.e. the service cost during the unidentified interrupted state.
- C4 - Holding cost of the customer in the queue.

Therefore the total expected cost, EC = C1∗(expected service rate) + C2∗ (fraction of time the customer in the interrupted state) + C3∗ (fraction of time the customer in the unidentified interrupted state) + C4∗(Expected number of customers in the queue).

Effect of number of interruption on Expected cost

Due to the complexity of the cost function we are unable to compute the optimal K explicitly. By fixing the values of the parameters:
$\lambda = 3, \mu = 8, n = 5,\ \alpha_1 = 7, \alpha_2 = 7, \alpha_3 = 6, \alpha_4 = 6, \alpha_5 = 5,\ \delta_1 = 0.1, \delta_2 = 0.2, \delta_3 = 0.2, \delta_4 = 0.3, \delta_5 = 0.2,\ \gamma_1 = 2, \gamma_2 = 2, \gamma_3 = 2, \gamma_4 = 2, \gamma_5 = 2,\ \eta_1 = 6, \eta_2 = 5, \eta_3 = 5, \eta_4 = 4, \eta_5 = 4,\ \mu_1 = 5, \mu_2 = 6, \mu_3 = 5, \mu_4 = 6, \mu_5 = 7,$

$C1 = \$100, C2 = \$300, C3 = \$20, C4 = \$20.$
The values for Expected cost corresponding to the variation in K and β are represented graphically (See Figure 2.2). By taking $\gamma_1 = \gamma_2 = \gamma_3 =$

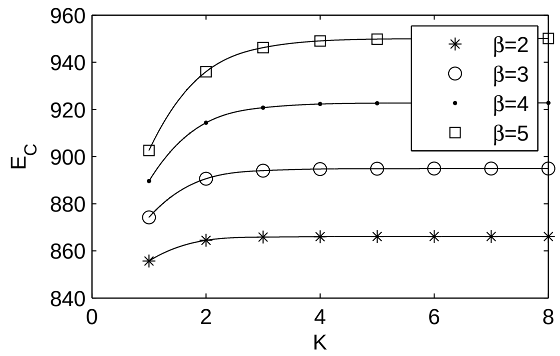

Figure 2.2: Effect of change in K and β on (EC)

$\gamma_4 = \gamma_5 = \gamma$, $\beta = 3$ and all other values as above,the expected cost is as in the Figure 2.3. From Figure 2.2 and Figure 2.3 it is clear that as the value of K and β increases the expected cost increases. But as the realization rate of super clock increases the expected cost decreases.

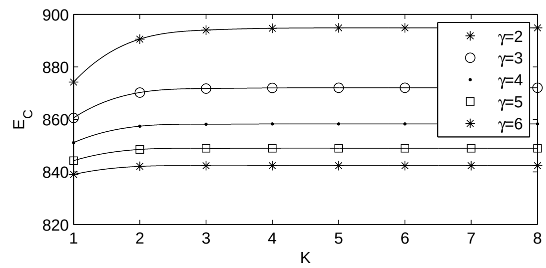

Figure 2.3: Effect of change in K and γ on (EC)

Effect of change in number of interruptions on $E(S)$

Table 2.1: **Effect of change in K on E(S)**

K	E(S)
1	1.0243
2	1.1877
3	1.2188
4	1.2240
5	1.2249
6	1.2250

As the value of K increases, expected number of customers in the system $E(S)$ increases which is on expected lines (see Table 2.1).

2.5.1 Effect of change in μ on various performance measures

Take λ =3, n=5, K=5, β = 3. As the value of μ increases, expected

Table 2.2: **Effect of change in μ on E(NI) ,E(S)and P(s)**

μ	E(NI)	E(S)	P(s)
4	3.9897	8.7787	0.5714
5	1.7007	3.9433	0.6250
6	1.0810	2.6057	0.6667
7	0.7928	1.9690	0.7000
8	0.6261	1.5925	0.7273
9	0.5175	1.3419	0.7500
10	0.4410	1.1621	0.7692

number of customers with no interruption E(NI) and expected number of customers in the system $E(S)$ decrease but probability $P(s)$ for service completion without any interruption increases (see Table 2.2).

Effect of change in λ on E(NI) and E(S)

Assuming μ=8, n=5, k=5, $\beta = 3$ and all other values as above we get the following values for E(NI) and E(S). As the value of λ increases, expected

Table 2.3: **Effect of change in λ on E(NI) and E(S)**

λ	E(NI)	E(S)
1	0.1047	0.2502
2	0.2722	0.6952
3	0.6261	1.5925
4	1.6788	4.0123

number $E(NI)$ of customers with no interruption and expected number $E(S)$ of customers in the system increase (see Table 2.3).
So far we concentrated on the study of the CTMC with underlying distributions exponential. However this distribution does not fit into several contexts, for example the service time.

2.6 Analysis of service process when the service time and the super clock follow Erlang Distribution

Here we consider a special case of the main system we analyzed above. Assume that the service time and super clock are Erlang distributed. The initial service is Erlang distributed with shape and scale parameters μ and a respectively and the services after interruption fixation are Erlang distributed with parameter μ_i and b_i if the i^{th} factor is the cause of interruption. The super clock is Erlang distributed with shape and scale parameters γ and c respectively. On identification of random clock the fixing time starts. All other assumptions are as in the previous model. The service process $X = \{X(t), t \geq 0\}$ where $X(t) = (S(t), I_1(t), I_2(t), S_1(t), I_3(t))$, is a continuous time Markov chain. Here $S(t)$ denotes the status of the server at time t:

$$S(t) = \begin{cases} 0, & \text{if the service going on at } t \text{ has not} \\ & \text{undergone any interruption so far} \\ 1, & \text{if the server is in the unidentified interrupted state} \\ 2, & \text{if the server is in the fixing state} \\ 3, & \text{if the service at } t \text{ has undergone interruption ;} \end{cases}$$

$I_1(t)$ denotes the environmental factor causing interruption. It varies from 1 to n. $I_2(t)$ denotes the number of interruptions occurred until time t to the current customer in service. $I_2(t)$ varies from 0 to K. $S_1(t)$ denotes the phase of service process. It varies from 1 to a if the service is without interruption and varies from 1 to b_i for the service after the completion of an interruption caused by the i^{th} factor. $I_3(t)$ denotes phase of super clock which varies from 1 to c. The state space of the Markov chain is $E = \{(0, m)/m = 1, \ldots, a\} \cup \{(1, l, j, m, r)/j = 1, 2, \ldots, K; l = 1, \ldots, n; m = 1, \ldots, a, \text{ or } 1, \ldots, b_l; r = 1, \ldots, c\} \cup \{(2, l, j, r)/j = 1, 2, \ldots, K; l = 1, \ldots, n; r = 1, \ldots, c\} \cup \{(3, l, j, m, r)/j = 1, 2, \ldots, K; l = 1, \ldots, n; m = 1, \ldots, b_l; r = 1, \ldots, c\} \cup$ {one absorbing state}. The infinitesimal generator is $A = \begin{bmatrix} \Omega & \Omega^0 \\ 0 & 0 \end{bmatrix}$

with initial probability vector $\boldsymbol{\zeta}=(1, 0, \ldots, 0)$, where

$$\Omega = \begin{bmatrix} H_{00} & H_{01} & 0 & 0 \\ 0 & H_{11} & H_{12} & 0 \\ 0 & 0 & H_{22} & H_{23} \\ 0 & H_{31} & 0 & H_{33} \end{bmatrix} \text{ and } \Omega^0 = \begin{bmatrix} H^{0(0)} \\ H^{0(1)} \\ H^{0(2)} \\ H^{0(3)} \end{bmatrix}$$

$$H_{00}(i,j) = \begin{cases} -\mu - \beta & \text{if } i = j; i,j = 1,2\ldots,a \\ \mu, & j = i+1; i = 1,2\ldots,a-1 \\ 0, & \text{otherwise;} \end{cases}$$

$$H_{01} = \begin{bmatrix} R_1 & R_2 & \cdots & R_n \end{bmatrix}$$

where each R_l is a matrix of order $a \times (ac + \sum_{l=1}^{n}(K-1)b_l c)$ and

$$R_l(i,j) = \begin{cases} \beta\delta_l, & \text{if } j = (i-1)c+1;; \\ 0, & \text{otherwise;} \end{cases}$$

and $l = 1,\ldots,n$.

$$H_{11} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$$

Here each A_l is a square matrix of order$(ac + \sum_{l=1}^{n}(K-1)b_l c)$ and

$$A_l = \begin{bmatrix} -\gamma-\eta_l & \gamma & 0 & \cdots & 0 \\ 0 & -\gamma-\eta_l & \gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & -\gamma-\eta_l & \gamma \\ 0 & 0 & \cdots & 0 & -\gamma-\eta_l \end{bmatrix} \text{,with } l = 1,2,\ldots,n$$

and $H_{12} = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & B_n \end{bmatrix}$ with;

$$B_l = \begin{bmatrix} B_l^1 & 0 & \cdots & 0 \\ 0 & B_l^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & B_l^K \end{bmatrix}_{ac+\sum_{l=1}^{n}(K-1)b_l c \times Kc}$$

where $B_l^1 = [\boldsymbol{e} \otimes \eta_l I_c]_{(ac\times c)}$ in which $\boldsymbol{e}$ is of dimension $a \times 1$;
and $B_l^r = [\boldsymbol{e} \otimes \eta_l I_c]_{(b_l c\times c)}$ with $\boldsymbol{e}$ of dimension $b_l \times 1; r = 2, \ldots, K; I_c$ is an identity matrix of order c.

Further $H_{22} = \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & C_n \end{bmatrix}$ where

$$C_j = \begin{bmatrix} -\gamma-\alpha_j & \gamma & 0 & \cdots & 0 \\ 0 & -\gamma-\alpha_j & \gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & -\gamma-\alpha_j & \gamma \\ 0 & 0 & \cdots & 0 & -\gamma-\alpha_j \end{bmatrix}_{(Kc\times Kc)} ;$$

for $j = 1, \ldots, n$;

$$H_{23} = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & D_n \end{bmatrix}$$

where each $D_l = \begin{bmatrix} D_l^1 & 0 & \cdots & 0 \\ 0 & D_l^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & D_l^K \end{bmatrix}_{Kc\times Kb_l c}$ where each D_l^r is of dimension $c \times b_l c$ and $l = 1, \ldots, n; r = 1, \ldots, K$

$$D_l^r(i,j) = \begin{cases} \alpha_l, & \text{if } j = i; \& \ i = 1, 2 \dots c; j = 1, \dots, b_l c \\ 0, & \text{otherwise;} \end{cases}$$

$H_{31} = \begin{bmatrix} I_1 & I_2 & \cdots & \cdots & I_n \end{bmatrix}$, where

$$I_j = \boldsymbol{e}_n \otimes S_j \text{ and } S_j = \begin{bmatrix} 0 & U_j & 0 & \cdots & 0 \\ \vdots & \ddots & U_j & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & U_j \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{Kb_jc \times (ac+(K-1)b_jc)}$$

$U_j = [diag(\beta\delta_j)]_{b_jc}$ and $j = 1, \dots, n$.

The sub matrix H_{33} is given by $H_{33} = \begin{bmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & E_n \end{bmatrix}$; here

$$E_j = \begin{bmatrix} -\mu_j - \beta & \mu_j & \cdots & \cdots & 0 \\ 0 & -\mu_j - \beta & \mu_j & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & -\mu_j - \beta & \mu_j \\ 0 & 0 & \cdots & 0 & -\mu_j - \beta \end{bmatrix},$$

for $j = 1, \dots, n$

$$H^{0(0)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mu \end{bmatrix}_{a \times 1} ; \ H^{1(0)} = \begin{bmatrix} \overline{H}_1 \\ \overline{H}_2 \\ \vdots \\ \overline{H}_n \end{bmatrix} \text{ where}$$

$$\overline{H}_j(i,1) = \begin{cases} \gamma, & \text{for } i = rc; r = 1, 2.., a + (K-1)b_j; \\ & i = 1, .., ac + (K-1)b_j c \\ 0, & \text{otherwise,} \end{cases}$$

for $j = 1, \ldots, n$.

$H^{2(0)} = \overline{H}' \boldsymbol{e}_n$.

here $\overline{H}'(i,1) = \begin{cases} \gamma, & \text{for } i = rc; r = 1, 2 \ldots, K; i = 1, \ldots, Kc; \\ 0, & \text{otherwise.} \end{cases}$

$$H^{3(0)} = \begin{bmatrix} \widehat{H}_1 \\ \widehat{H}_2 \\ \vdots \\ \widehat{H}_n \end{bmatrix} \text{ where } \widehat{H}_j = \begin{bmatrix} \widehat{H}_{j1} \\ \widehat{H}_{j2} \\ \vdots \\ \widehat{H}_{jK} \end{bmatrix} \text{ for } j = 1, \ldots, n;$$

and $\widehat{H}_{jl}(i,1) = \begin{cases} \mu_j, & \text{for } i = (b_j - 1)c + 1, \ldots, b_j c; \\ 0, & \text{otherwise.} \end{cases}$

where i varies from 1 to $b_j c$ and $l = 1, \ldots, K-1$.

$\widehat{H}_{jK}(i,1) = \begin{cases} \mu_j + \beta, & \text{for } i = (b_j - 1)c + 1, \ldots, b_j c; \\ \beta, & \text{otherwise.} \end{cases}$

where i varies from 1 to $b_j c$

2.6.1 performance measures

We arrive at the following performance measures of the system:

- The expected time for service completion/customer leaving the system without completing service due to realization of super clock or attaining maximum K is $E(ST) = \boldsymbol{\zeta}(-\Omega)^{-1}e$.
- The expected departure rate is $\mu_s = 1/E(ST)$
- Probability for service completion without any interruption is

$\boldsymbol{\zeta}(-H_{00})^{-1}H^{0(0)}$.

2.6.2 Numerical Results

Table 2.4: **Number of interruptions & Expected cost(EC)**

K	β=2	3	4	5
1	658.2127	662.2811	665.5660	668.0791
2	687.8693	694.9911	700.3799	704.4870
3	693.9890	702.3967	708.6655	713.3887
4	697.0032	705.5458	711.8078	716.4700
5	697.0478	705.6397	711.9537	716.6656
6	697.1248	705.7776	712.1441	716.8970

Table 2.5: **Effect of change in μ on E(NI) and E(S)**

μ	E(NI)	E(s)
2	0.4556	1.4341
3	0.4380	1.3862
4	0.4188	1.3331
5	0.3993	1.2788
6	0.3804	1.2252
7	0.3622	1.1733

With the following input values for parameters,
$\lambda = 2, \mu = 2, n = 3, \gamma = 2, a = 3, b = 3, c = 3,\ \alpha_1 = 7, \alpha_2 = 7, \alpha_3 = 6, \delta_1 = 0.3, \delta_2 = 0.4, \delta_3 = 0.3,\ \eta_1 = 6, \eta_2 = 5, \eta_3 = 5, \mu_1 = 5, \mu_2 = 6, \mu_3 = 5,\ C1 = \$100, C2 = \$300, C3 = \$20, C4 = \$20$, and cost function constructed in section 2.5, for distinct values of K and β we get the following values for Expected cost.

From the numerical results exhibited in Table 2.4, as the value of K and β increase the expected cost also increases. Table 2.5 depicts the effect of μ on E(NI) and E(s). Notice that E(NI) and E(s) steadily decrease with increasing value of μ. These are on expected lines.

Chapter 3

Queues with interruption in Markovian Environment

Introduction

In chapter 2 we discussed a queueing model with service interruption in which interruption is caused by different environmental factors which are not inter-related. In some cases the environmental factors are inter-related. i.e.The interruption caused by some factors can influence some other factors to cause interruption in future to the same customer in service. As a real life example consider the case of a patient undergoing medical treatment. During the course of the treatment medicines consumed for a particular illness can affect some other systems of the body which may be identified at a later stage. On identification corrective actions are taken. But due to the effect of medicines consumed, interruptions can again occur during the same treatment at a later stage and it may affect the treatment success. Such a model is considered in this chapter. All the assumptions on the model in this chapter are same as that in chapter 2 except the additional condition that the interruption causing environmental factors are the states of a Markov chain.

3.1 Model description

The queueing system that we consider is such that arrivals occur according to a Poisson process with parameter λ. The service time is exponentially distributed with parameter μ. During service, interruption occurs according to a Poisson process with parameter β. There are n environmental factors causing interruption. The interruption causing environmental factors are the states of a Markov chain with initial probability vector p_i; $i = 1, 2, \ldots, n$ and transition probability matrix $\mathbf{P} = (p_{ij}); i, j = 1, 2, \ldots, n$

For a certain duration of time the server is unaware of the interruption. At the onset of the interruption, a random clock which is exponentially distributed with parameter η_i, starts ticking when the interruption is caused by the i^{th} environmental factor. The random clock measures the time elapsed from the epoch of occurrence of interruption until the identification of interruption. The realization of the random clock indicates the identification of interruption. The fixing time is exponentially distributed with parameter α_i, if the i^{th} factor is the cause of interruption. On fixing the interruption, a new service starts which is exponentially distributed with parameter μ_i, provided the i^{th} factor caused the interruption.

The super clock, which is Erlang distributed with shape and scale parameters γ and a respectively, starts at the beginning of the first interruption that strikes the customer in service. It will freeze when the new service starts after interruption and again starts ticking from the position where it stopped, when another interruption occurs to the same customer in service. Sometimes the super clock may be realized before the random clock. The number of interruptions during the service of a customer is limited to K. On realization of the super clock or when the number of interruptions exceeds K, whichever occurs first to the customer in service, he goes out of the system without completing the service. When the current customer leaves the system the clock is reset to zero position.

3.2 Analysis of service process with interruptions (Response time)

The service process $\{X(t), t \geq 0\}$ where X(t)= $(S(t), I_1(t), I_2(t), I_3(t), I_4(t))$ is a Markov Chain with $3naK+1$ transient states and one absorbing state. The absorbing state represents the customer moving out of the system, either after service completion or without completing service. Here $S(t)$ denotes the status of the server at time t:

$$S(t) = \begin{cases} 0, & \text{if the service is going on and} \\ & \text{has not undergone any interruption so far,} \\ 1, & \text{if the service is interrupted,} \\ 2, & \text{if service is in interruption fixing state ,} \\ 3, & \text{if service is continuing after interruption ;} \end{cases}$$

$I_1(t)$ denotes the number of interruptions occurred until time t to the current customer in service. $I_1(t)$ varies from 0 to K;

$I_2(t)$ corresponds to the environmental factor that caused the current interruption to the service. In this model we consider n environmental factors. Thus $I_2(t)$ has values varying from 1 to n.

$I_3(t)$ represents the status of the random clock:

$$I_3(t) = \begin{cases} 0, & \text{if the random clock is realized} \\ 1, & \text{otherwise;} \end{cases}$$

$I_4(t)$ corresponds to the phase of the super clock. $I_4(t)$ varies from 1 to a. The states of the process is $\{0\} \cup \{(1, m, i, j, l)/m = 1, 2, \ldots, K; i = 1, \ldots, n; j = 1; l = 1, \ldots, a\} \cup \{(2, m, i, j, l)/m = 1, 2, \ldots, K; i = 1, \ldots, n; j = 0; l = 1, \ldots, a\} \cup \{(3, m, i, j, l)/m = 1, 2, \ldots, K; i = 1, \ldots, n; j = 0; l = 1, \ldots, a\} \cup \{*\}$, where $\{*\}$ is the absorbing state. The infinitesimal generator of the process is given by

$$\bar{Q} = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix} \text{ where } T = \begin{bmatrix} C_0 & C_1 & 0 & 0 \\ 0 & C_2 & C_3 & 0 \\ 0 & 0 & C_4 & C_5 \\ 0 & C_6 & 0 & C_7 \end{bmatrix}_{(3naK+1)}.$$

Let $\bar{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ then $C_0 = [-(\mu + \beta)]$;

$C_1 = \begin{bmatrix} \beta \mathbf{p} \otimes \boldsymbol{e}'_a(1) & 0 \end{bmatrix}_{(1 \times naK)}$; $C_2 = I_K \otimes G_0$;

$$G_0 = \begin{cases} -\gamma - \eta_r, & \text{if } i = j; i, j = (r-1)a + l; r = 1, \ldots, n; l = 1, \ldots, a; \\ \gamma, & \text{if } j = i + 1; i, j = (r-1)a + l; r = 1, \ldots, n; l = 1, \ldots, a - 1; \\ 0, & \text{otherwise;} \end{cases}$$

$C_3 = I_K \otimes G_1$, where

$$G_1 = \begin{cases} \eta_r, & \text{if } i = j; i, j = (r-1)a + l; r = 1, \ldots, n; l = 1, \ldots, a; \\ 0, & \text{otherwise;} \end{cases}$$

$C_4 = I_K \otimes G_2$ where

$$G_2 = \begin{cases} -\gamma - \alpha_r, & \text{if } i = j; i,j = (r-1)a + l; r = 1,\ldots,n; l = 1,\ldots,a; \\ \gamma, & \text{if } j = i+1; i,j = (r-1)a + l; r = 1,\ldots,n; l = 1,\ldots,a-1; \\ 0, & \text{otherwise;} \end{cases}$$

$C_5 = I_K \otimes G_3$ where

$$G_3 = \begin{cases} \alpha_r, & \text{if } i = J; i,j = (r-1)a + l; r = 1,\ldots,n; l = 1,\ldots,a; \\ 0, & \text{otherwise;} \end{cases}$$

$$C_6 = \begin{bmatrix} 0 & I_{K-1} \otimes G_4 \\ 0 & 0 \end{bmatrix}$$

where $G_4 = \beta \mathbf{P} \otimes I_a$, $\mathbf{P} = (p_{ij})$;for $i,j = 1,\ldots,n$.

$C_7 = I_K \otimes G_5$ and

$$G_5 = \begin{cases} -\mu_r - \beta, & \text{if } i = J; i,j = (r-1)a + l; r = 1,\ldots,n; l = 1,\ldots,a \\ 0, & \text{otherwise;} \end{cases}$$

$$. \ T^0 = \begin{bmatrix} C_{03} \\ C_{13} \\ C_{23} \\ C_{33} \end{bmatrix}_{(3nK+1)\times 1} \quad \text{with } C_{03} = \begin{bmatrix} \mu \end{bmatrix}; C_{13} = \begin{bmatrix} \boldsymbol{e}_{nK} \otimes \gamma \boldsymbol{e}_a(a) \end{bmatrix}.$$

$$C_{23} = C_{13},\ C_{33} = \begin{bmatrix} \boldsymbol{e}_{K-1} \otimes (\bar{\mu} \otimes \boldsymbol{e}_a) \\ \bar{\mu} \otimes \boldsymbol{e}_a + \boldsymbol{e}_{an}\beta \end{bmatrix}_{(naK\times 1)}$$

where e is a column vector with all its entries equal to 1 and of appropriate order.

In matrix T, $T_{ii} < 0, 1 \leq i \leq 3nak + 1$ and $T_{ij} \geq 0$ for $i \neq j$. Also $T\boldsymbol{e} + T^0 = 0$. The initial probability vector is $\boldsymbol{\zeta} = (1, 0, \ldots)$ which means that at first the service starts. Here the service process with interruption follows PH distribution. So using the property that residual service time in a phase type distributed service process is also phase type we see that

- Probability for service completion without any interruption is, $P(s) = \boldsymbol{\zeta}(-C_0)^{-1}C_{03}$.

- Probability for customer leaving the system due to realization of super clock,
$P_{RS} = \boldsymbol{\zeta}[(-C_0)^{-1}C_1][(-C_2)^{-1}C_{13}] + \boldsymbol{\zeta}[(-C_0)^{-1}C_1][(-C_2)^{-1}C_3][(-C_4)^{-1}C_{23}]$.

- Probability for customer leaving the system due to occurrence of maximum number of interruption,
$P_{MI} = \boldsymbol{\zeta}[(-C_0)^{-1}C_1][(-C_2)^{-1}C_3][(-C_4)^{-1}C_5][(-C_6)^{-1}C'_{33}]$
where $C'_{33} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \boldsymbol{e}_{an}\beta \end{bmatrix}_{(naK\times 1)}$.

Lemma: The expected time for service completion/customer leaving the system without completing service due to realization of super clock or attaining maximum interruption K is $E(ST) = \boldsymbol{\zeta}(\text{-T})^{-1}\text{e}$, and hence the expected service rate is $\mu_s = 1/E(ST)$.

3.2.1 Expected number of interruptions

The expected number of interruption during the service of a customer can be calculated by considering the Markov chain $\{Y(t), t \geq 0\}$ where $Y(t) = (I_1(t), S(t), I_2(t), I_3(t), I_4(t))$ with state space $\{0\} \cup \{(m, 1, i, j, l) | m = 1, 2, \ldots, K; i = 1, \ldots, n; j = 1; l = 1, \ldots, a, \} \cup \{(m, 2, i, j, l) | m = 1, 2, \ldots, K; i = 1, \ldots, n; j = 0; l = 1, \ldots, a, \} \cup \{(m, 3, i, j, l) | m = 1, 2, \ldots, K; i = 1, \ldots, n; j = 0; l = 1, \ldots, a, \} \cup \{\Omega\}$, where $\{\Omega\}$ is the absorbing state. The infinitesimal generator of the process is given by $Q_I = \begin{bmatrix} U & U^0 \\ 0 & 0 \end{bmatrix}$

where $U = \begin{bmatrix} D_0 & D_1 & & & & \\ & D_2 & D_3 & & & \\ & & D_2 & D_3 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \\ & & & & & D_2 \end{bmatrix}_{(3naK+1)}$. $U^0 = \begin{bmatrix} D_0' \\ D_1' \\ D_1' \\ . \\ . \\ D_2' \end{bmatrix}$.

where $D_0 = C_0$, $D_1 = \begin{bmatrix} \beta \mathbf{p} \otimes \boldsymbol{e}_a'(1) & 0 \end{bmatrix}_{(1\times na)}$

$D_2 = \begin{bmatrix} G_0 & G_1 & 0 \\ 0 & G_2 & G_3 \\ 0 & 0 & G_5 \end{bmatrix}_{(3na)}$, $D_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ G_4 & 0 & 0 \end{bmatrix}_{(3na)}$

$D_0' = C_{03}$, $D_1' = \begin{bmatrix} \boldsymbol{e}_n \otimes \gamma \boldsymbol{e}_a(a) \\ \boldsymbol{e}_n \otimes \gamma \boldsymbol{e}_a(a) \\ \bar{\mu} \otimes \boldsymbol{e}_a \end{bmatrix}$, $D_2' = \begin{bmatrix} \boldsymbol{e}_n \otimes \gamma \boldsymbol{e}_a(a) \\ \boldsymbol{e}_n \otimes \gamma \boldsymbol{e}_a(a) \\ \bar{\mu} \otimes \boldsymbol{e}_a + \boldsymbol{e}_{an}\beta \end{bmatrix}$. The initial probability vector $\boldsymbol{\zeta} = \begin{pmatrix} 1 & 0 & 0 & . & . \end{pmatrix}$.

From the above matrices we get the following system characteristics:

- Probability for absorption after $r (r \leq K)$ interruption is given by
$$(N_r) = \begin{cases} (-D_0)^{-1} D_0', & \text{for } r = 0, \\ (-D_0^{-1} D_1)(-D_2^{-1} D_3)^{(r-1)}(-D_2^{-1} D_2'), & \text{for } r = K, \\ (-D_0^{-1} D_1)(-D_2^{-1} D_3)^{(r-1)}(-D_2^{-1} D_1'), & \text{otherwise.} \end{cases}$$

- Expected number of interruptions E(I) for a customer $= \sum_{r=1}^{K} r N_r$.

- Probability for service completion after $r (1 \leq r \leq K)$ interruptions,
$M_r = \zeta(-D_0^{-1} D_1)(-D_2^{-1} D_3)^{(r-1)}(-D_{10}^{-1} D_3')$ where $D_3' = \begin{bmatrix} 0 \\ 0 \\ \bar{\mu} \otimes e_a \end{bmatrix}$.

- Expected number of interruptions E(IS) before service completion for a single service= $\sum_{r=1}^{K} rM_r$.

- Probability for service completion without any interruption $P(s) = \zeta(-D_0)^{-1}D_0'$.

Having computed the measures indicated above, we describe the queueing model and the condition for it to be stable.

3.3 The queueing model

Consider the queueing model $Z = \{Z(t), t \geq 0\}$, where $Z(t) = (N(t), S(t), I_1(t), I_2(t),\ I_3(t), I_4(t))$ where $N(t)$ is the number of customers in the system. Here $S(t)$ and $I_j(t), j = 1, 2, 3, 4$ are as defined in section 3.2. Z is a continuous time Markov chain with state space $\{0\} \cup \{(q, m, 1, i, j, l)|q = 1, 2 \ldots; m = 1, 2, \ldots, K; i = 1, \ldots, n; j = 1; l = 1, \ldots, a, \} \cup \{(q, m, 2, i, j, l)|q = 1, 2 \ldots; m = 1, 2, \ldots, K; i = 1, \ldots, n; j = 0; l = 1, \ldots, a, \} \cup \{(q, m, 3, i, j, l)|q = 1, 2 \ldots; m = 1, 2, \ldots, K; i = 1, \ldots, n; j = 0; l = 1, \ldots, a, \}$. Its infinitesimal generator Q is given by

$$
Q = \begin{bmatrix} B_0 & B_1 & & & & \\ B_2 & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}
$$

where $B_0 = [-\lambda]$, $B_1 = \begin{bmatrix} \lambda & \mathbf{0} \end{bmatrix}$, $B_2 = T^0$, $A_0 = \lambda I$, $A_1 = T - \lambda I$, $A_2 = \begin{bmatrix} T^0 & \mathbf{0} \end{bmatrix}_{(3nak+1)}$

Theorem: The system Z is stable when $\lambda < \mu_s$.

Proof: A queueing system is stable when arrival rate is less than service rate. Here the arrival rate is λ and effective service rate is μ_s. So the condition for stability of this queueing system is $\lambda < \mu_s$.

3.3.1 Stationary distribution

The stationary distribution, under the condition of stability, $\lambda < \mu_s$ of the model, has Matrix Geometric solution. Let χ=$(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ be the steady state probability vector of the Markov chain $\{Z(t), t \geq 0\}$. Each $\mathbf{x}_i, i > 0$ are vectors with $3naK + 1$ elements. We assume that $\mathbf{x}_2 = \mathbf{x}_1.R$, and $\mathbf{x}_i = \mathbf{x}_1.R^{i-1}, i \geq 2$,

where R is the minimal non- negative solution to the matrix quadratic equation

$R^2A_2 + RA_1 + A_0$=0.

From χQ=0 we get

$\mathbf{x}_0 B_0 + \mathbf{x}_1 B_2$=0.

$\mathbf{x}_0 B_1 + \mathbf{x}_1(A_1 + RA_2)$=0.

Solving the above two equations we get $\mathbf{x}_0$ and $\mathbf{x}_1$ subject to the normalizing condition $\mathbf{x}_0 e + \mathbf{x}_1(I - R)^{-1}e$=1.

3.4 Performance measures

In this section we list a number of key system performance measurebs to bring out the qualitative aspects of the model under study. The measures are listed below along with their formulae for computation.

3.4.1 Expected waiting time

The expected waiting time of a particular customer who joins the queue as the m^{th} customer, can be computed by considering the Markov Chain $Z=\{(M(t), S(t), I_1(t), I_2(t), I_3(t), I_4(t)), t \geq 0\}$ where $M(t)$ is the rank of the tagged customer. The tagged customers rank will decrease to 1 as the customers ahead of him leave the system. The rank of the customer is not affected by the arrival of customers following the tagged customer. Here $S(t)$ and $I_j(t), j = 1, 2, 3, 4$ are as defined section 3.2. Z is a Markov chain with state space $\{(m, 0)|m = r, \ldots, 1\} \cup \{(m, 1, s, i, j, l)|m = r, \ldots, 1; s = 1, 2, \ldots, K; i = 1, \ldots, n; j = 1; l = 1, \ldots, a\} \cup \{(m, 2, s, i, j, l)|m = r, \ldots, 1; s = 1, 2, \ldots, K; i = 1, \ldots, n; j = 0; l = 1, \ldots, a\} \cup \{(m, 1, s, i, j, l)|m = r, \ldots, 1; s = 1, 2, \ldots, K; i = 1, \ldots, n; j = 0; l = 1, \ldots, a\} \cup \{\Phi\}$, where $\{\Phi\}$ is the absorbing state. The infinitesimal generator matrix Q_2 is given by $Q_2 = \begin{bmatrix} W & W^0 \\ 0 & 0 \end{bmatrix}$ where

$$W= \begin{bmatrix} T & T^0\boldsymbol{\zeta} & & & \\ & T & T^0\boldsymbol{\zeta} & & \\ & & \ddots & \ddots & \\ & & & T & T^0\boldsymbol{\zeta} \\ & & & & T \end{bmatrix} \text{ and } W^0 = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ T^0 \end{bmatrix}.$$

The expected waiting time of the tagged customer, according to the position of the customer being served at the time of arrival of the tagged customer, is a column vector which is obtained from the formula
$E_W^r = -T^{-1}(I - (T^0\boldsymbol{\zeta}T^{-1})^{(r-1)})(I - T^0\boldsymbol{\zeta}T^{-1})^{-1}e.$
Hence the expected waiting time of a customer waiting in the queue is
$E(W) = \sum_{r=1}^{\infty} \mathbf{x}_r E_W^r$.

3.4.2 Other important performance characteristics

- Expected number of customers completing service without interruption E(NI)=$\sum_{i=1}^{\infty} i\mathbf{x}_{i,0}$.

- Probability that there are i $(i \geq 0)$ customers in the system, P_i=$\mathbf{x}_i\boldsymbol{e}$.

- Expected number of customers in the system, $E(s)$= $\sum_{i=1}^{\infty} iP_i$.

- Fraction of time the server in the interrupted state, FT(I)=$\sum_{i=1}^{\infty}(\mathbf{x}_{i,1}\boldsymbol{e} + \mathbf{x}_{i,2}\boldsymbol{e})$.

- Fraction of time the server is busy, FT(B)=$\sum_{i=1}^{\infty}(\mathbf{x}_{i,0}\boldsymbol{e} + \mathbf{x}_{i,1}\boldsymbol{e} + \mathbf{x}_{i,3}\boldsymbol{e})$.

- Fraction of time the server in the unidentified interrupted state, FT(NI)=$\sum_{i=1}^{\infty} \mathbf{x}_{i,1}\boldsymbol{e}$.

- Fraction of time the server in fixing state, FT(FS)=$\sum_{i=1}^{\infty} \mathbf{x}_{i,2}\boldsymbol{e}$.

- Fraction of time the super clock is freezed, FT(SF)=$\sum_{i_1=1}^{\infty}\mathbf{x}_{i,3}\boldsymbol{e}$.

- Fraction of time the super clock is on, $= \sum_{i=1}^{\infty}(\mathbf{x}_{i,1}\boldsymbol{e} + \mathbf{x}_{i,2}\boldsymbol{e})$.

3.5 Numerical illustrations

The performance of the system depending on the change in various parameters are numerically illustrated below. Assume $\lambda = 3, \mu = 6, n = 3, a = 3; \beta = 0.5; \gamma = 0.5,\ \alpha_1 = 4, \alpha_2 = 3, \alpha_3 = 3,,\ \eta_1 = 4, \eta_2 = 4, \eta_3 = 3,,$ $\mu_1 = 3, \mu_2 = 2, \mu_3 = 2,\ \mathbf{p} = (0.3, 0.4, 0.3)$ and the transition probability matrix $\mathbf{P} = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.4 & 0.1 & 0.5 \\ 0.5 & 0.4 & 0.1 \end{bmatrix}$

Effect of K on various performance measures

Table 3.1: **Effect of change in K on** $E(s)\&E(W)$

K	$E(s)$	$E(W)$
1	0.6792	0.0585
2	0.7175	0.0645
3	0.7238	0.0655
4	0.7248	0.0657
5	0.7250	0.0657
6	0.7250	0.0657
7	0.7250	0.0657

As the value of K increases expected number of customers in the system$E(s)$ and expected waiting time$E(W)$ increase (see Table 3.1).

Effect of β on various performance measures

By considering $\mu = 8$ and $K = 5$ we have the following values. As

Table 3.2: **Effect of change in β on various performance measures**

β	$P(s)$	μ_s	$FT(B)$	$FT(I)$	$E(s)$	$E(I)$	$E(W)$
0.2	0.9756	0.0727	0.3713	0.0054	0.4445	0.0191	0.0321
0.4	0.9524	0.0739	0.3677	0.0106	0.5175	0.0304	0.042
0.6	0.9302	0.0751	0.3643	0.0155	0.5934	0.0376	0.0538
0.8	0.9091	0.0763	0.3611	0.0202	0.6712	0.0425	0.0653
1	0.8889	0.0774	0.3580	0.0247	0.7453	0.0465	0.0772

the value of β increases probability for service completion with interruption $P(s)$, fraction of time the server is busy $FT(B)$ decreases, but effective service rate μ_s, fraction of time the server getting interrupted $FT(I)$,expected number of customers in the system $E(s)$, expected number of interruptions$E(I)$ and expected waiting time $E(W)$ begins to increase which are on expected lines (Refer Table 3.2).

Effect of μ on various performance measures

By considering $\beta = .5$ and $K = 5$ we have the following values. From Ta-

Table 3.3: **Effect of change in μ on various performance measures**

μ	$P(s)$	μ_s	$FT(B)$	$FT(I)$	$E(s)$	$E(I)$	$E(W)$
6	0.9231	0.0864	0.4786	0.0171	0.7250	0.0450	0.0657
7	0.9333	0.0800	0.4148	0.0148	0.6290	0.0390	0.0557
8	0.9412	0.0745	0.3660	0.0131	0.5551	0.0344	0.0482
9	0.9474	0.0697	0.3275	0.0117	0.4967	0.0308	0.0423
10	0.9524	0.0655	0.2963	0.0106	0.4494	0.0278	0.0376
11	0.9565	0.0617	0.2705	0.0097	0.4103	0.0254	0.0338
12	0.9600	0.0584	0.2489	0.0089	0.3775	0.0234	0.0306

ble 3.3 as the value of μ increases probability for service completion with

interruption $P(s)$increases but fraction of time the server is busy $FT(B)$, effective service rate μ_s, fraction of time the server getting interrupted $FT(I)$,expected number of customers in the system $E(s)$, expected number of interruptions$E(I)$ and expected waiting time $E(W)$ increases which are on expected line

Effect of γ on various performance measures

By considering $\beta = .5$, $\mu = 8$ and $K = 5$ we have the following values. Inference from Table 3.4 is that the realization rate of the super clock in-

Table 3.4: **Effect of change in γ on various performance measures**

γ	$FT(B)$	$FT(I)$	$E(s)$	$E(I)$	$E(W)$
0.2	0.0132	0.5572	0.3661	0.0343	0.0485
0.4	0.0131	0.5561	0.3661	0.0344	0.0483
0.6	0.0130	0.5538	0.3660	0.0344	0.0480
0.8	0.0129	0.5507	0.3658	0.0345	0.0475
1	0.0127	0.5468	0.3656	0.0347	0.0469
1.2	0.0125	0.5423	0.3655	0.0350	0.0462
1.4	0.0123	0.5376	0.3653	0.0352	0.045

creases expected number of customers in the system and expected waiting time decrease. This is due to customers leaving the system when super clock realizes. Expected number of interruptions increases as γ increases.

Effect of λ on various performance measures

By considering $\mu = 14$ and $K = 5$ we have the following values. As the value of λ increases fraction of time the server is busy $FT(B)$, fraction of

Table 3.5: **Effect of change in λ on various performance measures**

λ	$FT(B)$	$FT(I)$	$E(s)$	$E(W)$
1	0.0830	0.0030	0.1258	0.0205
2	0.1659	0.0059	0.2517	0.0270
3	0.2489	0.0089	0.3775	0.0306
4	0.3319	0.0119	0.5032	0.0329
5	0.4148	0.0148	0.6272	0.0345
6	0.4974	0.0183	0.7336	0.0357
7	0.5689	0.0480	0.7287	0.0359

time the server getting interrupted $FT(I)$,expected number of customers in the system $E(s)$, expected number of interruptions$E(I)$ and expected waiting time $E(W)$ increase (see Table 3.5).

Chapter 4

Queue with partially ignored interruption in Markovian environment

Introduction

In chapter 2 and 3 we considered queueing models with delayed identification of interruption in random environment and Markovian environment. In this chapter we consider a queueing model in which the interruption is identified on the instant it strikes. But the interruption is ignored for sometime so that the service of the current customer may be completed without break. Depending on the environmental factor causing interruption, some times the interruption has to be attended immediately. i.e. some interruptions will not allow the server to continue service due to major damage. As a real life example we consider the case of a computer providing service. Customers are waiting in queue with jobs to be done by the computer. During the operation of the computer there can be software problem, hardware problem, power failure, virus problem, etc. The computer may complete its job even if there are such problems. On some occasions if it continues working with interruption there is a possibility for increase in the severity of damage. In that case the computer may become irreparable or repairing may become a time consuming and costly process. In such situations the replacement of the computer is the only possible solution for the smooth functioning of the system.

4.1 Model description

Consider a single server queueing system in which arrival occurs according to a Poisson Process with parameter λ. On arrival if the customer finds the server busy, he joins the tail of the queue otherwise he gets service immediately. The service is Erlang distributed with shape and scale parameters μ and a respectively. During service there is a possibility for interruption in service due to different factors. Here we assume that there are $n+1$ environmental factors causing interruption to the service. These factors are numbered 1 to $n+1$ depending on the ascending order of severity of interruption caused by them. The interruption occurs according to a Poisson Process with parameter β. When the interruption due to i^{th} factor occurs the rate of service changes from μ to μ_i. On the onset of interruption one random clock which is exponentially distributed with parameter α and one interruption clock which is PH distributed with representation (δ, U) of order m are started. Only forward phase change is allowed for the interruption clock, i.e. $U_{ij} = 0$ for $i > j$. When the interruption occurs due to any one of the first n factors, it is not attended immediately. It is ignored in the beginning and service is continued with interruption. When the interruption clock realizes the service is stopped and the server is taken for repair. The repair time is exponentially distributed with parameter η. After repair the service to the interrupted customer is resumed if the interruption clock is realized before the random clock; else the service is restarted. There is a possibility for customer completing service with interruption. In that case the server goes for repair after the service completion. If the interruption is due to $(n+1)^{th}$ factor the customer goes out of the system and the server is replaced immediately. Once the interruption starts getting attended both the clocks are reset to zero position. As the duration of ignored interruption increases the severity of interruption also increases. After some duration, the cause of interruption changes from i^{th} factor to j^{th} factor, where $j \geq i$ and i^{th} factor is the one causing initial interruption. Then the service rate also changes from μ_i to μ_j. Again if the interruption remains unattended for some more time, the cause of interruption changes from j^{th} factor to k^{th} factor, where $k \geq j$. The server is replaced on being interrupted by the $(n+1)^{th}$ factor. The customer in service is also lost when the interruption is due to $(n+1)^{th}$ factor.

The $n+1$ environmental factors are the states of a Markov chain with initial probability vector $p_i, i = 1, 2, \dots, n+1$ and transition probability matrix $P = (p_{ij}), i, j = 1, 2, \dots, n+1$. Graphical representation of the model is given in Figure 4.1.

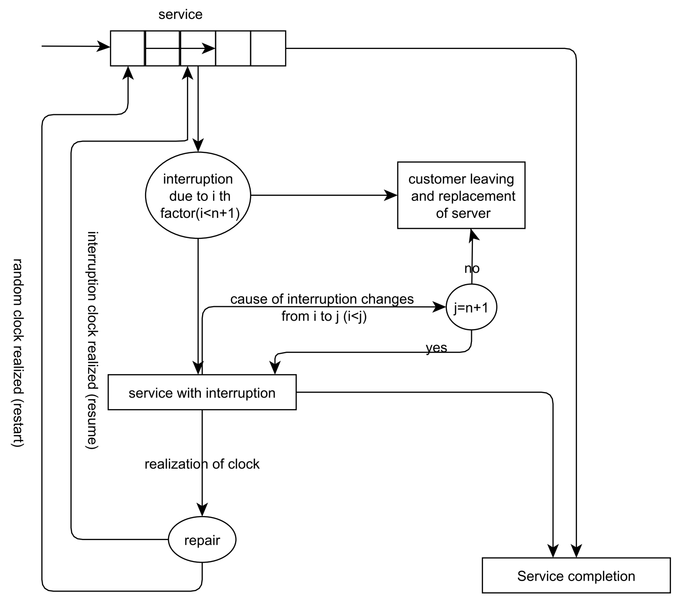

Figure 4.1: model description

4.2 Analysis of service process with interruption

The service process $\{Y(t), t \geq 0\}$ where $Y(t) = (S(t), I_1(t), I_2(t), I_3(t), I_4(t))$ is a Markov chain with $(2 + mn)a + 1$ transient states given by $\{(0, j) \cup (1, i, j, l, 1) \cup (2, j, 1) \cup (2, 0)\}$ with $i = 1, 2, \ldots, n; j = 1, 2, \ldots, a; l = 1, 2, \ldots, m$; and one absorbing state denoted by $*$. The absorbing state represents the customer moving out from the system either after service completion or due to interruption caused by $(n+1)^{th}$ environmental factor. $S(t)$ denotes the status of the server at time t:

$$S(t)\begin{cases} 0, & \text{if the ongoing service is without interruption.} \\ 1, & \text{if the ongoing service is with interruption.} \\ 2, & \text{if server is under repair;} \end{cases}$$

$I_1(t)$ corresponds to the environmental factor that caused the current interruption to the service. In this model we consider $n+1$ environmental factors;

$I_2(t)$ denotes the phase of service. It varies from 1 to a;

$I_3(t)$ denotes the phase of interruption clock. It varies from 1 to m;

$I_4(t)$ denotes the phase of random clock. It takes the value 0 if clock is realized and 1 if it is functioning;

The infinitesimal generator of the process is given by $Q = \begin{bmatrix} W & W^0 \\ 0 & 0 \end{bmatrix}$

with initial probability vector $\boldsymbol{\vartheta} = (1, \mathbf{0}, \mathbf{0})$

where $W = \begin{bmatrix} D_0 & D_1 & 0 \\ 0 & D_2 & D_3 \\ D_4 & 0 & D_5 \end{bmatrix}$ and $W^0 = \begin{bmatrix} D_{01} \\ D_{11} \\ 0 \end{bmatrix}$

Let $\delta = (\delta_1, \delta_2, \ldots, \delta_m)$

then $D_0(i,j) = \begin{cases} -\mu - \beta, & \text{if } i = j; \\ \mu, & \text{if } i = j-1 \quad i,j = 1, \ldots, a; \\ 0, & \text{otherwise;} \end{cases}$

$D_1 = \beta[p' \otimes (I_a \otimes \delta)]_{1\times mna}$ with $p' = (p_1, p_2, \ldots, p_n)$;

$D_2 = I_{na} \otimes U - \overline{\mu} \otimes I_{ma} + \widehat{P} \otimes I_{am} - \alpha I_{mna} + J$;

where $\widehat{P} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ & P_{22} & \cdots & P_{2n} \\ & & \ddots & \\ & & & P_{nn} \end{bmatrix}$. $\overline{\mu} = \begin{bmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \ddots \\ & & & \mu_n \end{bmatrix}$.

and J is a square matrix of order mna

$$J(i,j) = \begin{cases} \mu_r, & \text{for } j = i+m, i = (r-1)am+1, \ldots, (ra-1)m; \\ & \quad r = 1, \ldots, n \\ 0, & \text{otherwise;} \end{cases}$$

$D_3 = I_n \otimes A$; with $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$, where $A_1 = I_a \otimes U^0$, $A_2 = \alpha e_{ma}$

D_4 is a matrix of order $((a+1) \times a)$ with

$$D_4(i,j) = \begin{cases} \eta, & \text{for } i = j = 1 \ \& \ j = i+1 \\ 0, & \text{otherwise;} \end{cases}$$

$D_5 = -\eta I_{a+1}$

D_{01} is a column vector, $D_{01}(i,1) = \begin{cases} \beta p_{n+1} & \text{for } i = 1, 2, \ldots, a-1 \\ \mu + \beta p_{n+1} & \text{for } i = a; \end{cases}$

$D_{11} = [K + \overline{P} \otimes e_{ma}]_{mna \times 1}$; where $\overline{P} = \begin{bmatrix} P_{1n+1} \\ P_{2n+1} \\ \\ P_{nn+1} \end{bmatrix}$.

K is an $(mna \times 1)$ matrix and

$$K(i,1) = \begin{cases} \mu_r, & \text{for } i = (ra-1)m+1, \ldots, ram; r = 1, \ldots, n \\ 0, & \text{otherwise;} \end{cases}$$

- Expected time for service completion, $E(ST) = \vartheta(-W)^{-1}e$. Hence expected service rate $\mu_s = \frac{1}{E(ST)}$

- Rate of replacement of server due to the interruption caused by $(n+1)^{th}$ factor,

  $R_{replacement} = \vartheta(-W)^{-1}W^{0'}$ where $W^{0'} = \begin{bmatrix} e_a \beta p_{n+1} \\ \overline{P} \otimes e_{ma} \\ 0 \end{bmatrix}$

4.2.1 Expected number of interruptions during the service of a customer

For calculating the expected number of interruptions during the service of a customer we consider the Markov chain $\left\{\widehat{Y}(t), t \geq 0\right\}$ where $\widehat{Y}(t) = (M(t), S(t), I_1(t), I_2(t), I_3(t), I_4(t))$ where $M(t)$ is the number of interruptions occurred until time t. $S(t), I_1(t), I_2(t), I_3(t)$ and $I_4(t)$ are as defined above. The state space of $\widehat{Y}(t)$ is $\{(r, 0, j) \cup (r, 1, i, j, l, 1) \cup (r, 2, j, 1) \cup (r, 2, 0)\}$ with $r = 0, 1, 2, \ldots, \infty; i = 1, 2, \ldots, n; j = 1, 2, \ldots, a; l = 1, 2, \ldots, m;$ and one absorbing state ∇. ∇ represents the customer moving out from the system either after service completion or due to interruption caused by $(n+1)^{th}$ environmental factor. The infinitesimal generator matrix associated with $\widehat{Y}(t)$ is $\widehat{Q} = \begin{bmatrix} Y & Y^0 \\ 0 & 0 \end{bmatrix}$

where $Y = \begin{bmatrix} \widehat{B}_0 & \widehat{B}_1 & & & & \\ & \widehat{A}_1 & \widehat{A}_0 & & & \\ & & \widehat{A}_1 & \widehat{A}_0 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \end{bmatrix}$ and $Y^0 = \begin{bmatrix} \widehat{B}_2 \\ \widehat{A}_2 \\ \widehat{A}_2 \\ \vdots \\ \vdots \end{bmatrix}$.

Here $\widehat{B}_0 = D_0$; $\widehat{B}_1 = D_1$; $\widehat{B}_2 = D_{01}$;

$\widehat{A}_1 = \begin{bmatrix} D_2 & D_3 & 0 \\ 0 & D_5 & D_4 \\ 0 & 0 & D_0 \end{bmatrix}$; $\widehat{A}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D_1 & 0 & 0 \end{bmatrix}$; $\widehat{A}_2 = \begin{bmatrix} D_{11} \\ 0 \\ D_{01} \end{bmatrix}$.

Let a_r be the probability for absorption after r interruptions, then

$$a_0 = \vartheta((-\widehat{B}_0^{-1}\widehat{B}_2).$$

$a_r = \vartheta(-\widehat{B}_0^{-1}\widehat{B}_1)((\widehat{A}_1^{-1}\widehat{A}_0))^{r-1}(-\widehat{A}_1^{-1}\widehat{A}_2)$, r$= 1, 2, \ldots$

Expected number of interruptions before absorption,

$E(NI) = \sum_{r=1}^{\infty} r a_r = \vartheta(-\widehat{B}_0^{-1}\widehat{B}_1)(I - (\widehat{A}_1^{-1}\widehat{A}_0))^{-2}(-\widehat{A}_1^{-1}\widehat{A}_2)$

4.3 The queueing model

Let $N(t)$ be the number of customers in the system. Then $X = \{X(t), t \geq 0\}$, where $X(t) = (N(t), S(t), I_1(t), I_2(t), I_3(t), I_4(t))$, is a CTMC with state space $\{r, 0, j) \cup (r, 1, i, j, l, 1) \cup (r, 2, j, 1) \cup (r, 2, 0)\}$ with $r = 1, 2, \ldots, ; i = 1, 2, \ldots, n; j = 1, 2, \ldots, a; l = 1, 2, \ldots, m$. $S(t), I_1(t), I_2(t), I_3(t)$ and $I_4(t)$ are as defined in section 4.2. The infinitesimal generator matrix associated with the model $\widehat{Q} = \begin{bmatrix} B_0 & B_1 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$ where $B_0 = [-\lambda]$;
$B_1 = \begin{bmatrix} \lambda & 0 & 0 \end{bmatrix}$
$B_2 = W^0$.
$A_0 = [\lambda I]$; $A_1 = W - \lambda I$; $A_2 = [W^0 \ \mathbf{0}]$ is a square matrix.
Theorem: The queueing system is stable when $\lambda < \mu_s$.
Proof: Let $\boldsymbol{\pi}$ denote the steady-state probability vector of the generator $A_0 + A_1 + A_2$. That is, $\boldsymbol{\pi}(A_0 + A_1 + A_2) = 0$; $\boldsymbol{\pi}\boldsymbol{e} = 1$: The LIQBD description of the model indicates that the queueing system is stable if and only if $\boldsymbol{\pi} A_0 \boldsymbol{e} < \boldsymbol{\pi} A_2 \boldsymbol{e}$ That is, the rate of drift to the left has to be higher than that to the right. Right drift rate means the arrival rate λ and left drift rate is the effective service rate μ_s. Hence the queueing system is stable when $\lambda < \mu_s$.

4.3.1 Stationary distribution

Let χ=($\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...$) be the steady state probability vector of the Markov chain $\{X(t), t \geq 0\}$. Each $\mathbf{x}_i, i > 0$ are vectors with $(nm+1)a$ elements. We assume that $\mathbf{x}_2 = \mathbf{x}_1.R$, and $\mathbf{x}_i = \mathbf{x}_1.R^{i-1}, i \geq 2$, where R is the minimal non-negative solution to the matrix quadratic equation
$R^2A_2 + RA_1 + A_0$=0.
From χQ=0 we get
$\mathbf{x}_0 B_0 + \mathbf{x}_1 B_2$=0.
$\mathbf{x}_0 B_1 + \mathbf{x}_1(A_1 + RA_2)$=0.
Solving the above two equations we get $\mathbf{x}_0$ and $\mathbf{x}_1$ subject to the normalizing condition $\mathbf{x}_0 e + \mathbf{x}_1(I-R)^{-1}e$=1.

4.4 Performance measures

After calculating the steady state probability vector we now calculate some important performance measures of the system to bring out the qualitative aspects of the model under study. These are listed below along with their formula for computation.

4.4.1 Expected waiting time

The expected waiting time of the customer who joins as the r^{th} customer $(r > 0)$ in the queue can be obtained from the Markov chain $X'(t) = (M'(t), S(t), I_1(t), I_2(t), I_3(t), I_4(t))$ where $M'(t)$ is the rank of the tagged customer.

$S(t), I_1(t), I_2(t), I_3(t)$ and $I_4(t)$ are as defined in section 4.2. The waiting time distribution of the tagged customer follows phase type distribution with representation $(\boldsymbol{\theta}, S)$ and is given by

$$S = \begin{bmatrix} W & W^0\boldsymbol{\vartheta} & & & \\ & W & W^0\boldsymbol{\vartheta} & & \\ & & \cdots & \cdots & \\ & & & & \\ & & & & W \end{bmatrix}, \quad S^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ W^0 \end{bmatrix}.$$

The expected waiting time of the tagged customer is a column vector given by

$$E_w^r = -W^{-1}(I - W^0\boldsymbol{\vartheta}W^{-1})^{(r-1)}(I - W^0W^{-1})^{-1}\boldsymbol{e}.$$

Depending on the state of the server at the time of joining of the r^{th} customer we get different values for expected waiting time of the tagged customer.
So the expected waiting time of a customer who waits in the queue is

$$E(W) = \sum_{r=1}^{\infty} \mathbf{X}_r E_w^r.$$

4.4.2 Important Performance measures

- Mean number of customers in the system, $E(s) = \sum_{i=1}^{\infty} i\mathbf{x}_i\boldsymbol{e}$.

- Mean number of customers in the queue, $E(q) = \sum_{i=1}^{\infty} (i-1)\mathbf{x}_i\boldsymbol{e}$.

- Probability that there is no customer in the system, $P_{idle} = \mathbf{x}_0$.
- Probability that the system is under interruption, $P(I) = \sum_{i=1}^{\infty} \mathbf{x}_{i1}\boldsymbol{e}$.
- Probability that the system is under repair, $P(r) = \sum_{i=1}^{\infty} \mathbf{x}_{i2}\boldsymbol{e}$.
- Effective interruption rate, $R_{int} = \beta \sum_{i=1}^{\infty} \mathbf{x}_{i0}\boldsymbol{e}$.
- Effective repair rate, $R_{Rep} = \eta \sum_{i=1}^{\infty} \mathbf{x}_{i1}\boldsymbol{e}$.
- Effective rate of repetition of service, $R_{rpt} = \alpha \sum_{i=1}^{\infty} \mathbf{x}_{i1}\boldsymbol{e}$.
- Rate at which service completion with interruption occurs before the random clock is realized is $\sum_{i=1}^{\infty}\sum_{j=1}^{n}\sum_{r=1}^{m} \mathbf{x}_{i1jar}\mu_j$.
- Probability of a customer completing service without any interruption, $P(s)$=P(Service time < exponentially distributed random variable with parameter β)= $\frac{\mu^a}{(\mu+\beta)^{a+1}}$.
- Probability that at least one interruption in service is $1 - \frac{\mu^a}{(\mu+\beta)^{a+1}}$.
- probability that the interruption is attended before the random clock is realized,P_{ia} =P(interruption random variable < random clock variable)= $\delta(\alpha I_m - U)^{-1}U^0$.
- Probability for restart of service is $1 - \delta(\alpha I_m - U)^{-1}U^0$.

4.5 Cost function

To compute the expected cost, we construct a cost function depending on certain important performance measures.The total cost for running the system $E(C)$=$\mu.a * C_0 + R_{replacement} * C_1 + E(q) * C_2 + R_{rpt} * C_3 + C_4 * R_{Rep}$, where

- C_0-Unit time Cost of service;
- C_1-Unit time Cost for replacing the server;
- C_2- Holding cost per customer in the queue;
- C_3-Unit time Cost for restarting the service;
- C_4-Unit time Cost for repairing the server

4.6 Numerical examples

The performance of the queueing system is numerically illustrated in this section.
Let $n = m = a = 3$, $\lambda = 1$, $C_0 = \$100$, $C_1 = \$15000$, $C_2 = \$10$, $C_3 = \$200$, $C_4 = \$2000$, $\mu = 5, \mu_1 = 3, \mu_2 = 2, \mu_3 = 2$, $p' = (0.2, 0.2, 0.2)$, $p_{n+1} = 0.4$, $\delta = (0.3, 0.3, 0.4)$,

$$P = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.1 \\ 0 & 0.5 & 0.3 & 0.2 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}; U = \begin{bmatrix} -30 & 15 & 5 \\ 0 & -20 & 10 \\ 0 & 0 & -10 \end{bmatrix}, U^0 = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix}$$

By taking $\beta = 2$, $\eta = 4$ the numerical values of various performance measures are calculated here.

Effect of the rate of realization of random clock on various performance measures

Table 4.1: **Effect of α on various performance measures**

α	$E(s)$	$R_{replacement}$	μ_S	P_{idle}	P_{ia}
1	2.2246	1.0493	1.5736	0.2849	0.1609
2	2.2422	1.0458	1.5630	0.2851	0.8333
3	2.2472	1.0428	1.5546	0.2862	0.7692
4	2.2447	1.0401	1.5478	0.2876	0.7143
5	2.2379	1.0378	1.5423	0.2892	0.6667
6	2.2288	1.0356	1.5376	0.2909	0.6250

From Table 4.1 as the rate of realization of random clock increases expected service rate decreases. When random clock realizes the server goes for repair and after the repair of server the service of the customer in service restarts. This time lag reduces the service rate. Here the replacement rate decreases. This is because when random clock realizes the service is stopped and repair begins. So the progress in interruption is stopped. This reduces the chance of replacement of server. But the rate of restart of service increases. This is due to restart of service after every realization of random clock.

Table 4.2: **Effect of α and β on $E(C)$**

α	$E(C)$	β	$E(C)$
1	17262	1	1685
2	17210	2	17120
3	17165	3	17328
4	17124	4	17495
5	17089	5	17635
6	17057	6	17777

Effect of rate of realization of random clock and rate of interruption on $E(C)$

From Table 4.2 as rate of realization of random clock increases expected cost is also increases. when rate of interruption increases expected cost also increases which are on expected lines.

Effect of rate of interruption on various performance measures

By taking $\mu = 5$, $\alpha = 4$, $\eta = 5$ the numerical values of various performance measures are calculated here. From Table 4.3 as occurrence of interruption increases expected number of customers in the system, rate of replacement of server increases and effective service rate decreases. This happens due to delay in service caused by repair of server or the reduced service rate of interrupted server. Increase in the occurrence of interruption reduces the probability for service completion without interruption. By taking $\alpha = 1$, $\beta = 1$ the numerical values of various performance measures are calculated here.

Table 4.3: **Effect of β on various performance measures**

β	$E(s)$	$R_{replacement}$	μ_S	P_{idle}	$P(s)$
1	1.4802	1.0226	1.6311	0.3651	0.3517
2	1.8160	1.0401	1.6182	0.3295	0.1609
3	2.2718	1.0537	1.6183	0.2894	0.0863
4	2.9943	1.0643	1.6259	0.2407	0.0514
5	4.4084	1.0727	1.6378	0.1786	0.0331
6	8.5368	1.0794	1.6521	0.0993	0.0225

Effect of repair rate on various performance measures

Table 4.4: **Effect of η on various performance measures**

η	$E(s)$	μ_S	P_{idle}	$E(C)$
2	32.8330	1.1848	0.0244	17104
3	5.9106	1.2793	0.1195	16769
4	3.9102	1.3324	0.1686	16749
5	3.2094	1.3664	0.1974	16742
6	2.8571	1.3901	0.2162	16738
7	2.6461	1.4075	0.2294	16736
8	2.5060	1.4208	0.2391	16735

Inference from Table 4.4 is that as the repair rate increases expected number of customers in the system and expected service rate decreases. But probability for idleness increases. The increased repair rate reduces the waiting time of customers due to which the queue length reduces. When queue length reduces the probability of idleness increases. the repair rate increases the expected cost decreases. This is due to the decrease in waiting time of customer in the queue.

Rate of interruption clock realization & efficiency of system

Table 4.5: **rate of interruption clock realization & efficiency of system**

rate of interruption clock realization	efficiency of system
0.0480	1.6307
0.0653	1.6058
0.0674	1.6031
0.0778	1.5894
0.0799	1.5863
0.0813	1.5848
0.1000	1.5630
0.1408	1.5195
0.1519	1.5081

From table 4.5 as the rate of realization of interruption clock increases rate of repair increases which causes reduction in the efficiency of the system.

Chapter 5

Queue with ignored interruption in random environment and self correction.

Introduction

In the previous chapter we discussed queueing models with interruption in Markovian environment with partially ignored interruption. In this chapter we analyze two queueing models. In the first model we consider a single server queueing system with arrival following Poisson process and service time Erlang distributed. At times there is a possibility for interruption in service process. Interruption occurs according to a Poisson process and each interruption duration is exponentially distributed. Assume the interruption does not affect the customer in service. Further at a time at most one interruption affects any service. The service continues ignoring the interruption. During interrupted service there is a scope for self correction of interruption. Self correction occurs according to Poisson process. On the onset of interruption an interruption clock is started which is Erlang distributed. If the interruption clock is realized before service completion the server goes for repair and after repair the service is resumed. Repair time is exponentially distributed. If service is completed before the realization of interruption clock the next customer in the head of the queue enters for service.

In the second model the arrival process and the service process are as in the first model. During service interruption may occur. Interruption to service occurs according to a Poisson process. There are n environmental factors causing interruption. Interruption due to i^{th} environmental factor occurs with probability p_i. If the interruption is due to first m factors it is ignored and service continues. But the service will be at lower rate. The duration up to which the server works without breakdown is assessed with the help of an interruption clock. This clock starts ticking with the initiation of the first interruption to the service of a customer. The duration of the clock is exponentially distributed. During that period there is a possibility for self correction of interruption. This self correction duration is exponentially distributed. If self correction occurs the service rate changes. On realization of the interruption clock the server goes for repair. The repair time is exponentially distributed. After repair the interrupted service is resumed. If the service of a customer is completed with interruption the next customer in the head of the queue enter for service in the interrupted server. If the interruption is due to the remaining $n - m$ factors the server directly goes for repair. Taking into account the severity of interruption caused by these $n - m$ factors, protection for remaining service is provided at the epoch of resumption of service after repair. The stability of both the systems are analyzed. Steady state probability vector is calculated using matrix analytic method. Important performance measures are numerically substantiated.

5.1 Model Description (Model I)

Consider a single server queueing system in which arrival occurs according to a Poisson process with parameter λ. The service time is Erlang distributed with shape and scale parameters μ and a, respectively. During service there is a possibility for occurrence of interruption to service. The duration of interruption is exponentially distributed with parameter β. The service is continued without attending the interruption. A clock called interruption clock, is started on the onset of interruption. The interruption clock is Erlang distributed with shape and scale parameters δ and b respectively. Sometimes the interruption gets self corrected. The self correction occurs according to a Poisson process with parameter γ. If the customer in service completes service with interruption the next customer in the head of the queue enters for service. If the interruption clock realizes before completion of service of the customer, the server goes for repair and after repair the service to the interrupted customer is resumed. The repair time is exponentially distributed with parameter η. The model is pictorially represented in FIgure 5.1.

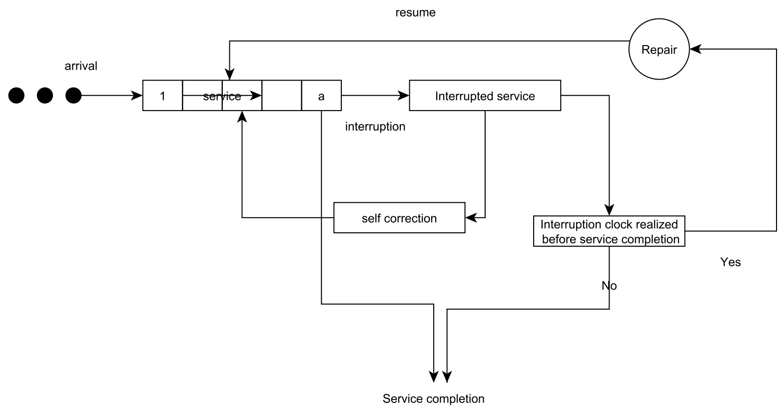

Figure 5.1: Model Description

5.2 Analysis of the model

The Markov process associated with the queueing model, $X = \{X(t), t \geq 0\}$ is a continuous time Markov Chain, where $X(t) = (N(t), S(t), I_1(t), I_2(t))$.
Here at time t:
$N(t)$ - Number of customers in the system;
$I_1(t)$ - Phase of interruption clock. It varies from 1 to b;
$I_2(t)$ - Phase of service. It varies from 1 to a;
$S(t)$ - Status of server;

$$S(t) = \begin{cases} 0, & \text{if a service is going on without interruption;} \\ 1, & \text{if service is going on with interruption;} \\ 2, & \text{if server is under repair.} \end{cases}$$

The state space associated with X is $\{0\} \cup \{n, 0, j\} \cup \{n, 1, i, j\} \cup \{n, 2, j\}; n = 1, 2, \ldots; i = 1, \ldots, b; j = 1, \ldots, a$.The infinitesimal generator matrix, Q associated with the queueing model is $Q = \begin{bmatrix} B_0 & B_1 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$.

where $B_0 = [-\lambda]$, $B_1 = \begin{bmatrix} \lambda & 0 \end{bmatrix}_{1\times a(2+b)}$.

B_2 is a column matrix of order $(2+b)a \times 1$.

$$B_2(i,1) = \begin{cases} \mu, & \text{for } i = ra, r = 1,2,\dots b+1; \\ 0, & \text{otherwise;} \end{cases}$$

A_2, A_1 and A_0 are square matrices of order $a(b+2)$.

$A_0 = \lambda I_{a(b+2)}$.

$$A_2(i,j) = \begin{cases} \mu, & \text{for } i = ra, j = (r-1)a+1, r = 1,2,\dots b+1; \\ 0, & \text{otherwise;} \end{cases}$$

$A_1 = \begin{bmatrix} C_0 & C_1 & 0 \\ C_2 & C_3 & C_4 \\ C_5 & 0 & C_6 \end{bmatrix}$ where

$$C_0(i,j) = \begin{cases} -\mu-\lambda-\beta, & \text{for } i = j = 1,\dots,a; \\ \mu, & \text{for } j = i+1, i = 1,\dots,a-1; \\ 0, & \text{otherwise;} \end{cases} \quad i,j = 1,\dots,a$$

$$C_1(i,j) = \begin{cases} \beta, & \text{for } i = j; i = 1,\dots,a; \\ 0, & \text{otherwise;} \end{cases} \quad j = 1,\dots,ab$$

$C_2 = e_b \otimes \gamma I_a$.

C_3 is a square matrix of order ab.

$$C_3(i,j) = \begin{cases} \omega, & \text{if } i = j; \\ \delta, & \text{for } j = i+a; i = 1,\dots,(b-1)a; \\ \mu, & \text{for } i = (r-1)a+l; j = i+1, r = 1,\dots,b, l = 1,\dots,a-1; \\ \mu, & \text{for } i = ra; j = (r-1)a+1, r = 1,\dots,b; \end{cases}$$

where $\omega = -\mu - \lambda - \beta - \delta$.

$$C_4 = \begin{bmatrix} 0 \\ \delta I_a \end{bmatrix}_{ab \times ab};$$

$$C_5 = \eta I_a,\ C_6 = (-\eta - \lambda) I_a.$$

5.2.1 Steady-state analysis

Let $\boldsymbol{\pi}$ denote the steady-state probability vector of the generator $A_0 + A_1 + A_2$. That is, $\boldsymbol{\pi}(A_0 + A_1 + A_2) = 0$; $\boldsymbol{\pi e} = 1$: The LIQBD description of the model indicates that the queueing system is stable if and only if $\boldsymbol{\pi} A_0 \boldsymbol{e} < \boldsymbol{\pi} A_2 \boldsymbol{e}$ The vector, $\boldsymbol{\pi}$, cannot be obtained explicitly in terms of the Here $\boldsymbol{\pi} A_0 e = \lambda$, $\boldsymbol{\pi} A_2 e = (\sum_{i=a}^{(b+1)a} \pi_i)\mu$

and the condition for stability is $\lambda < (\sum_{i=a}^{(b+1)a} \pi_i)\mu$,

where $$\sum_{i=a}^{(b+1)a} \pi_i = \frac{1}{a}\left(1 - \frac{1}{\left[\frac{\eta}{\beta}(1+\frac{\gamma}{\delta})^b + \frac{\eta}{\delta}\sum_{r=1}^{b-1}(1+\frac{\gamma}{\delta})^{b-r} + (1+\frac{\eta}{\delta})\right]}\right).$$

5.2.2 Stationary distribution

Let χ=($\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...$) be the steady state probability vector of the Markov chain $\{X(t), t \geq 0\}$. Each $\mathbf{x}_i = (\mathbf{x}_{i0}, \mathbf{x}_{i1}, \mathbf{x}_{i2})$, $i > 0$. $\mathbf{x}_{i0}$ is a vector with a elements, $\mathbf{x}_{i1}$ is a vector with ab elements and $\mathbf{x}_{i2}$ is a vector with a elements. We assume that $\mathbf{x}_2 = \mathbf{x}_1.R$, and $\mathbf{x}_i = \mathbf{x}_1.R^{i-1}, i \geq 2$, where R is the minimal non-negative solution to the matrix quadratic equation

$R^2A_2 + RA_1 + A_0$=0.
From χQ=0 we get
$\mathbf{x}_0B_0 + \mathbf{x}_1B_2$=0.
$\mathbf{x}_0B_1 + \mathbf{x}_1(A_1 + RA_2)$=0.
Solving the above two equations we get $\mathbf{x}_0$ and $\mathbf{x}_1$ subject to the normalizing condition $\mathbf{x}_0e + \mathbf{x}_1(I - R)^{-1}e$=1.

5.3 Performance measures

In this section we list a number of key system performance measures to bring out the qualitative aspects of the model under study. These are listed below along with their formula for computation.

5.3.1 Expected Service Rate

Let $Y = \{Y(t), t \geq 0\}$, where $Y(t) = (S(t), I_1(t), I_2(t))$ is a continuous time Markov chain representing the service process with interruption with $a(b + 2)$ transient states and one absorbing state. The state space corresponding to Y is $\{0, j\} \cup \{1, i, j\} \cup \{2, j\} \cup *$ where $*$ is the absorbing state, $i = 1, 2 \ldots, b; j = 1, 2, \ldots, a$. The infinitesimal generator associated with the service process $\overline{Q} = \begin{bmatrix} S & S^0 \\ 0 & 0 \end{bmatrix}$ where

$$S = \begin{bmatrix} C_0 + \lambda & C_1 & 0 \\ C_2 & C_3 + \lambda & C_4 \\ C_5 & 0 & C_6 + \lambda \end{bmatrix}$$

The important results obtained from the analysis of the service process are

- The time until absorption $E(S) = -\boldsymbol{\vartheta} S^{-1} e$ where $\boldsymbol{\vartheta} = (1, 0, \ldots, 0)$.
- The time spent in each of the $a(b+2)$ phases is given by $\boldsymbol{\vartheta} S^{-1}$.
- The expected service rate $\mu_s = \frac{1}{E(S)}$.

5.3.2 Expected waiting time

The waiting time of the particular customer who joined as the m^{th} customer in the queue is the time until absorption of the Markov Chain $W = \{W(t), t \geq 0\}$ where $W(t) = \big(\overline{N}(t), S(t), I_1(t), I_2(t)\big)$. $\overline{N}(t)$ is the rank of the tagged customer and all other random variables are as defined in section 5.2. The state space corresponding to W is $\{r, 0, j\} \cup \{r, 1, i, j\} \cup \{r, 2, j\} \cup \Omega$ where Ω is the absorbing state, $i = 1, 2 \ldots, b; j = 1, 2, \ldots, a; r = m, m-1, \ldots, 1$. The infinitesimal generator associated with the waiting time is $Q' = \begin{bmatrix} \overline{W} & \overline{W}^0 \\ 0 & 0 \end{bmatrix}$
The waiting time of a tagged customer follows a phase type representation $(\boldsymbol{\sigma}, \overline{W})$ where $\overline{W} = \begin{bmatrix} S & S^0\boldsymbol{\vartheta} & & & \\ & S & S^0\boldsymbol{\vartheta} & & \\ & & S & S^0\boldsymbol{\vartheta} & \\ & & & \ddots & \ddots \\ & & & & \ddots \end{bmatrix}$ and $\overline{W}^0 = \begin{bmatrix} 0 \\ \vdots \\ \\ S^0 \end{bmatrix}$
$\boldsymbol{\sigma}$ is the initial probability vector which indicates that the chain is starting from level r.

- The expected waiting time of the r^{th} customer is

$$E_w^r = -S^{-1}(I - (S^0\boldsymbol{\vartheta}S^{-1})^{(r-1)})(I - S^0\boldsymbol{\vartheta}S^{-1})^{-1}e.$$

- The expected waiting time of general customer is $E_w = \sum_{r=1}^{\infty} \mathbf{x}_r E_w^r$.

5.3.3 Expected number of interruptions during the service of a single customer

For calculating the expected number of interruptions we consider the Markov chain $Z = \{Z(t), t \geq 0\}$ where $Z(t) = \left(\widehat{N}(t), S(t), I_1(t), I_2(t)\right)$. $\widehat{N}(t)$ is the number of interruptions occurred until time t and all other random variables are as defined in section 5.2. The state space corresponding to Z is $(0,0,j) \cup (r,1,i,j) \cup (r,2,j) \cup (r,0,j) \cup \nabla$ where ∇ represents the absorbing state, $i = 1,2\ldots,b; j = 1,2,\ldots,a; r = 1,2,\ldots$. The infinitesimal generator associated with the Markov chain is $\widetilde{Q} = \begin{bmatrix} U & U_0 \\ 0 & 0 \end{bmatrix}$

where $U = \begin{bmatrix} \overline{B_0} & \overline{B_1} & & & \\ & \overline{A_1} & \overline{A_0} & & \\ & & \overline{A_1} & \overline{A_0} & \\ & & & \ddots & \ddots \\ & & & & \ddots & \ddots \\ & & & & & \end{bmatrix}$ and $U_0 = \begin{bmatrix} \overline{A_2} \\ \overline{B_2} \\ \vdots \\ \vdots \\ \overline{B_2} \end{bmatrix}$

$\overline{B_0}$ is a matrix of dimension a.

$$\overline{B_0}(i,j) = \begin{cases} -\mu - \beta, & \text{for } i = j = 1,\ldots,a; \\ \mu, & \text{for } j = i+1, i = 1,\ldots,a-1; \\ 0 & \text{otherwise}; \end{cases}.$$

$$\overline{B_1} = \begin{bmatrix} \beta I_a & 0 \end{bmatrix}_{a \times a(b+2)}, \quad \overline{A_2} = \begin{bmatrix} 0 \\ \vdots \\ \\ \mu \end{bmatrix}_{a \times 1},$$

$$\overline{B_2}(i,1) = \begin{cases} \mu, & \text{for } i = ra, i = a(b+2), r = 1, \ldots, b; \\ 0 & \text{otherwise}; \end{cases}.$$

$\overline{A_1}$ and $\overline{A_0}$ are of dimension $a(b+2)$

$$\overline{A_1}(i,j) = \begin{cases} -\mu - \gamma - \delta, & \text{for } i = j = 1, \ldots, ab \\ -\eta, & \text{for } j = i = 1 + ab, \ldots, ab + a \\ -\mu - \beta, & \text{for } j = i = ab + a + 1, \ldots, a(b+2) \\ \delta, & \text{for } j = i + 1, i = 1, \ldots, ab; j = a + 1, \ldots, a(b+1) \\ \mu, & \text{for } j = i + 1, i = (r-1)a + l, l = 1, \ldots, a - 1 \\ 0 & \text{otherwise}; \end{cases}$$

$$\overline{A_0}(i,j) = \begin{cases} \beta, & \text{for } i = a + ab + j, j = 1, \ldots, a; \\ 0 & \text{otherwise}. \end{cases}$$

$Z(t)$ follows a phase type distribution with representation$(\boldsymbol{\vartheta}, U)$ where $\boldsymbol{\vartheta}$ is the initial probability vector.

Let y_j be the probability that there are exactly j interruptions during the service of acustomer. Then

$$y_j = \begin{cases} \boldsymbol{\vartheta}(\overline{B}_0^{-1}\overline{A}_2), & \text{for } j = 0; \\ \boldsymbol{\vartheta}(\overline{B}_0^{-1}\overline{B}_1)(\overline{A}_1^{-1}\overline{A}_0)^{j-1}(\overline{A}_1^{-1}\overline{B}_2) & \text{otherwise}. \end{cases}$$

5.3.4 Other important performance measures

- Probability that the system is idle $P_{Idle} = \mathbf{x}_0$.

- Probability that there are i customers in the system $P_I = \mathbf{x}_i \boldsymbol{e}$

- Expected number of customers in the system $E(C) = \sum_{i=1}^{\infty} i\mathbf{x}_i \boldsymbol{e}$.

- Probability for self correction of interruption before interruption clock realization $P_{selfcorrection} = \sum_{n=0}^{b-1} \frac{\gamma\delta^n}{(\gamma+\delta)^{n+1}}$.

- Probability for service completion without any interruption $P(s) = \left[\frac{\mu^a}{(\mu+\beta)^a}\right]$.

- Probability that the system is working with interruption $P_{Int} = \sum_{i=1}^{\infty} \mathbf{x}_{i1}\boldsymbol{e}$

- Effective self correction rate $E_{selfcorr} = \eta \sum_{i=1}^{\infty} \mathbf{x}_{i1}\boldsymbol{e}$

- Probability that the server is under repair $P_{rep} = \sum_{i=1}^{\infty} \mathbf{x}_{i2}\boldsymbol{e}$

- Effective interruption rate,$E_{Int} = \beta \sum_{i=1}^{\infty} \mathbf{x}_{i0}\boldsymbol{e}$

- Effective repair rate,$E_{rep} = \eta \sum_{i=1}^{\infty} \mathbf{x}_{i1}\boldsymbol{e}$

5.4 Numerical illustrations

The results obtained in the previous sections are numerically illustrated in this section. Assume arbitrary values for parameters $\mu = 5$, $\gamma = .2$, $\delta = 3$, $\eta = 5$ and $\beta = 3$.

Effect of λ on various performance measures

Table 5.1: **Effect of λ on various performance measures**

λ	$E(C)$	P_{Idle}	$E_{selfcorr}$
.5	0.2275	0.8056	0.0150
1	0.4761	0.6540	0.0277
1.5	0.7441	0.5388	0.0380
2	1.0280	0.4519	0.0460
2.5	1.3243	0.3860	0.0523
3	1.6299	0.3353	0.0572
3.5	1.9424	0.2956	0.0611
4	2.2600	0.2639	0.0643

From Table 5.1 as the value of λ increases expected number of customers in the system $E(C)$ and expected self correction rate begins to increase but probability for idleness decreases which are on expected lines.

Effect of μ on various performance measures

Taking $\lambda = 2$ and all other values as above we get the following values for different performance measures on the variation in μ As service rate μ

Table 5.2: **Effect of μ on various performance measures**

μ	$E(S)$	$E(C)$	P_{Idle}	$P(s)$
3	.5670	1.2243	0.4125	0.1250
4	0.4935	1.1237	0.4310	0.1866
5	0.4341	1.0280	0.4519	0.2441
6	0.3861	0.9394	0.4743	0.2963
7	0.3467	0.8588	0.4974	0.3430
8	0.3141	0.7862	0.5206	0.3847

increases expected service time $E(S)$ and expected number of customers in the system decreases $E(C)$, but probability for idleness and probability for service completion without any interruption increases(refer Table 5.2).

Effect of η on various performance measures

Assuming $\mu = 4$ we get the following values for different performance measures corresponding to the variation in η

As the repair rate increases expected service time $E(S)$ and expected

Table 5.3: **Effect of η on various performance measures**

η	$E(S)$	$E(C)$	P_{Idle}	E_{Int}
1	0.5397	1.5271	0.3986	0.2839
2	0.5108	1.2491	0.4180	0.5943
3	0.5012	1.1754	0.4251	0.9054
4	0.4964	1.1423	0.4287	1.2166
5	0.4935	1.1237	0.4310	1.5278
6	0.4916	1.1118	0.4325	1.8391

number of customers in the system $E(C)$ decrease, but probability for idleness and effective interruption rate increase which are on expected lines (Table 5.3).

Effect of δ on various performance measures

From Table 5.4, as the realization rate of interruption clock increases expected service time $E(S)$, expected number of customers in the system $E(C)$ and effective interruption rate E_{Int} increase, but probability for

Table 5.4: **Effect of δ on various performance measures**

δ	$E(s)$	$E(C)$	P_{Idle}	$E_{selfcorr}$	E_{Int}
1	0.5000	1.1187	0.4265	0.1331	0.6033
2	0.5125	1.2216	0.4134	0.1291	0.6088
3	0.5328	1.3637	0.3955	0.1237	0.6162
4	0.5577	1.5246	0.3761	0.1178	0.6242
5	0.5850	1.6946	0.3569	0.1119	0.6322

idleness and effective self correction rate $E_{selfcorr}$ decrease which are on expected lines. When interruption clock realization rate increases rate of repair increases which causes the increase in effective service time of a customer. This leads to the increase in number of customers in the system. As a result the probability for idleness decreases.

5.5 Model description (Model II)

We consider a single server queueing model in which customers arrive according to a Poisson process with rate λ. Service time is Erlang distributed with shape and scale parameters μ and a respectively. During service there is a chance for interruption. There are n environmental factors causing interruption to service. These n factors are numbered 1 to n depending on the ascending order of severity of interruption caused by these factors. Interruption to service occurs according to Poisson process with parameter β. Interruption due to i^{th} environmental factor occurs with probability p_i. If the interruption is due to first $m(m < n)$ factors it is ignored and service is continued. But the service is at a lower rate $\mu_i, i = 1, 2, \ldots, m$. This interruption clock is started simultaneously with the occurrence of interruption. It is exponentially distributed with

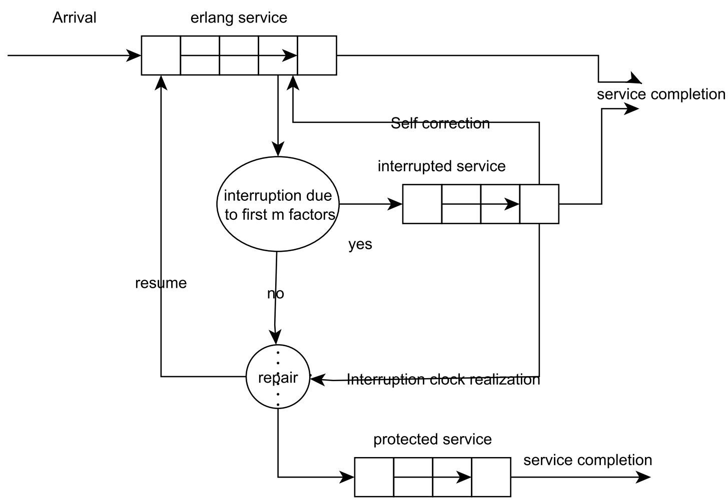

Figure 5.2: Model description

parameter $\delta_i, i = 1, 2 = \ldots, m$. During interrupted service period there is a possibility for self correction of interruption. This self correction is exponentially distributed with parameter $\gamma_i, i = 1, 2, \ldots, m$. If self correction occurs the service rate changes from μ_i to μ. On realization of the interruption clock the server goes for repair. The repair time is exponentially distributed with parameter $\eta_i, i = 1, 2, \ldots, m$. After repair the interrupted service is resumed. If the service of a customer is completed with interruption the next customer in the head of the queue enter for service with the server in interruption.

If the interruption is due to the remaining $n - m$ factors the server directly goes for repair. Taking into account the severity of interruption caused by these $n-m$ factors, protection for remaining service is provided at the epoch of resumption of service after repair.

5.6 Mathematical description

The queueing model described above can be mathematically formulated as a Markov chain. Let $\mathbf{X} = \{X(t), t \geq 0\} = \{(N(t), S(t), I_1(t), I_2(t)), t \geq 0\}$ where $N(t)$ is the number of customers in the system, $S(t)$ is the status of the server, $I_1(t)$ is the environmental factor causing interruption and $I_2(t)$ is the phase of service:

$$S(t) = \begin{cases} 0, & \text{when a service facing so far no interruption ;} \\ 1, & \text{if interrupted service going on;} \\ 2, & \text{if server under repair;} \\ 3, & \text{if protected service is going on.} \end{cases}$$

The state space of the process is
$\{(r, 0, i) \cup (r, 1, j, i) \cup (r, 2, k, i) \cup (r, 3, i); r = 1, \ldots, \infty; i = 1, \ldots, a; j = 1, \ldots, m; k = 1, \ldots, n; k = 1, \ldots, n\} \cup \nabla$ where ∇ represents the absorbing state. Absorbing state means the customer moving out from the system after service completion. The infinitesimal generator matrix of the process is given by

$$Q = \begin{bmatrix} B_0 & B_1 & & & & \\ B_2 & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix} \text{ where } B_0 = [-\lambda], \;\; B_1 = \begin{bmatrix} \lambda & 0 \end{bmatrix}$$

B_2 is a column matrix of order $(2 + m + n)a \times 1$ and

$$B_2(i, 1) = \begin{cases} \mu, & \text{for } i = a, \&i = (2 + m + n)a; \\ \mu_j, & \text{for } i = a(r + 1); r = 1, 2, \ldots, m; \\ 0, & \text{otherwise.} \end{cases}$$

$$A_0 = \lambda I_{(2+m+n)a} \quad A_1 = \begin{bmatrix} C_0 & C_1 & C_2 & 0 \\ C_3 & C_4 & C_5 & 0 \\ C_6 & 0 & C_7 & C_8 \\ 0 & 0 & 0 & C_9 \end{bmatrix}_{(2+m+n)a\times(2+m+n)a}$$

C_0 is a matrix of order a,

$$C_0(i,j) = \begin{cases} -\lambda-\beta-\mu, & \text{for } i=j; \\ \mu, & \text{for } j=i+1; i=1,\dots,a-1 \\ 0, & \text{otherwise.} \end{cases}$$

$C_1 = (\beta p' \otimes I_a)_{a\times am}$, $C_2 = \begin{bmatrix} 0 & \beta p'' \otimes I_a \end{bmatrix}_{a\times am}$ where $p = (p', p'')$ with $p' = (p_1, \dots, p_m)$ and $p'' = (p_{m+1}, \dots, p_n)$.

Let $\boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \\ \\ \gamma_m \end{bmatrix}$ then $C_3 = \boldsymbol{\gamma} \otimes I_a$ and C_4 is a matrix of order ma,

$$C_4(i,j) = \begin{cases} \theta_r, & \text{for } i=j; i=(r-1)a+l; r=1,\dots,m; l=1,\dots,a; \\ \mu_r, & \text{for } i+1=j, i=(r-1)a+l; r=1,\dots,m; l=1,\dots,a-1; \\ 0, & \text{otherwise.} \end{cases}$$

where $\theta_r = -\lambda-\mu_r-\gamma_r-\delta_r$.

C_5 is a matrix of order $ma \times na$,

$$C_5(i,j) = \begin{cases} \delta_t, & \text{for } i=j; i=(t-1)a+k, t=1,\dots,m; k=1,\dots,a \\ 0, & \text{otherwise;} \end{cases}$$

Let $\boldsymbol{\eta} = \begin{bmatrix} \boldsymbol{\eta'} \\ \boldsymbol{\eta''} \end{bmatrix}$ with $\boldsymbol{\eta'} = (\eta_1, \eta_2, \dots, \eta_m)^T$ and $\boldsymbol{\eta''} = (\eta_{m+1}, \dots, \eta_n)^T$

then $C_6 = \begin{bmatrix} \boldsymbol{\eta'} \otimes I_a \\ 0 \end{bmatrix}_{(na\times a)}$.

C_7 is a matrix of order na,

$$C_7 = \begin{cases} -\lambda - \eta_r, & \text{for } i = j; i = (r-1)a + k, r = 1, \ldots, n; k = 1, \ldots, a \\ 0, & \text{otherwise.} \end{cases}$$

$$C_8 = \begin{bmatrix} 0 \\ \boldsymbol{\eta}'' \otimes I_a \end{bmatrix}_{(na \times a)}. \ C_9 \text{ is a matrix of order } a,$$

$$C_9 = \begin{cases} -\lambda - \mu, & \text{for } i = j; i = (r-1)a + k, k = 1, \ldots, a; \\ \mu, & \text{for } j = i + 1; i = (r-1)a + k, k = 1, \ldots, a - 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$A_2(i, j) = \begin{cases} \mu, & \text{for } i = a; j = 1; \text{and } i = (m+n+2)aj = 1, ; \\ \mu_r, & \text{for } i = (r+1)a; j = ra + 1, r = 1, \ldots, m; \\ 0, & \text{otherwise.} \end{cases}$$

5.7 Analysis of service process

The service time follows PH distribution with representation $(\boldsymbol{\alpha}, S)$ where $\boldsymbol{\alpha} = (1, 0, \ldots, 0)_{1 \times (m+n+2)a}$ and $S = \begin{bmatrix} C_0' & C_1 & C_2 & 0 \\ C_3 & C_4' & C_5 & 0 \\ C_6 & 0 & C_7' & C_8 \\ 0 & 0 & 0 & C_9' \end{bmatrix}$.
$C_0' = C_0 + \lambda I$, $C_4' = C_4 + \lambda I$, $C_7' = C_7 + \lambda I$ and $C_9' = C_9 + \lambda I$. The absorbing state is represented by $S^0 = B_2$ which is a column matrix.

- The response time of the service process, $E(S) = -\boldsymbol{\alpha} S^{-1} e$.

- Hence the expected service rate $\mu_s = \frac{1}{E(S)}$.

- **Theorem**: *The queueing system is stable when* $\lambda < \mu_s$.

5.7.1 Stationary distribution

The stationary distribution, under the condition of stability, $\lambda < \mu_s$ of the model, has Matrix Geometric solution. Let χ=($\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...$) be the steady state probability vector of the Markov chain $\{Z(t), t \geq 0\}$. Each $\mathbf{x}_i, i > 0$ are vectors with $(2 + m + n)a$ elements. We assume that $\mathbf{x}_2 = \mathbf{x}_1.R$, and $\mathbf{x}_i = \mathbf{x}_1.R^{i-1}, i \geq 2$, where R is the minimal non- negative solution to the matrix quadratic equation
$R^2A_2 + RA_1 + A_0$=0.
From χQ=0 we get

$$\mathbf{x}_0B_0 + \mathbf{x}_1B_2 = 0.$$

$$\mathbf{x}_0B_1 + \mathbf{x}_1(A_1 + RA_2) = 0.$$

Solving the above two equations we get $\mathbf{x}_0$ and $\mathbf{x}_1$ subject to the normalizing condition $\mathbf{x}_0e + \mathbf{x}_1(I - R)^{-1}e$=1.

Expected number of interruptions during the service of any customer

Let $N'(t)$ be the number of interruptions due to first m environmental factors during the service of a particular customer at time t. $S(t)$ be the status of the server at time t.

$$S(t) = \begin{cases} 0, & \text{when service is going on ;} \\ 1, & \text{if interrupted service going on;} \\ 2, & \text{if server under repair} \end{cases}$$

$I_1(t)$ is the environmental factor causing interruption and $I_2(t)$ is the phase of service. Then $\{(N'(t), S(t), I_1(t), I_2(t)), t \geq 0\}$ is a Markov chain with state space $\{(r, 0, i) \cup (r, 1, j, i) \cup (r, 2, j, i); r = 1, 2, \ldots; i = 1, \ldots, a; j = 1,$

$\ldots, m\} \cup *$ where $*$ represents the absorbing state. The infinitesimal generator matrix of the process is given by $\widehat{Q} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & & & \\ U_2' & U_1' & U_0' & & & \\ U_2 & \mathbf{0} & U_1 & U_0 & & \\ U_2 & \mathbf{0} & \mathbf{0} & U_1 & U_0 & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$

where U_2' is a column matrix of order $a \times 1$. $U_2' = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \mu \end{bmatrix}$,

$$U_1' = \begin{cases} -\beta \sum_{r=1}^{m} p_r - \mu, & \text{for } i = j; i, j = 1, \ldots, a \\ \mu, & \text{for } j = i+1; i = 1, \ldots, a-1 \\ 0, & \text{otherwise.} \end{cases}$$

$U_0' = (\beta p' \otimes I_a)_{a \times am}$.

U_2 is a column matrix of order $(1+2m)a \times 1$ and

$$U_2(i,1) = \begin{cases} \mu_j, & \text{for } i = ar; r = 1, 2, \ldots, m; \\ \mu, & \text{for } i = a(2m+1); \\ 0, & \text{otherwise.} \end{cases}$$

$$U_1 = \begin{bmatrix} D_0 & D_1 & D_2 \\ 0 & D_3 & D_4 \\ 0 & 0 & D_5 \end{bmatrix}_{(1+2m)a \times (1+2m)a}$$

$$D_0(i,j) = \begin{cases} \vartheta_r, & \text{for } i = j; i = ra + l; r = 0, \ldots, m-1; l = 1, \ldots, a; \\ \mu_r, & \text{for } i+1 = j, i = ra + l; r = 0, \ldots, m-1; \\ & l = 1, \ldots, a-1; \\ 0, & \text{otherwise.} \end{cases}$$ where

$\vartheta_r = -\mu_r - \gamma_r - \delta_r$.

$$D_1(i,j) = \begin{cases} \gamma_t, & \text{for } i=j; i=(t-1)a+k, t=1,\ldots,m; k=1,\ldots,a \\ 0, & \text{otherwise.} \end{cases}$$

$D_2 = \delta \otimes I_a$ and

$$D_3(i,j) = \begin{cases} -\eta_r, & \text{for } i=j; i=(r-1)a+k, r=1,\ldots,m; k=1,\ldots,a \\ 0, & \text{otherwise.} \end{cases}$$

$$D_4(i,j) = \begin{cases} \eta_r, & \text{for } i=a+ma+(r-1)a+j; r=1,\ldots,m; \&j=1,\ldots,a; \\ 0, & \text{otherwise.} \end{cases}$$

$$D_5(i,j) = \begin{cases} -\beta-\mu, & \text{for } i=j; \\ \mu, & \text{for } j=i+1; \\ 0, & \text{otherwise.} \end{cases}$$

$$U_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D_6 & 0 & 0 \end{bmatrix}_{(1+2m)a\times(1+2m)a} . \; D_6 = (\beta p' \otimes I_a)_{a\times am}$$

Let Z_k be the probability that there are exactly k interruptions during the service of a customer due to first m environmental factors.

Then $Z_k = \begin{cases} \boldsymbol{\alpha}(-U_1')^{-1}U_2', & \text{for } k=0 \\ \boldsymbol{\alpha}[(-U_1')^{-1}U_0'](-U_1)^{-1}U_0]^{k-1}(-U_1)^{-1}U_2, & \text{for } k=1,2,3,\ldots. \end{cases}$

So the expected number of interruptions due to first m environmental factors during single service $E(I) = \sum_{k=0}^{\infty} kZ_k$.

5.8 Performance measures

After finding the steady state probability vector we find the important performance measures of the system. The important measures are as follows.

5.8.1 Expected waiting time

We consider the customer who joined as the m^{th} in the queue. During the time of arrival of $,^{th}$ customer one customer in the system may be in service or the server may be in repair and other customers are waiting in the queue. So the waiting time of the tagged customer is the time until absorption of the Markov chain $W = \{(M(t), S(t), I_1(t), I_2(t)), t \geq 0\}$ where $M(t)$ is the rank of the tagged customer, $S(t), I_1(t)$ and $I_2(t)$ are as defined in earlier sections. The waiting time of the tagged customer follows phase type distribution with representation $(\boldsymbol{\omega}, T)$ where

$$T = \begin{bmatrix} S & S^0\boldsymbol{\alpha} & & & & \\ & S & S^0\boldsymbol{\alpha} & & & \\ & & S & S^0\boldsymbol{\alpha} & & \\ & & & S & S^0\boldsymbol{\alpha} & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$

$\boldsymbol{\omega}$ is the initial probability vector.

Depending on the state of the server at the time of joining, the expected waiting time of the tagged customer, $E_W^m = -S^{-1}(I - (S^0\boldsymbol{\alpha}S^{-1})^{(r-1)})(I - S^0\boldsymbol{\alpha}S^{-1})^{-1}e$.

The expected waiting time of any customer who waits in the queue,

$$E(W) = \sum_{m=1}^{\infty} \mathbf{x}_m E_W^m$$

5.8.2 Other important performance measures

- Probability that the system is idle, $P(I) = \mathbf{x}_0$.

- Probability that the system is working without interruption,

$$P(WI) = \sum_{i=1}^{\infty} (\mathbf{x}_{i0}\boldsymbol{e} + \mathbf{x}_{i3}\boldsymbol{e})$$

.

- Probability that the system is under repair $P(R) = \sum_{i=1}^{\infty} \mathbf{x}_{i2}\boldsymbol{e}$.

- Probability that the system is under protection $P(p) = \sum_{i=1}^{\infty} \mathbf{x}_{i3}\boldsymbol{e}$.

- Expected number of customers in the system, $E(C) = \sum_{i=1}^{\infty} i\mathbf{x}_i\boldsymbol{e}$.

- Effective interruption rate,$E_{int} = \beta \sum_{i=1}^{\infty} \mathbf{x}_{i1}\boldsymbol{e}$.

- Effective rate of self correction,$E_{selfcorr} = \sum_{i=1}^{\infty} \sum_{j=1}^{m} \delta_j \mathbf{x}_{i1j}\boldsymbol{e}$.

- Effective rate of protection,$E_{protection} = \sum_{i=1}^{\infty} \sum_{j=m+1}^{n} \eta_j \mathbf{x}_{i2j}\boldsymbol{e}$.

5.9 Numerical Illustrations

In this section assuming arbitrary values for the parameters, subject to stability, we obtained the numerical values for important performance measures. Let $n = 4, m = 2, \mu = 7, \mu_1 = 5, \mu_2 = 4; \eta_1 = 4, \eta_2 = 3, \eta_3 = 2, \eta_4 = 1, \beta = .5; \gamma_1 = 1, \gamma_2 = 2; \delta_1 = 1, \delta_2 = .5; p_1 = p_2 = p_3 = p_4 = 0.25$.
The conclusion drawn are purely based on the values of input parameters.

Effect of λ on various performance measures

Table 5.5: **Effect of λ on various performance measures**

λ	$E(C)$	$P(I)$
1	0.9230	0.5012
1.5	1.9157	0.3134
2	3.5376	0.1910
2.5	6.1867	0.1166
3	10.5083	0.0716
3.5	17.1259	0.0443
4	24.7322	0.0276

As the arrival rate λ increases the expected number of customers in the system $E(C)$ increase, but probability for idleness of the server $P(I)$ decrease which are on expected lines (refer Table 5.5).

Effect of μ on various performance measures

Assuming $\lambda = 2$ and varying μ we get the following values for different performance measures.

As the initial service rate μ increases the expected service time $E(s)$, the expected number of customers in the system $E(C)$, probability for repair $P(R)$, expected rate of interruption E_{int} and expected rate of self correction $E_{selfcorr}$ decrease but probability for idleness of the server $P(I)$ increase which are on expected lines. μ increases means number of service completion in unit time increases. So rate of self correction, rate of interruption and probability for repair in unit time reduces (see Table 5.6).

Table 5.6: **Effect of μ on various performance measures**

μ	$E(S)$	$E(C)$	$P(I)$	$P(R)$	E_{int}	$E_{selfcorr}$
3	2.2897	29.2040	0.0175	0.1422	0.3322	0.0983
4	1.7508	15.5352	0.0471	0.1437	0.3364	0.0978
5	1.4180	8.0176	0.0891	0.1398	0.3281	0.0938
6	1.1917	5.0105	0.1384	0.1338	0.3147	0.0885
7	1.0279	3.5376	0.1910	0.1267	0.2986	0.0827
8	0.9037	2.6963	0.2437	0.1192	0.2814	0.0769
9	0.8063	2.16207	0.2946	0.1117	0.2641	0.0713

Effect of β on various performance measures

Table 5.7: **Effect of β on various performance measures**

β	$E(S)$	$E(C)$	$P(I)$	$P(R)$	E_{int}	$E_{selfcorr}$
.5	1.4180	8.0176	0.0891	0.1398	0.3281	0.0938
1	1.5863	11.0407	0.0688	0.2230	0.5230	0.1502
2	1.8247	16.4010	0.0476	0.31577	0.7394	0.2140
3	1.9818	20.3997	0.0373	0.3648	0.8539	0.2484
4	2.0906	23.1333	0.0314	0.3946	0.9231	0.2695
5	2.1691	24.9636	0.0277	0.4141	0.9685	0.2834

From Table 5.7 we note that as the interruption rate β increases effective service time $E(S)$, the expected number of customers in the system $E(S)$, probability for repair $P(R)$, expected rate of interruption E_{int} and expected rate of self correction $E_{selfcorr}$ increases but probability for idleness of the server $P(I)$ decrease which are on expected lines.

Effect of γ on various performance measures

Assuming $\gamma_1 = \gamma_2 = \gamma$ and varying over its value we get the following table for different performance measures. As the interruption clock realization rate γ increases effective service time $E(S)$, the expected number of customers in the system $E(C)$, probability for repair $P(R)$, expected rate of interruption E_{int} increase but probability for idleness of the server $P(I)$ and expected rate of self correction $E_{selfcorr}$ decrease which are on expected lines. Rate of protection decrease with increase in γ. As the realization rate of interruption clock increase the server immediately goes for repair reducing the chance for self correction (see Table 5.8).

Table 5.8: **Effect of γ on various performance measures**

γ	$E(S)$	$E(C)$	$P(I)$	$P(R)$	E_{int}	$E_{selfcorr}$	$E_{protection}$
0.5	1.4069	7.7491	0.0915	0.1303	0.3254	0.1184	0.1627
1	1.4136	7.9371	0.0898	0.1367	0.3283	0.1030	0.1642
1.5	1.4192	8.0705	0.0887	0.1413	0.3302	0.0919	0.165
2	1.4240	8.1718	0.0878	0.1448	0.3315	0.0832	0.1657
2.5	1.4282	8.2523	0.0871	0.1475	0.3324	0.0763	0.1662
3	1.4318	8.3184	0.0865	0.1498	0.3331	0.0705	0.1665
3.5	1.4349	8.3739	0.0860	0.1517	0.33367	0.0656	0.1668

Chapter 6

An $M/M/1$ queue with multiple vacation, vacation interruption and vacation controlled by random environment

introduction

In the previous chapters we discussed queues with environment dependent interruption. In this chapter we consider a queueing model with environment depended vacation and interruption of vacation. In vacation queueing system the server is unavailable for a random duration of time at a stretch. Vacation is the absence of server from the service center. There are different types of vacations like vacation taken at the end of a busy period, vacation taken when the number of customers in the system is less than a predetermined number, working vacation, vacation taken after a long duration of service even if customers are present in the queue, etc. The vacation of server can be considered as interruption to service if the vacation is taken while customers are present in the queue or when an arriving customer finds the absence of server.

In contrast in working vacation server stays at service station and also provides service, may be at a reduced pace, if customers are available. The vacation queueing system is very useful as we can utilize the idle time for maintenance of server without affecting the customers. During working vacation risk of loosing customers and the dissatisfaction of customers are less, the utilization of server for other works is not possible. The vacation queueing models have wide range of application in the field of communication networks and computer systems.

The idea of server vacation was first introduced by Levy and Yechiali [30]. The main survey papers on vacation model are Doshi [6] and Zhang [51]. The books of Takagi [46] and Tian [47] provide great insight into the idea of vacation. In multiple vacation queueing system, on returning from a vacation , if the server finds no customer in the system, it goes for another vacation. The idea of multiple vacation provides more flexibility for optimal utilization of available free time.

The concept of vacation interruption was introduced and developed by Li and Tian [33] [36]. Zhang and Hou [51] studied an $M/G/1$ queue with multiple working vacation and vacation interruption. Ibe and Isijola [14] discuss two types of vacations: Type I vacation is taken after a non zero busy period of serving at least one customer and type II vacation is taken after a zero busy period, on completing type I vacation. The distribution of both vacations are different. The authors extend the idea in paper [15] by introducing two new concepts, partial vacation interruption and total vacation interruption. In partial vacation interruption only type II vacation is interrupted. The interruption occurs when the number of customers in the system reaches a threshold value K. In total vacation interruption, type I and type II vacations are interrupted when the number of customers in the system reaches the threshold values K_1 and K_2 respectively, where $K_1 \geq K_2$.

In this chapter we consider a queueing system with two types of vacations. The server goes for a vacation of type I after a non-zero busy period. There are n kinds of type I vacation depending on the environment. These type I vacations are numbered 1 to n based on the descending order of duration of vacation in the distributed stochastic sense. After a non-zero busy period the server goes for a vacation of i^{th} kind with probability $p_i, 1 \leq i \leq n$. The server goes for type II vacation after a zero busy period, provided there is no customer in the system. The type II vacation is numbered as the $(n+1)^{th}$ kind of vacation. All the vacations can be interrupted depending on the threshold value $K_i, 1 \leq i \leq n+1$ as described: If the number of customers in the system reaches $K_i, 1 \leq i \leq n$, while the server is in the i^{th} kind of vacation, that vacation is interrupted. When the number of customers in the system reaches K_{n+1}, type two vacation gets interrupted. The main assumption is that $K_{n+1} < K_n < ... < K_1$.

As a real life example we consider the case of a physician attending patients. After attending all patients in the queue the he goes for a vacation. During vacation he can perform different jobs. He can visit the inpatients, go to take food, go for reading etc. Depending on the importance of the jobs piling up during vacation the queue length for interrupting a vacation is fixed. When the queue length reaches the respective thresholds that kind of vacation is interrupted.

6.1 Model Description

We consider a single server queueing system in which arrival occurs according to a Poisson process with parameter λ. The service time is exponentially distributed with parameter μ. There are two type of vacations in this model. The type I vacation is taken at the end of a non-zero busy period. There are n different kinds of type I vacation based on n environmental factors. These n kinds of vacation are numbered 1 to n based on the descending order of duration of vacation. On returning from type I vacation, if the server finds the queue empty, it goes for type II vacation. The type II vacation is numbered as the $(n+1)^{th}$ kind of vacation. The i^{th} vacation duration is exponentially distributed with parameter γ_i, $1 \leq i \leq n+1$. At the end of a non-zero busy period, depending on the environment, the server opts for a vacation of i^{th} kind with probability $p_i, 1 \leq i \leq n$. When the number of customers in the system exceeds K_{n+1} during the $n+1^{th}$ kind of vacation, the server returns from vacation and starts serving customers. The i^{th} type of vacation is interrupted when the number of customers in the queue reaches K_i, where $K_1 > K_2 > \ldots > K_{n+1}$. The pictorial representation of the model is shown in Figure 6.1, 6.1, 6.1, 6.1.

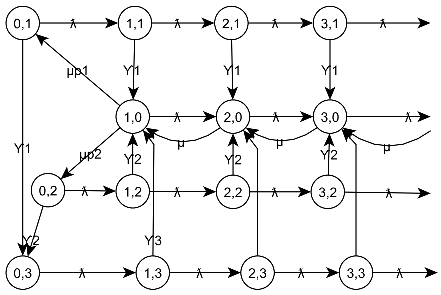

Figure 6.1: Model description

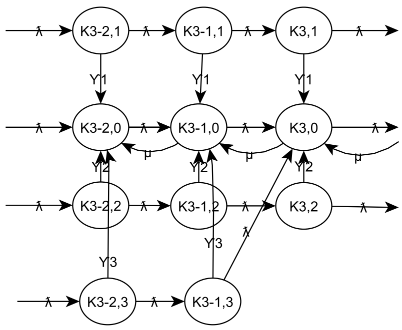

Figure 6.2: Model description

6.2 Analysis of the model

We define the state of the system as (n, k), when there are n customers in the system and k is the status of the server:

$$k = \begin{cases} 0 & \text{when service is going on ;} \\ i & \text{If the server is on the } i^{th} \text{ kind of type I vacation;} \\ & \text{i=1,2,...,n;} \\ n+1 & \text{If the server is on the } n+1^{th} \text{ kind of vacation(type II);} \end{cases}$$

Let $P_{i,k}$ denote the steady state probability that the system contains i

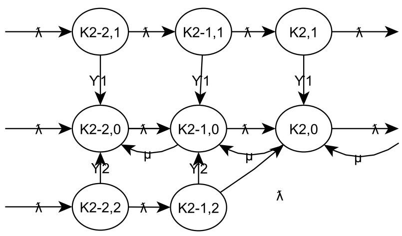

Figure 6.3: Model description

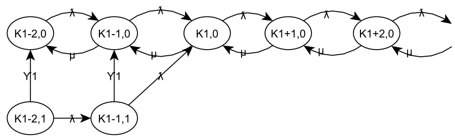

Figure 6.4: Model description

customers when the server is in the k^{th} state. First consider the case of $n = 2$.

From the global balance in the above figures we have the following equations,

$$(\lambda + \gamma_1)P_{0,1} = \mu p_1 P_{1,0}, \tag{6.1}$$

$$(\lambda + \gamma_2)P_{0,2} = \mu p_2 P_{1,0}, \tag{6.2}$$

$$\lambda P_{03} = \gamma_1 P_{0,1} + \gamma_2 P_{0,2}, \tag{6.3}$$

$$(\lambda + \gamma_3)P_{k,3} = \lambda P_{k-1,3},\ 1 \leq k < K_3, \tag{6.4}$$

$$(\lambda + \gamma_2)P_{k,2} = \lambda P_{k-1,2},\ 1 \leq k < K_2, \tag{6.5}$$

$$(\lambda + \gamma_1)P_{k,1} = \lambda P_{k-1,1},\ 1 \leq k < K_1. \tag{6.6}$$

From the local balance we have the following equations,

$$\lambda(P_{k,0} + P_{k,1} + P_{k,2} + P_{k,3}) = \mu P_{k+1,0},\ 1 \le k < K_3, \tag{6.7}$$

$$\lambda(P_{k,0} + P_{k,1} + P_{k,2}) = \mu P_{k+1,0},\ K_3 \le k < K_2, \tag{6.8}$$

$$\lambda(P_{k,0} + P_{k,1}) = \mu P_{k+1,0},\ K_2 \le k < K_1, \tag{6.9}$$

$$\lambda(P_{k,0}) = \mu P_{k+1,0},\ k \ge K_1. \tag{6.10}$$

From (6.1) $\Rightarrow P_{0,1} = \beta_1 P_{10}$.
From (6.2) $\Rightarrow P_{0,2} = \beta_2 P_{10}$.
From (6.3) $\Rightarrow P_{0,3} = \beta_3 P_{10}$.
In the above $\beta_1 = \frac{\mu p_1}{\lambda+\gamma_1}$, $\beta_2 = \frac{\mu p_2}{\lambda+\gamma_2}$ and $\beta_3 = (\frac{\gamma_1}{\lambda}\frac{\mu p_1}{\lambda+\gamma_1} + \frac{\gamma_2}{\lambda}\frac{\mu p_2}{\lambda+\gamma_2})$.
(6.4) $\Rightarrow P_{k,3} = \alpha_3^k \beta_3 P_{10}$, for $1 \le k \le K_3 - 1$.
(6.5) $\Rightarrow P_{k,2} = \alpha_2^k \beta_2 P_{10}$, for $1 \le k \le K_2 - 1$.
(6.6) $\Rightarrow P_{k,1} = \alpha_1^k \beta_1 P_{10}$, for $1 \le k \le K_1 - 1$.
(6.7) $\Rightarrow P_{k,0} = \left[\rho^{k-1} + \sum_{i=1}^{3} \rho\alpha_i\beta_i \left[\frac{\rho^{k-1} - \alpha_i^{k-1}}{\rho - \alpha_i}\right]\right] P_{10}$, for $2 \le k \le K_3$.
From(6.8)

$$P_{k,0} = \left[\rho^{k-1} + \sum_{i=1}^{2} \rho\alpha_i\beta_i \left[\frac{\rho^{k-1} - \alpha_i^{k-1}}{\rho - \alpha_i}\right] + \rho^{k-K_3+1}\alpha_3\beta_3 \left[\frac{\rho^{K_3-1} - \alpha_3^{K_3-1}}{\rho - \alpha_3}\right]\right] P_{10},$$

for $K_3 + 1 \le k \le K_2$.
From (6.9)

$$P_{k,0} = \left[\rho^{k-1} + \rho\alpha_1\beta_1 \left[\tfrac{\rho^{k-1}-\alpha_1^{k-1}}{\rho-\alpha_1}\right] + \sum_{i=2}^{3} \rho^{k-K_i+1}\alpha_i\beta_i \frac{\rho^{K_i-1} - \alpha_i^{K_i-1}}{\rho - \alpha_i}\right] P_{10},$$

for $K_2 + 1 \le k \le K_1$.
From(6.10)

$$P_{k,0} = [\rho^{k-1} + \sum_{i=1}^{3} \rho^{k-K_i}\alpha_i\beta_i \frac{\rho^{K_i-1} - \alpha_i^{K_i-1}}{\rho - \alpha_i}] P_{10}, \text{ for } k \ge K_1 + 1.$$

Since the total probability equals one,

$$\sum_{k=1}^{\infty} P_{k,0} + \sum_{k=0}^{K_1-1} P_{k,1} + \sum_{k=0}^{K_2-1} P_{k,2} + \sum_{k=0}^{K_3-1} P_{k,3} = 1$$
$\Rightarrow [A_1 + A_2 + A_3 + A_4]P_{10} = 1,$
$P_{10} = \frac{1}{[A_1+A_2+A_3+A_4]}$. where $A_1 = \frac{1}{1-\rho}$, $A_2 = \sum_{i=1}^{3}\left[\frac{\rho^2\alpha_i\beta_i}{1-\rho}\right]\left[\frac{\rho^{K_i-1}-\alpha_i^{K_i-1}}{\rho-\alpha_i}\right]$,
$A_3 = \sum_{i=1}^{3}\left[\frac{\alpha_i\beta_i\rho}{\rho-\alpha_i}\right]\left[\frac{\alpha_i-\alpha_i^{K_i}}{1-\alpha_i} - \frac{\rho-\rho^{K_i}}{1-\rho}\right]$, $A_4 = \sum_{i=1}^{3}\sum_{k=1}^{K_i}\alpha_i^{k-1}\beta_i$.
Also the expected number of customers in the system,
$$E(N) = \sum_{k=1}^{\infty} kP_{k,0} + \sum_{k=0}^{K_1-1} kP_{k,1} + \sum_{k=0}^{K_2-1} kP_{k,2} + \sum_{k=0}^{K_3-1} kP_{k,0} =$$
$[B_1 + B_2 + B_3 + B_4]P_{10}$ where $B_1 = \frac{1}{(1-\rho)^2}$,
$$B_2 = \sum_{i=1}^{3}\frac{\alpha_i\beta_i\rho}{(\alpha_i-\rho)}\left[\frac{K_i\alpha_i^{K_i+1} - (K_i+1)\alpha_i^{K_i} - \alpha_i^2 + 2\alpha_i}{(1-\alpha_i)^2}\right] -$$
$$\sum_{i=1}^{3}\frac{\alpha_i\beta_i\rho}{(\alpha_i-\rho)}\left[\frac{K_i\rho^{K_i+1} - (K_i+1)\rho^{K_i} - \rho^2 + 2\rho}{(1-\rho)^2}\right],$$
$$B_3 = \sum_{i=1}^{3}\frac{\alpha_i\beta_i}{(\alpha_i-\rho)}\left[\alpha_i^{K_i-1} - \rho^{K_i-1}\right]\left[\frac{(K_i\rho^2)}{(1-\rho)} + \frac{\rho^2}{(1-\rho)^2}\right],$$
$$B_4 = \sum_{i=1}^{3}\beta_i\left[\frac{(K_i-1)\alpha_i^{K_i+1} - K_i\alpha_i^{K_i} + \alpha_i}{(1-\alpha_i)^2}\right].$$
Now consider the case of $n = 3$.

Then we have $[C_1 + C_2 + C_3 + C_4]P_{10} = 1$,

where $C_1 = \frac{1}{1-\rho}$, $C_2 = \sum_{i=1}^{4}\left[\frac{\rho^2\alpha_i\beta_i}{1-\rho}\right]\left[\frac{\rho^{K_i-1}-\alpha_i^{K_i-1}}{\rho-\alpha_i}\right]$,
$C_3 = \sum_{i=1}^{4}\left[\frac{\alpha_i\beta_i\rho}{\rho-\alpha_i}\right]\left[\frac{\alpha_i-\alpha_i^{K_i}}{1-\alpha_i} - \frac{\rho-\rho^{K_i}}{1-\rho}\right]$, $C_4 = \sum_{i=1}^{4}\sum_{k=1}^{K_i}\alpha_i^{k-1}\beta_i$.
$P_{10} = \frac{1}{[C_1+C_2+C_3+C_4]}$
Also the expected number of customers in the system,
$E(N) = [D_1 + D_2 + D_3 + D_4]P_{10}$ where $D_1 = \frac{1}{(1-\rho)^2}$,

$$D_2 = \sum_{i=1}^{4} \frac{\alpha_i\beta_i\rho}{(\alpha_i-\rho)} \left[\frac{K_i\alpha_i^{K_i+1}-(K_i+1)\alpha_i^{K_i}-\alpha_i^2+2\alpha_i}{(1-\alpha_i)^2}\right] -$$
$$\sum_{i=1}^{4} \frac{\alpha_i\beta_i\rho}{(\alpha_i-\rho)} \left[\frac{K_i\rho^{K_i+1}-(K_i+1)\rho^{K_i}-\rho^2+2\rho}{(1-\rho)^2}\right],$$
$$D_3 = \sum_{i=1}^{4} \frac{\alpha_i\beta_i}{(\alpha_i-\rho)} \left[\alpha_i^{K_i-1}-\rho^{K_i-1}\right] \left[\frac{(K_i\rho^2)}{(1-\rho)} + \frac{\rho^2}{(1-\rho)^2}\right] \text{ and}$$
$$D_4 = \sum_{i=1}^{4} \beta_i \left[\frac{(K_i-1)\alpha_i^{K_i+1}-K_i\alpha_i^{K_i}+\alpha_i}{(1-\alpha_i)^2}\right].$$

So depending on the environmental factor, for n kinds of type I vacation,
$$\sum_{k=1}^{K_{n+1}} P_{k,0} + \sum_{i=2}^{n+1}\sum_{k=K_i+1}^{K_{i-1}} P_{k,0} + \sum_{k=K_1+1}^{\infty} P_{k,0} + \sum_{i=1}^{n+1}\sum_{k=0}^{K_i-1} P_{k,i} = 1.$$
$\Rightarrow [S_1+S_2+S_3+S_4]P_{10} = 1.$
$P_{10} = \frac{1}{[S_1+S_2+S_3+S_4]}$ where $S_1 = \frac{1}{1-\rho}$, $S_2 = \sum_{i=1}^{n+1}\left[\frac{\rho^2\alpha_i\beta_i}{1-\rho}\right]\left[\frac{\rho^{K_i-1}-\alpha_i^{K_i-1}}{\rho-\alpha_i}\right]$,
$S_3 = \sum_{i=1}^{n+1}\left[\frac{\alpha_i\beta_i\rho}{\rho-\alpha_i}\right]\left[\frac{\alpha_i-\alpha_i^{K_i}}{1-\alpha_i} - \frac{\rho-\rho^{K_i}}{1-\rho}\right]$, $S_4 = \sum_{i=1}^{n+1}\sum_{k=1}^{K_i}\alpha_i^{k-1}\beta_i$.
Expected number of customers in the system,
$$E(N) = \sum_{k=1}^{K_{n+1}} kP_{k,0} + \sum_{i=2}^{n+1}\sum_{k=K_i+1}^{K_{i-1}} kP_{k,0} + \sum_{k=K_1+1}^{\infty} kP_{k,0} + \sum_{i=1}^{n+1}\sum_{k=0}^{K_i-1} kP_{k,i}.$$
$= [I_1+I_2+I_3+I_4]P_{10}.$
$I_1 = \frac{1}{(1-\rho)^2}$,
$$I_2 = \sum_{i=1}^{n+1} \frac{\alpha_i\beta_i\rho}{(\alpha_i-\rho)} \left[\frac{K_i\alpha_i^{K_i+1}-(K_i+1)\alpha_i^{K_i}-\alpha_i^2+2\alpha_i}{(1-\alpha_i)^2}\right] -$$
$$\sum_{i=1}^{n+1} \frac{\alpha_i\beta_i\rho}{(\alpha_i-\rho)} \left[\frac{K_i\rho^{K_i+1}-(K_i+1)\rho^{K_i}-\rho^2+2\rho}{(1-\rho)^2}\right],$$
$$I_3 = \sum_{i=1}^{n+1} \frac{\alpha_i\beta_i}{(\alpha_i-\rho)} \left[\alpha_i^{K_i-1}-\rho^{K_i-1}\right] \left[\frac{(K_i\rho^2)}{(1-\rho)} + \frac{\rho^2}{(1-\rho)^2}\right],$$

$I_4 = \sum_{i=1}^{n+1} \beta_i \left[\frac{(K_i - 1)\alpha_i^{K_i+1} - K_i \alpha_i^{K_i} + \alpha_i}{(1-\alpha_i)^2} \right].$

Using Little's Law $E(N) = \lambda E(W)$, expected waiting time in the system, $E(W) = \frac{E(N)}{\lambda}$.

Variance of the number of customers in the system,
$V(N) = E(N^2) - (E(N))^2$.

$E(N^2) = \sum_{k=1}^{K_{n+1}} k^2 P_{k,0} + \sum_{i=2}^{n+1} \sum_{k=K_i+1}^{K_{i-1}} k^2 P_{k,0} + \sum_{k=K_1+1}^{\infty} k^2 P_{k,0} + \sum_{i=1}^{n+1} \sum_{k=0}^{K_i-1} k^2 P_{k,i}$

$= [R_1 + R_2 + R_3 + R_4]P_{10}$, where $R_1 = \frac{(1+\rho)}{(1-\rho)^3}$;

$R_2 = \sum_{i=1}^{n+1} \frac{\alpha_i \beta_i \rho}{(\alpha_i - \rho)} \left[\frac{K_i^2 \alpha_i^{K_i+2} - (2K_i^2 + 2K_i - 1)\alpha_i^{K_i+1} + (K_i+1)^2 \alpha_i^{K_i} - \alpha_i^3 + 3\alpha_i^2 - 4\alpha}{(\alpha_i - 1)^3} \right.$
$\left. - \frac{K_i^2 \rho^{K_i+2} - (2K_i^2 + 2K_i - 1)\rho^{K_i+1} + (K_i+1)^2 \rho^{K_i} - \rho^3 + 3\rho^2 - 4\rho}{(\rho-1)^3} \right],$

$R_3 = \sum_{i=1}^{n+1} \left[\frac{\alpha_i \beta_i \rho^2}{(1-\rho)} \right] \left[\frac{\alpha_i^{K_i-1} - \rho^{K_i-1}}{\alpha_i - \rho} \right] \left[K_i^2 + \frac{2K_i}{(1-\rho)} + \frac{1+\rho}{(1-\rho)^2} \right],$

$R_4 = \sum_{i=1}^{n+1} \beta_i \left[\frac{(K_i^2 - 2K_i + 1)\alpha_i^{K_i+2} - (2K_i^2 - 2K_i - 1)\alpha_i^{K_i+1} + (K_i)^2 \alpha_i^{K_i} - \alpha_i^2 - \alpha_i}{(\alpha_i - 1)^3} \right].$

6.3 Optimization problem

For the effective utilization of the model discussed, optimization of the threshold values ($K_i's$) is inevitable. So an optimization problem is discussed in this section and Numerical illustrations are provided.

- C_0 be the unit time revenue obtained from providing service.
- $C_i, 1 \le i \le n+1$, be the unit time revenue obtained from i^{th} kind of vacation.
- C be the holding cost per unit time per customer.

- C_i' be the fixed cost for switching the service from i^{th} kind of vacation to normal service.

So the expected total profit $TP = T_1 + T_2 - \overline{C} - \widehat{C}$ where T_1 is the total revenue from service, T_2 is the revenue from vacation, $\overline{C}$ is the holding cost of waiting customers and $\widehat{C}$ is the total switching cost. Here

$T_1 = \frac{1}{\mu-\lambda}C_0$, $\overline{C} = C.E(N)$.

$$T_2 = \sum_{i=1}^{n} p_i C_i \left[\left(\frac{\lambda}{\lambda+\gamma_i}\right)^{K_i} \frac{K_i}{\lambda} + \sum_{r=0}^{K_i-1} \frac{\lambda^r}{(\lambda+\gamma_i)^{r+1}} \right] + \sum_{i=1}^{n} \frac{p_i\gamma_i C_{n+1}}{\lambda+\gamma_i} \left[\left(\frac{\lambda}{\lambda+\gamma_{n+1}}\right)^{K_{n+1}} \frac{K_{n+1}}{\lambda} + \sum_{s=1}^{\infty} \frac{s}{\gamma_{n+1}} \frac{(\gamma_{n+1})^s}{(\lambda+\gamma_{n+1})^{s+1}} \right].$$

$$\widehat{C} = \sum_{i=1}^{n} p_i C_i' \left[\left(\frac{\lambda}{\lambda+\gamma_i}\right)^{K_i} \frac{1}{K_i} + \sum_{r=1}^{K_i-1} \frac{\lambda^r \gamma_i}{(\lambda+\gamma_i)^{r+1}} \right] (\mu-\lambda) + \sum_{i=1}^{n} p_i C_i' \frac{\gamma_i}{(\lambda+\gamma_i)} \left[\left(\frac{\lambda}{\lambda+\gamma_{n+1}}\right)^{K_{n+1}} \frac{1}{K_{n+1}} + \left(1 - \left(\frac{\lambda}{\lambda+\gamma_{n+1}}\right)^{K_{n+1}}\right) \right] (\mu-\lambda).$$

6.4 Numerical Illustrations

In order to bring out the qualitative nature of the model under study, we present a few representative examples.

The effect of various values of ρ, K_1, K_2, and K_3 on $E(N)$

As an example we consider the case when $\gamma_1 = 0.1$, $\gamma_2 = 0.2$, $\gamma_3 = 0.3$, $p_1 = 0.6$, $p_2 = 0.4$. Then the effect of various values of traffic intensity (ρ), K_1, K_2, and K_3 on the expected number of customers in the system and expected waiting time are plotted below (fig 6.5-fig 6.7). From Fig

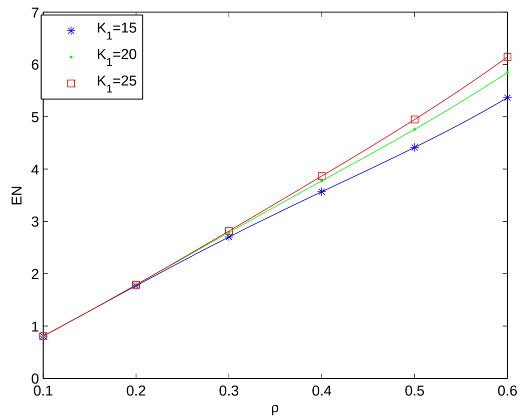

Figure 6.5: Effect of various values of K_1 and ρ on $E(N)$ when $K_2 = 10, K_3 = 5$

6.5, Fig 6.6 and Fig 6.7 we note that as ρ and K_1 increase the expected number of customers in the system also increases. Increase in ρ means either arrival rate increases or service rate decreases. When arrival rate increases the number of customers in the system also increases. When service rate decreases then also the number of customers in the system increases due to slow service. When K_1 increases it is trivially seen that the number of customers in the system will increase as the customers should wait for the return of the server from vacation until the threshold value K_1 is reached.

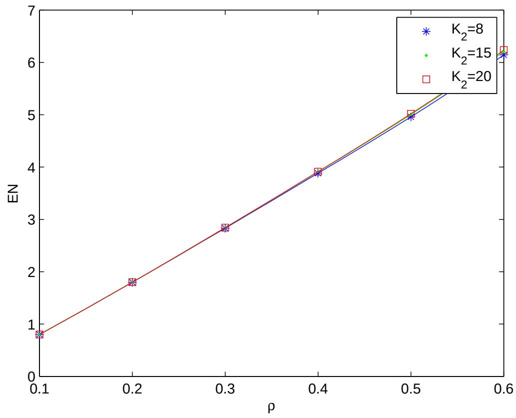

Figure 6.6: Effect of various values of K_2 and ρ on $E(N)$ when $K_1 = 25, K_3 = 8$

Then the effect of various values of ρ, K_1, K_2, and K_3 on $E(N)$

From Fig 6.8, Fig 6.9 and Fig 6.10 it is clear that as ρ increases expected waiting time also increases. From Fig 6.8 it is clear that for small values of ρ the value of K_1 does not make much difference in the expected waiting time. As K_1 increases the expected waiting time also increases. This is due to the delay of the server return from vacation due to the increased threshold value K_1. From Fig 6.10, we see that for small values of ρ, as K_3 increases waiting time also increases but as the value of ρ increases, the waiting time is greater for smaller values of K_3. As the duration of vacations decrease, expected waiting time increases with increase in the value of K_3.

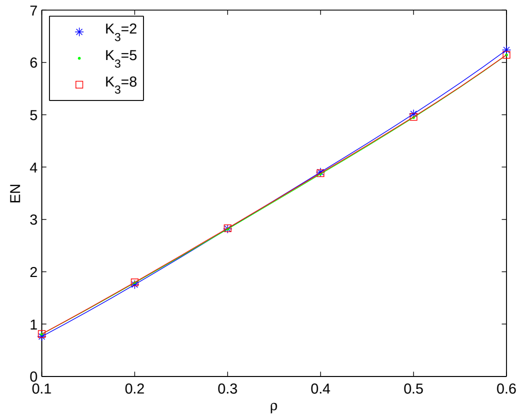

Figure 6.7: Effect of various values of K_3 and ρ on $E(N)$ when $K_1 = 25, K_2 = 10$

The effect of various values of ρ, K_1, K_2, and K_3 on EP

By assuming C_0 = \$250, C_1 = \$250, C_2 = \$100, C_3 = \$50, C = \$25 and $C_1' = C_2' = C_3'$ = \$100 the effect of various values of traffic intensity(ρ), K_1, K_2, and K_3 on expected profit EP are plotted below (Fig 6.11 - Fig 6.13).

From Fig 6.11, Fig 6.12 and Fig 6.13 it is clear that as ρ increases the expected profit decreases, reaches a minimum value and then begins to increase. As ρ increases either arrival rate increases or service rate decreases. Increase in arrival rate causes frequent interruption of vacation and switching on/off of service which is very expensive and it reduces the profit. Also the increase in arrival rate or the decrease in service rate reduces the chance of occurrence of vacation. This reduces the loss due to switching off of service.

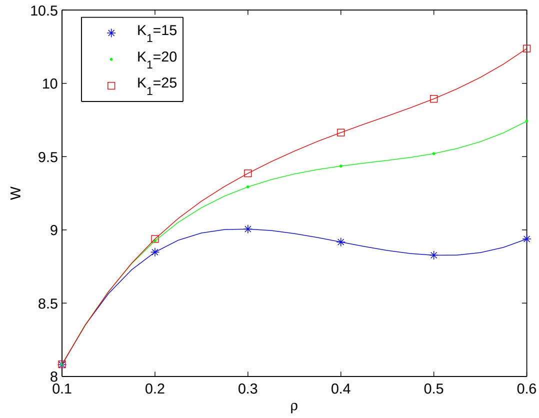

Figure 6.8: Effect of various values of K_1 and ρ on W when $K_2 = 10, K_3 = 5$

Effect of various values of λ, μ and threshold values on EP

For small values of λ expected profit shows convexity (Fig6.14). As λ increases the expected profit decreases(Fig6.15).

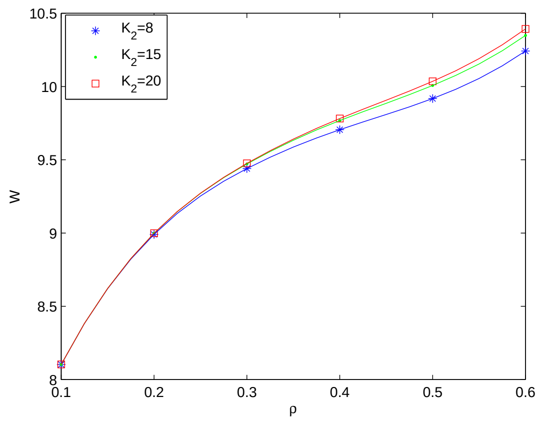

Figure 6.9: Effect of various values of K_2 and ρ on W when $K_1 = 25, K_3 = 8$

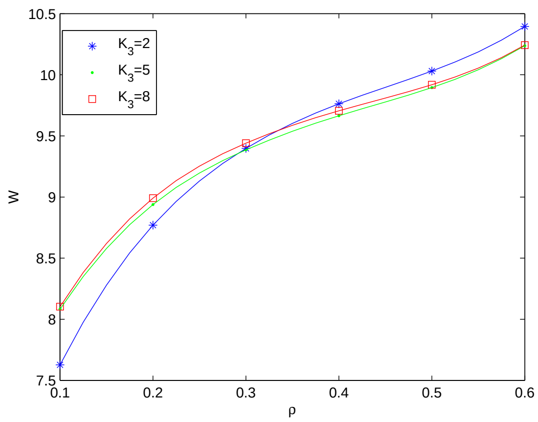

Figure 6.10: Effect of various values of K_3 and ρ on W when $K_2 = 10, K_1 = 25$

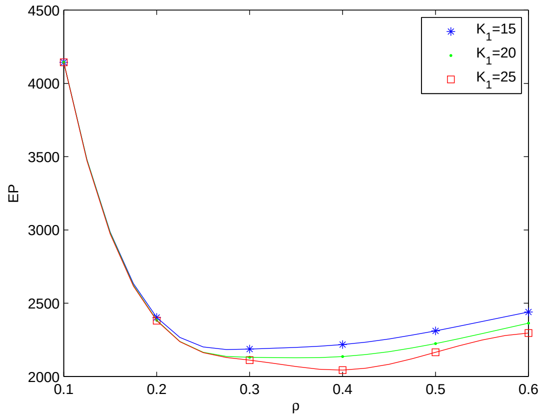

Figure 6.11: Effect of various values of K_1 and ρ on expected profit when $K_2 = 10, K_3 = 5$

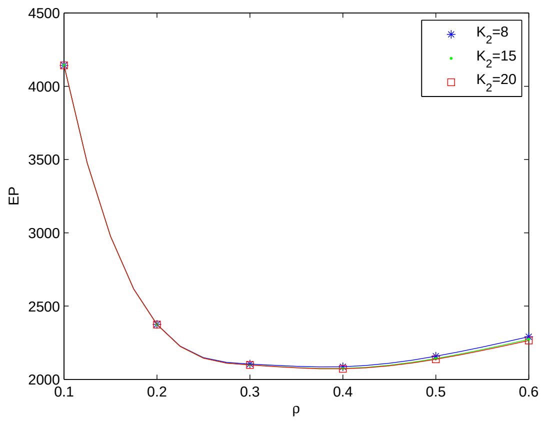

Figure 6.12: Effect of various values of K_2 and ρ on expected profit when $K_1 = 25, K_3 = 8$

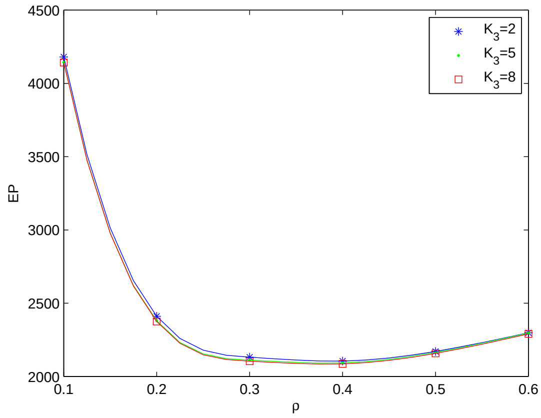

Figure 6.13: Effect of various values of K_3 and ρ on expected profit when $K_2 = 10, K_1 = 25$

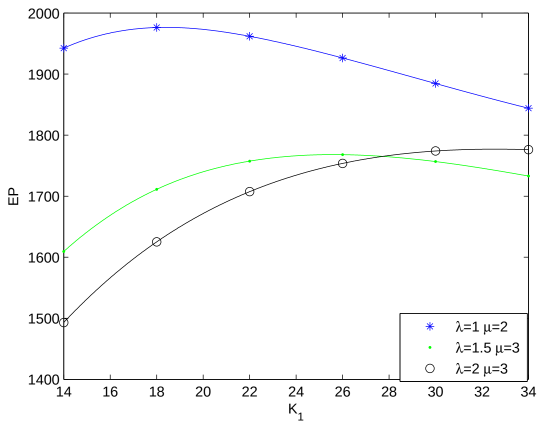

Figure 6.14: Effect of threshold values and traffic intensity on expected profit

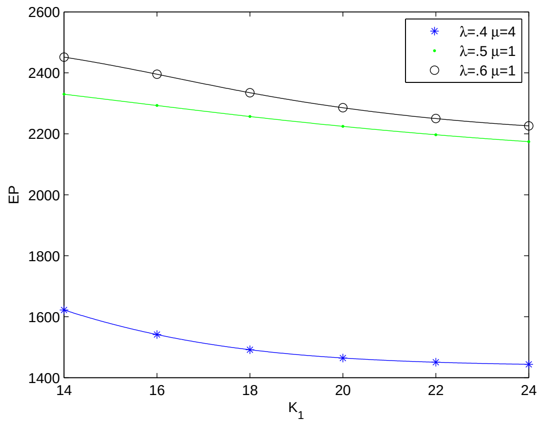

Figure 6.15: Effect of threshold values and traffic intensity on expected profit

Chapter 7

Stochastic decomposition of the M/M/1 queue with environment dependent working vacation

Introduction

If a queue is empty the server remains idle. policy we may think of a vacation with server working during that time in slow mode, if customers are available. The idle time of the server can be utilized for some other work. Instead of a complete vacation If the customers in the queue is less, the functioning of the server in a slow rate will reduce the operating cost, energy consumption and the start up cost. These advantages are pointing

towards working vacation. Working vacation is an extension of regular vacation. In working vacation, instead of completely stopping the service, the server provides service at a slow rate. Working vacation reduces the chance of reneging of the customers compared to normal vacation. In this era of high demand for commodities and services which are available in a short spell, the concept of working vacation is very useful. This may be the main reason of the extensive research work going on in working vacation queueing models.
In this chapter we consider a single server queueing system with working vacation. On completion of a service if the server finds the system empty, he goes for a working vacation. There are n types of working vacations. Depending on the environment, after a busy period, the server goes for i^{th} type of vacation with probability $p_i, 1 \leq i \leq n$. During vacation if customers arrive, the server provides service at a lower rate . On completion of service during vacation, if there is no customer in the system the server continues to be on vacation. Otherwise the vacation is interrupted, i.e. the server returns to normal service without completing the vacation and starts service in the normal rate. On completion of vacation if the server finds the system empty, he remains in the corresponding vacation. We demonstrate stochastic decomposition of the queue length and waiting time processes using method of induction and Little's formula.

7.1 Model description

Consider a single server queueing system with working vacation in which arrival occurs according to a Poisson process with parameter λ. The service time is exponentially distributed with parameter μ. On completion

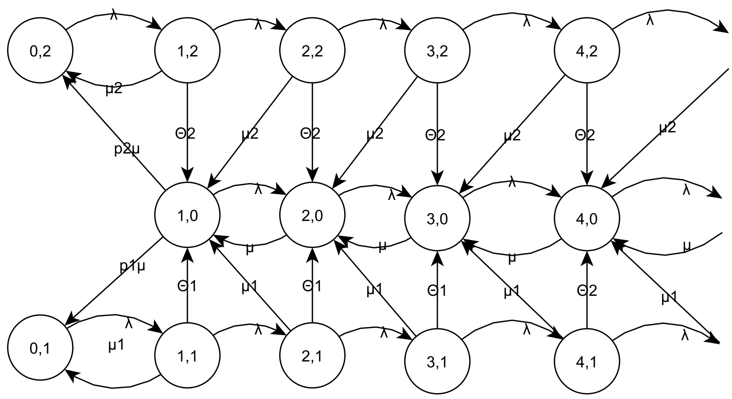

Figure 7.1: Model description

of a service if the server finds the system empty he goes for a working vacation. There are n types of working vacations. Depending on the environment, after a busy period, the server goes for i^{th} type of vacation with probability $p_i, 1 \leq i \leq n$. The duration of i^{th} type of vacation is exponentially distributed with parameter $\gamma_i, 1 \leq i \leq n$. During vacation if customers arrive, the server provides service at a lower rate μ_i, while in i^{th} type of vacation, $1 \leq i \leq n$. On completion of service during vacation, if there is no customer in the system the server continues to stay on vacation. Otherwise the vacation is interrupted, i.e. the server returns to normal service without completing the vacation and starts service in the normal rate μ. On completion of vacation if the server finds the system empty, he remains in the corresponding vacations. Figure 7.1 is a diagrammatic representation of the model.

7.2 Mathematical description

We establish the stochastic decomposition of the state space by induction on the number of environmental factors.

Case.1 First we consider the case of $n = 2$.

Let $N(t)$ be the number of customers in the system and $S(t)$ be the status of the server at time t:

$$S(t) = \begin{cases} 0, & \text{if the server is serving in normal mode;} \\ 1, & \text{if server is in the type I working vacation;} \\ 2, & \text{if server is in the type II working vacation;} \end{cases}$$

Then $X = \{X(t), t \geq 0\}$ where $X(t) = (N(t), S(t))$ is a continuous time Markov chain with state space $\{0, 1\} \cup \{0, 2\} \cup \{(j, k),\ j = 1, 2, \ldots; k = 0, 1, 2\}$. The infinitesimal generator associated with the Markov chain is

$$Q_1 = \begin{bmatrix} B_0 & B_1 & & & & \\ B_2 & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{bmatrix} \text{ where } -B_0 = A_0 = \lambda I,$$

$$B_1 = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, B_2 = \begin{bmatrix} \mu p_1 & \mu p_2 \\ \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \ A_2 = \begin{bmatrix} \mu & 0 & 0 \\ \mu_1 & 0 & 0 \\ \mu_2 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -\lambda - \mu & 0 & 0 \\ \theta_1 & -\lambda - \mu_1 - \theta_1 & 0 \\ \theta_2 & 0 & -\lambda - \mu_2 - \theta_2 \end{bmatrix}$$

Stability analysis

We have $A = A_0 + A_1 + A_2 = \begin{bmatrix} 0 & 0 & 0 \\ \theta_1 + \mu_1 & -\mu_1 - \theta_1 & 0 \\ \theta_2 + \mu_2 & 0 & -\mu_2 - \theta_2 \end{bmatrix}$

Then A is the infinitesimal generator of a Markov chain with state space $\{0, 1, 2\}$ which represents the status of the server. Let $\mathbf{y} = (y_0, y_1, y_2)$ be the invariant probability vector of A. Then $\mathbf{y}A = 0$ and $\mathbf{y}e = 1$. The left drift rate of the original Markov chain is $\mathbf{y}A_2e$ and that for right drift is $\mathbf{y}A_0e$. Left drift indicates a service completion and right drift represents arrival of customer. Thus the system is stable if and only if $\mathbf{y}A_0e < \mathbf{y}A_2e$. Here $\mathbf{y}A_0e = \lambda$ and $\mathbf{y}A_2e = \mu$.

Hence we have

Theorem: The system is stable if and only if $\lambda < \mu$.

7.2.1 Steady State Analysis

For the analysis of the model it is necessary to solve for the minimal non-negative solution R_1 of the matrix quadratic equation

$$R_1^2 A_2 + R_1 A_1 + A_0 = 0. \tag{7.1}$$

Since the Matrices A_2, A_1, A_0 are lower triangular R_1 is also lower triangular. Solving (7.1) we obtain R_1 as $R_1 = \begin{bmatrix} r_0 & 0 & 0 \\ r_1 & \overline{r}_1 & 0 \\ r_2 & 0 & \overline{r}_2 \end{bmatrix}$ where $r_0 = \rho$, $r_1 = \frac{\rho(\lambda+\theta_1)}{(\lambda+\mu_1+\theta_1)}$, $\overline{r}_1 = \frac{\lambda}{(\lambda+\mu_1+\theta_1)}$, $r_2 = \frac{\rho(\lambda+\theta_2)}{(\lambda+\mu_2+\theta_2)}$ and $\overline{r}_2 = \frac{\lambda}{(\lambda+\mu_2+\theta_2)}$.

Let $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots,)$ be the steady state probability vector associated with the Markov process X. Here $\mathbf{x}_0 = (x_{01}, x_{02})$ and $\mathbf{x}_i = (x_{i0}, x_{i1}, x_{i2}), i = 1, 2, \ldots, \infty$. Assume that $\mathbf{x}_i = \mathbf{x}_1 R_1^{i-1}, i = 2, 3, \ldots$, then $\mathbf{x}$ can be obtained by solving $\mathbf{x}Q = 0$ using the boundary condition

$$\mathbf{x}_0 e + \mathbf{x}_1 (I - R_1)^{-1} e = 1. \tag{7.2}$$

From $\mathbf{x}Q = 0$ we get

$$\mathbf{x}_0 B_0 + \mathbf{x}_1 B_2 = 0. \tag{7.3}$$

$$\mathbf{x}_0 B_1 + \mathbf{x}_1 (A_1 + R_1 A_2) = 0. \tag{7.4}$$

From (7.3) and (7.4) we will get

$$\mu p_1 x_{10} + \mu_1 x_{11} = (\lambda) x_{01}. \tag{7.5}$$

$$\mu p_2 x_{10} + \mu_2 x_{12} = (\lambda) x_{02}. \tag{7.6}$$

$$\mu x_{10} = (\lambda + \theta_1) x_{11} + (\lambda + \theta_2) x_{12}. \tag{7.7}$$

$$\lambda x_{01} = (\lambda + \mu_1 + \theta_1) x_{11}. \tag{7.8}$$

$$\lambda x_{02} = (\lambda + \mu_2 + \theta_2) x_{12}. \tag{7.9}$$

Assume $x_{01} = k_1$ and $x_{02} = k_2$, then from (7.8) and (7.9), $x_{11} = \overline{r}_1 k_1$, $x_{12} = \overline{r}_2 k_2$. Substituting the values of x_{11} and x_{01} in (7.5) we will get $x_{10} = \frac{k_1 \overline{r}_1}{p_1}$. Also

$$k_2 = \frac{\mu p_2 r_1}{p_1(\lambda - \mu_2 r_2)} k_1$$

To find the value of k_1 we use the normalizing condition
$\mathbf{x}_0 e + \mathbf{x}_1 (I - R_1)^{-1} e = 1$.

Let $r_0^{'} = 1-r_0, \overline{r}_1^{'} = 1-\overline{r}_1, \overline{r}_2^{'} = 1-\overline{r}_2$; then $(I-R_1)^{-1} = \begin{bmatrix} 1/r_0^{'} & 0 & 0 \\ -r_1/\overline{r}_0^{'}r_1^{'} & 1/\overline{r}_1^{'} & 0 \\ -r_2/r_0^{'}\overline{r}_2^{'} & 0 & 1/\overline{r}_2^{'} \end{bmatrix}$

Using (7.2)

$$k_1\left[1+\frac{r_1}{p_1 r_0^{'}}+\frac{\overline{r}_1}{\overline{r}_1^{'}}-\frac{r_1\overline{r}_1}{r_0^{'}\overline{r}_1^{'}}\right]+k_2\left[1+\frac{\overline{r}_2}{\overline{r}_2^{'}}-\frac{r_2\overline{r}_2}{r_0^{'}\overline{r}_2^{'}}\right]=1. \tag{7.10}$$

Substituting k_2 in (7.10)

$$k_1\left[1+\frac{r_1}{p_1 r_0^{'}}+\frac{\overline{r}_1}{\overline{r}_1^{'}}-\frac{r_1\overline{r}_1}{r_0^{'}\overline{r}_1^{'}}+\frac{\mu p_2 r_1}{p_1(\lambda-\mu_2 r_2)}\left[1+\frac{\overline{r}_2}{\overline{r}_2^{'}}-\frac{r_2\overline{r}_2}{r_0^{'}\overline{r}_2^{'}}\right]\right]=1. \tag{7.11}$$

From (7.11) $k_1 = \frac{1}{\left[1+\frac{r_1}{p_1 r_0^{'}}+\frac{\overline{r}_1}{\overline{r}_1^{'}}-\frac{r_1\overline{r}_1}{r_0^{'}\overline{r}_1^{'}}+\frac{\mu p_2 r_1}{p_1(\lambda-\mu_2 r_2)}\left[1+\frac{\overline{r}_2}{\overline{r}_2^{'}}-\frac{r_2\overline{r}_2}{r_0^{'}\overline{r}_2^{'}}\right]\right]}$.

Now $R_1^{k-1} = \begin{bmatrix} r_0^{(k-1)} & 0 & 0 \\ r_1\frac{(r_0^{k-1}-\overline{r}_1^{k-1})}{(r_0-\overline{r}_1)} & \overline{r}_1^{(k-1)} & 0 \\ r_2\frac{(r_0^{k-1}-\overline{r}_2^{k-1})}{(r_0-\overline{r}_2)} & 0 & \overline{r}_2^{(k-1)} \end{bmatrix}$ and

$\mathbf{x}_k\mathbf{e} = x_{10}r_0^{k-1} + x_{11}\left[\overline{r}_1^{(k-1)} + r_1\frac{(r_0^{k-1}-\overline{r}_1^{k-1})}{(r_0-\overline{r}_1)}\right] + x_{12}\left[\overline{r}_2^{(k-1)} + r_2\frac{(r_0^{k-1}-\overline{r}_2^{k-1})}{(r_0-\overline{r}_2)}\right]$ for $k>1$.

Let $Q_v(z)$ be the PGF associated with the number of customers in the system. Then $Q_v(z) = \sum_{n=0}^{\infty}\mathbf{x}_n\mathbf{e}z^n$

$= x_{01}+x_{02}+\frac{x_{10}z}{1-r_0z}+\frac{x_{11}z}{1-\overline{r}_1z}+\frac{x_{12}z}{1-\overline{r}_2z}+\frac{x_{11}r_1z}{r_0-\overline{r}_1}\left[\frac{1}{1-r_0z}-\frac{1}{1-\overline{r}_1z}\right]+\frac{x_{12}r_2z}{r_0-\overline{r}_2}\left[\frac{1}{1-r_0z}-\frac{1}{1-\overline{r}_2z}\right]$

$= \frac{1-r_0}{1-r_0z}\Big[x_{01}\frac{(1-r_0z)}{(1-r_0)}+x_{02}\frac{(1-r_0z)}{(1-r_0)}+\frac{x_{10}z}{1-r_0}+\frac{x_{11}z}{1-\overline{r}_1z}\frac{(1-r_0z)}{(1-r_0)}+\frac{x_{12}z}{1-\overline{r}_2z}\frac{(1-r_0z)}{(1-r_0)}+$
$\frac{x_{11}r_1z}{r_0-\overline{r}_1}\frac{(1-r_0z)}{(1-r_0)}\left[\frac{1}{1-r_0z}-\frac{1}{1-\overline{r}_1z}\right]+\frac{x_{12}r_2z}{r_0-\overline{r}_2}\frac{(1-r_0z)}{(1-r_0)}\left[\frac{1}{1-r_0z}-\frac{1}{1-\overline{r}_2z}\right]\Big]$.

$Q_v^{'}(z) = \frac{r_0}{1-r_0z}\Big[x_{01}\frac{(1-r_0z)}{(1-r_0)}+x_{02}\frac{(1-r_0z)}{(1-r_0)}+\frac{x_{10}z}{1-r_0}\frac{x_{11}z}{1-\overline{r}_1z}\frac{(1-r_0z)}{(1-r_0)}+\frac{x_{12}z}{1-\overline{r}_2z}\frac{(1-r_0z)}{(1-r_0)}+$
$\frac{x_{11}r_1z}{r_0-\overline{r}_1}\frac{(1-r_0z)}{(1-r_0)}\left[\frac{1}{1-r_0z}-\frac{1}{1-\overline{r}_1z}\right]+\frac{x_{12}r_2z}{r_0-\overline{r}_2}\frac{(1-r_0z)}{(1-r_0)}\left[\frac{1}{1-r_0z}-\frac{1}{1-\overline{r}_2z}\right]\Big]+\left(\frac{1-r_0}{1-r_0z}\right)\left(\frac{1}{1-r_0}\right)$
$\Big[-r_0x_{01}-r_0x_{02}+x_{10}+\frac{x_{11}r_1}{(r_0-\overline{r}_1)}+\frac{x_{12}r_2}{(r_0-\overline{r}_2)}+x_{11}\left(\frac{r_0-r_1-\overline{r}_1}{r_0-\overline{r}_1}\right)\left(\frac{1-2r_0z+r_0\overline{r}_1z^2}{(1-\overline{r}_1z)^2}\right)+$

$x_{12}\left(\frac{r_0-r_2-\overline{r}_2}{r_0-\overline{r}_2}\right)\left(\frac{1-2r_0z+r_0\overline{r}_2z^2}{(1-\overline{r}_2z)^2}\right)\Big]$

Expected queue length $E(\overline{L}) = Q_v'(1) = \frac{r_0}{1-r_0}+\left(\frac{1}{1-r_0}\right)[-r_0x_{01} - r_0x_{02} + x_{10}$
$+\frac{x_{11}r_1}{(r_0-\overline{r}_1)} + \frac{x_{12}r_2}{(r_0-\overline{r}_2)} + x_{11}\left(\frac{r_0-r_1-\overline{r}_1}{r_0-\overline{r}_1}\right)\left(\frac{1-2r_0+r_0\overline{r}_1}{(1-\overline{r}_1)^2}\right) + x_{12}\left(\frac{r_0-r_2-\overline{r}_2}{r_0-\overline{r}_2}\right)\left(\frac{1-2r_0+r_0\overline{r}_2}{(1-\overline{r}_2)^2}\right)\Big]$
$= \frac{r_0}{1-r_0}+\left(\frac{1}{1-r_0}\right)\Big[-r_0k_1 - r_0k_2 + \frac{k_1r_1}{p_1} + \frac{r_1k_1\overline{r}_1}{(1-\overline{r}_1)(r_0-\overline{r}_1)} + \frac{k_2\overline{r}_2r_2}{(r_0-\overline{r}_2)} + k_1\overline{r}_1\left(\frac{r_0-r_1-\overline{r}_1}{r_0-\overline{r}_1}\right)$
$\left(\frac{1-2r_0+r_0\overline{r}_1}{(1-\overline{r}_1)^2}\right) + k_2\overline{r}_2\left(\frac{r_0-r_2-\overline{r}_2}{r_0-\overline{r}_2}\right)\left(\frac{1-2r_0+r_0\overline{r}_2}{(1-\overline{r}_2)^2}\right)\Big]$
$= \frac{r_0}{1-r_0} + \left(\frac{k_1}{1-r_0}\right)\Big[-r_0 + \frac{r_1}{p_1} + \frac{r_1\overline{r}_1}{(1-\overline{r}_1)(r_0-\overline{r}_1)} + \overline{r}_1\left(\frac{r_0-r_1-\overline{r}_1}{r_0-\overline{r}_1}\right)\left(\frac{1-2r_0+r_0\overline{r}_1}{(1-\overline{r}_1)^2}\right)\Big] +$
$\left(\frac{k_2}{1-r_0}\right)$
$\Big[-r_0 + \frac{\overline{r}_2r_2}{(r_0-\overline{r}_2)} + \overline{r}_2\left(\frac{r_0-r_2-\overline{r}_2}{r_1-\overline{r}_2}\right)\left(\frac{1-2r_0+r_1\overline{r}_2}{(1-\overline{r}_2)^2}\right)\Big]$

Case.2

Now consider the case of $n = 3$. Then $S(t)$ has four states.

$$S(t) = \begin{cases} 0, & \text{if the server is serving in normal mode;} \\ 1, & \text{if server is in the type I working vacation;} \\ 2, & \text{if server is in the type II working vacation;} \\ 3, & \text{if server is in the type III working vacation;} \end{cases}$$

The state space of X is $\{(0,k)|k = 1,2,3\} \cup \{(j,k),\ j = 1,2,\ldots;k = 0,1,2,3\}$. The infinitesimal generator associated with the Markov chain

is $Q_2 = \begin{bmatrix} B_0 & B_1 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}$

where $-B_0 = A_0 = \lambda I_3,\ B_1 = \begin{bmatrix} 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, B_2 = \begin{bmatrix} \mu p_1 & \mu p_2 & \mu p_3 \\ \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix},$

$$A_2 = \begin{bmatrix} \mu & 0 & 0 & 0 \\ \mu_1 & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 \\ \mu_3 & 0 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -\lambda-\mu & 0 & 0 & 0 \\ \theta_1 & -\lambda-\mu_1-\theta_1 & 0 & 0 \\ \theta_2 & 0 & -\lambda-\mu_2-\theta_2 & 0 \\ \theta_3 & 0 & 0 & -\lambda-\mu_3-\theta_3 \end{bmatrix}.$$

$$A = A_0 + A_1 + A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \theta_1+\mu_1 & -\mu_1-\theta_1 & 0 & 0 \\ \theta_2+\mu_2 & 0 & -\mu_2-\theta_2 & 0 \\ \theta_3+\mu_3 & 0 & 0 & -\mu_3-\theta_3 \end{bmatrix}$$

We get $\lambda < \mu$ as the condition for stability.

$R_2 = \begin{bmatrix} r_0 & 0 & 0 & 0 \\ r_1 & \overline{r}_1 & 0 & 0 \\ r_2 & 0 & \overline{r}_2 & 0 \\ r_3 & 0 & 0 & \overline{r}_3 \end{bmatrix}$ where $r_0 = \rho,\ r_1 = \frac{\rho(\lambda+\theta_1)}{(\lambda+\mu_1+\theta_1)},\ \overline{r}_1 = \frac{\lambda}{(\lambda+\mu_1+\theta_1)},$

$r_2 = \frac{\rho(\lambda+\theta_2)}{(\lambda+\mu_2+\theta_2)},\ \overline{r}_2 = \frac{\lambda}{(\lambda+\mu_2+\theta_2)},\ r_3 = \frac{\rho(\lambda+\theta_3)}{(\lambda+\mu_3+\theta_3)}$ and $\overline{r}_3 = \frac{\lambda}{(\lambda+\mu_3+\theta_3)}$

Let $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots,)$ be the steady state probability vector associated with the Markov process X. Here $\mathbf{x}_0 = (x_{01}, x_{02}, x_{03})$ and $\mathbf{x}_i = (x_{i0}, x_{i1}, x_{i2}, x_{i3}), i = 1, 2, \ldots$. Then assuming $x_{01} = k_1\ x_{02} = k_2$ and $x_{03} = k_3$, we get $x_{11} = \overline{r}_1 k_1,\ \ x_{12} = \overline{r}_2 k_2\ \ x_{13} = \overline{r}_3 k_3.\ \ x_{10} = \frac{k_1 r_1}{p_1}$. Also $k_2 = \frac{\mu p_2 r_1}{p_1(\lambda-\mu_2 r_2)} k_1,\ k_3 = \frac{\mu p_3 r_1}{p_1(\lambda-\mu_3 r_3)} k_1$.

Let $r_0^{'} = 1 - r_0,\ \overline{r}_1^{'} = 1 - \overline{r}_1,\ \overline{r}_2^{'} = 1 - \overline{r}_2,\ \overline{r}_3^{'} = 1 - \overline{r}_3$ then

$$(I-R_2)^{-1}=\begin{bmatrix} 1/r_0^{'} & 0 & 0 & 0 \\ -r_1/\overline{r}_1^{'}r_0^{'} & 1/\overline{r}_1^{'} & 0 & 0 \\ -r_2/\overline{r}_2^{'}r_0^{'} & 0 & 1/\overline{r}_2^{'} & 0 \\ -r_3/\overline{r}_3^{'}r_0^{'} & 0 & 0 & 1/\overline{r}_3^{'} \end{bmatrix}$$

Using the normalizing condition $\mathbf{x}_0e+\mathbf{x}_1(I-R_2)^{-1}e=1$, we get
$k_1=\frac{1}{\left[1+\frac{\overline{r}_1}{\overline{r}_1^{'}}-\frac{r_1\overline{r}_1}{r_0^{'}\overline{r}_1^{'}}+\frac{r_1}{p_1r_0^{'}}+\sum_{j=2}^{3}\frac{\mu p_jr_1}{p_1(\lambda-\mu_jr_j)}\left(1+\frac{\overline{r}_j}{\overline{r}_j^{'}}-\frac{r_j\overline{r}_j}{r_0^{'}\overline{r}_j^{'}}\right)\right]}$

Now $R_2^{k-1}=\begin{bmatrix} r_1^{(k-1)} & 0 & 0 & 0 \\ r_2\frac{(r_1^{k-1}-r_3^{k-1})}{(r_1-r_3)} & r_3^{(k-1)} & 0 & 0 \\ r_4\frac{(r_1^{k-1}-r_5^{k-1})}{(r_1-r_5)} & 0 & r_5^{(k-1)} & 0 \\ r_6\frac{(r_1^{k-1}-r_7^{k-1})}{(r_1-r_7)} & 0 & 0 & r_7^{(k-1)} \end{bmatrix}$ and
$\mathbf{x}_k\mathbf{e}=x_{10}r_1^{k-1}+x_{11}\left[r_3^{(k-1)}+r_2\frac{(r_1^{k-1}-r_3^{k-1})}{(r_1-r_3)}\right]+x_{12}\left[r_5^{(k-1)}+r_4\frac{(r_1^{k-1}-r_5^{k-1})}{(r_1-r_5)}\right]+x_{13}\left[r_7^{(k-1)}+r_6\frac{(r_1^{k-1}-r_7^{k-1})}{(r_1-r_7)}\right]$ for $k>1$.
$Q_v(z)=\sum_{n=0}^{\infty}\mathbf{x}_n\mathbf{e}z^n$
$=x_{01}+x_{02}+x_{03}+\frac{x_{10}z}{1-r_1z}+\frac{x_{11}z}{1-r_3z}+\frac{x_{12}z}{1-r_5z}+\frac{x_{13}z}{1-r_7z}+\frac{x_{11}r_2z}{r_1-r_3}\left[\frac{1}{1-r_1z}-\frac{1}{1-r_3z}\right]+\frac{x_{12}r_4z}{r_1-r_5}\left[\frac{1}{1-r_1z}-\frac{1}{1-r_5z}\right]+\frac{x_{13}r_6z}{r_1-r_7}\left[\frac{1}{1-r_1z}-\frac{1}{1-r_7z}\right]$
Expected queue length $E(\overline{L})=Q_v^{'}(1)$
$=\frac{r_1}{1-r_1}+\left(\frac{k_1}{1-r_1}\right)\left[-r_1+\frac{r_2}{p_1}+\frac{r_2r_3}{(1-r_3)(r_1-r_3)}+r_3\left(\frac{r_1-r_2-r_3}{r_1-r_3}\right)\left(\frac{1-2r_1+r_1r_3}{(1-r_3)^2}\right)\right]+\left(\frac{k_2}{1-r_1}\right)\left[-r_1+\frac{r_5r_4}{(r_1-r_5)}+r_5\left(\frac{r_1-r_4-r_5}{r_1-r_5}\right)\left(\frac{1-2r_1+r_1r_5}{(1-r_5)^2}\right)\right]+\left(\frac{k_3}{1-r_1}\right)\left[-r_1+\frac{r_7r_6}{(r_1-r_7)}+r_7\left(\frac{r_1-r_6-r_7}{r_1-r_7}\right)\left(\frac{1-2r_1+r_1r_7}{(1-r_7)^2}\right)\right]$

Case.3

Now we consider the case where there are $n\geq 4$ distinct type of vacations.

Then

$S(t)$ has $n+1$ distinct values.

$$S(t) = \begin{cases} 0, & \text{if the server is serving in normal mode;} \\ i, & \text{if server is in the } i^{th} \text{ type working vacation, } 1 \le i \le n; \end{cases}$$

The state space of X is $\{(0,k)/k = 1,2,\ldots,n\} \cup \{(j,k)/j = 0,1,2,\ldots; k = 1,2,\ldots,n\}$ The infinitesimal generator associated with the Markov chain is

$$Q_n = \begin{bmatrix} B_0 & B_1 & & & & \\ B_2 & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \end{bmatrix} \text{ where } B_1 = \begin{bmatrix} 0 & \lambda & & & \\ & & \lambda & & \\ & & & \lambda & \\ & & & & \\ & & & & \lambda \end{bmatrix}_{n\times(n+1)},$$

$$B_2 = \begin{bmatrix} \mu p_1 & \mu p_2 & \cdots & \mu p_n \\ \mu_1 & & & \\ & \mu_2 & & \\ & & & \\ & & & \mu_n \end{bmatrix}_{(n+1)\times n} \quad A_2 = \begin{bmatrix} \mu & & \\ \mu_1 & & \\ \vdots & & \\ \mu_n & & \end{bmatrix}_{(n+1)\times(n+1)},$$

and $-B_0 = A_0 = \lambda I_n$

$$A_1 = \begin{bmatrix} -\lambda-\mu & & & & & \\ \theta_1 & -\lambda-\mu_1-\theta_1 & & & & \\ \theta_2 & & -\lambda-\mu_2-\theta_2 & & & \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \\ \theta_n & & & & & -\lambda-\mu_n-\theta_n \end{bmatrix}$$

As in the earlier sections

$$A_0 + A_1 + A_2 = \begin{bmatrix} -\lambda - \mu & & & & & \\ \theta_1 + \mu_1 & -\mu_1 - \theta_1 & & & & \\ \theta_2 + \mu_2 & & -\mu_2 - \theta_2 & & & \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \\ \theta_n + \mu_n & & & & & -\mu_n - \theta_n \end{bmatrix}$$

Let $\mathbf{y} = (y_0, y_1, y_2, \ldots, y_n)$ be the invariant probability vector of A satisfying $\mathbf{y}A = 0$ and $\mathbf{y}e = 1$. The system is stable if and only if $\mathbf{y}A_0e < \mathbf{y}A_2e$. Here $\mathbf{y}A_0e = \lambda$ and $\mathbf{y}A_2e = \mu$.

Theorem: The system is stable if and only if $\lambda < \mu$

$$R_n = \begin{bmatrix} r_0 & & & & & \\ r_1 & \overline{r}_1 & & & & \\ r_2 & 0 & \overline{r}_2 & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ r_n & & & & 0 & \overline{r}_n \end{bmatrix}$$ where $r_0 = \rho, r_i = \frac{\rho(\lambda+\theta_i)}{(\lambda+\mu_i+\theta_i)}$, $\overline{r}_i = \frac{\lambda}{(\lambda+\mu_i+\theta_i)}$.

Let $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots,)$ be the steady state probability vector associated with the Markov chain X. Here $\mathbf{x}_0 = (x_{01}, x_{02}, \ldots, x_{0n})$ and $\mathbf{x}_i = (x_{i0}, x_{i1}, x_{i2}, \ldots, x_{in}), i = 1, 2, \ldots$. Then assuming $x_{0j} = k_j, 1 \leq i \leq n$, we get $x_{1j} = \overline{r}_j k_j$, $x_{10} = \frac{k_1 \overline{r}_1}{p_1}$.

Also $k_j = \frac{\mu p_j \overline{r}_1}{p_1(\lambda - \mu_j \overline{r}_j)} k_1$. Let $\overline{r}_i^{'} = 1 - \overline{r}_i, 1 \leq i \leq n, r_0^{'} = 1 - r_0$,
$\ell_i = 1/\overline{r}_i^{'}, 0 \leq i \leq n, \chi_i = -r_i/(\overline{r}_i^{'}\overline{r}_0^{'}), 1 \leq i \leq n$ then,

$$(I-R_n)^{-1}=\begin{bmatrix} \ell_0 & & & & \\ \chi_1 & \ell_2 & & & \\ \chi_2 & & \ell_2 & & \\ & & & & \\ \chi_n & & & & \ell_n \end{bmatrix}$$

$$k_1=\frac{1}{\left[1+\frac{\overline{r}_1}{\overline{r}_1'}-\frac{r_1\overline{r}_1}{r_0'\overline{r}_1'}+\frac{r_1}{p_1r_0'}+\sum_{j=2}^{n}\frac{\mu p_jr_1}{p_1(\lambda-\mu_jr_j)}\left(1+\frac{\overline{r}_j}{\overline{r}_j'}-\frac{r_j\overline{r}_j}{r_0'\overline{r}_j'}\right)\right]}$$

Now $R_n^{k-1}=\begin{bmatrix} r_0^{(k-1)} & & & & \\ r_1\frac{(r_0^{k-1}-\overline{r}_1^{k-1})}{(r_0-\overline{r}_1)} & \overline{r}_1^{(k-1)} & & & \\ r_2\frac{(r_0^{k-1}-\overline{r}_2^{k-1})}{(r_0-\overline{r}_2)} & 0 & \overline{r}_2^{(k-1)} & & \\ \vdots & & & \ddots & \\ \vdots & & & & \ddots \\ r_n\frac{(r_0^{k-1}-\overline{r}_n^{k-1})}{(r_0-\overline{r}_n)} & & & & \overline{r}_n^{(k-1)} \end{bmatrix}$ and

$\mathbf{x}_k\mathbf{e}=x_{10}r_0^{k-1}+\sum_{i=1}^{n}x_{1i}\left[\overline{r}_i^{(k-1)}+r_i\frac{(r_0^{k-1}-\overline{r}_i^{k-1})}{(r_0-\overline{r}_i)}\right]$ for $k>1$.

Then $Q_v(z)=\sum_{n=0}^{\infty}\mathbf{x}_nz^n$

$$=\sum_{j=1}^{n}x_{0j}+\frac{x_{10}z}{1-r_0z}+\sum_{j=1}^{n}\frac{x_{1j}z}{1-\overline{r}_jz}+\sum_{j=1}^{n}\frac{x_{1j}r_jz}{r_0-\overline{r}_j}\left[\frac{1}{1-r_0z}-\frac{1}{1-\overline{r}_jz}\right]$$

Expected queue length $E(\overline{L})=Q_v'(1)$

$$=\frac{r_0}{1-r_0}+\sum_{j=1}^{n}\left(\frac{k_j}{1-r_0}\right)\left[-r_0+\frac{r_j}{p_1}+\frac{r_j\overline{r}_j}{(1-\overline{r}_j)(r_0-\overline{r}_j)}+\right.$$
$$\left.\overline{r}_j\left(\frac{r_0-r_j-\overline{r}_j}{r_0-\overline{r}_j}\right)\left(\frac{1-2r_0+r_0\overline{r}_j}{(1-\overline{r}_j)^2}\right)\right].$$

The above discussions lead to

***Theorem*(Stochastic decomposition):** The expected queue length $E(\overline{L})$ can be decomposed into the sum of the expectations of $n+1$ independent random variables as: $E(\overline{L})=E(L)+\sum_{i=1}^{n}E(L_{V_i})$ where $E(L)$

is the queue length of classical $M/M/1$ queue and $\sum_{i=1}^{n} E(L_{V_i})$ is the additional queue length due to n types of vacations.

7.2.2 Stationary waiting time

Using Little's formula the expected waiting time $E(\overline{W}) = \frac{E(L)}{\lambda}$.

$$E(\overline{W}) = (\frac{1}{\mu - \lambda} + \frac{1}{\lambda}\sum_{i=1}^{n} E(L_{V_i})) \tag{7.12}$$

From (7.12) it is clear that the expected waiting time can be decomposed into the sum of $n+1$ independent random variables: $E(\overline{W}) = E(W) + \sum_{i=1}^{n} E(W_{V_i})$. where $E(W)$ is the expected waiting time of a customer in the $M/M/1$ queue and $\sum_{i=1}^{n} E(L_{W_i})$ is the additional waiting time due to n types of vacations.

Chapter 8

On an M/G/1 queue with vacation in random environment

Introduction

In chapters 6 and 7 the service time and vacation time are assumed to follow exponential distribution. In this chapter we consider a single server queueing system with general service time distribution. Normal vacation and working vacation are also considered where both the vacations follow general distribution. The important features of the model discussed in this chapter are

---

Some results of this chapter are included in the following Manuscript. *A.Krishnamoorthy, Jaya.S, B.Lakshmy.* : On an M/G/1 queue with vacation in random environment, (Communicated).

- The server goes for vacation only when the queue becomes empty. i.e. the exhaustive service discipline has been applied.

- Both normal vacation(type I vacation) and multiple working vacation (type II vacation) are considered.

- During normal vacation, if a customer arrives, service is not provided until completion of vacation whereas while in working vacation service is provided at a slower rate.

- At the end of a busy period, depending on the environment, the server opts for normal vacation or working vacation.

- On completion of type I vacation, if the server finds the system empty he goes for type II vacation.

- On completion of type II vacation if the server finds the system empty he goes for another type II vacation.

- On completion of service in type II vacation, if the server finds one or more customers in queue he returns to normal service interrupting the vacation.

- A customer arriving during type I vacation, joins the queue with probability q or leaves the system with probability $1 - q$.

- A customer arriving during type II vacation, joins the queue with probability 1.

8.1 Model description

Consider an $M/G/1$ queue with Poisson arrival of rate λ. Vacation to server starts whenever the system turns empty at a service completion epoch. There are two types of vacations. Depending on the environment the server goes either for type I vacation with probability p_1 or for type II vacation with probability p_2 such that $p_1 + p_2 = 1$. During type I vacation the arriving customer joins the queue with probability q or leaves the system with probability $1 - q$. On completion of type I vacation if the server finds the system empty, he goes for type II vacation. Type II vacation is a working vacation in which a customer on arrival is served at a lower rate if the server is idle during vacation. On completion of type II vacation if the server finds the system empty it again goes for type II vacation. On completion of service in working vacation if the server finds one or more customers in the system it shifts to normal service, interrupting the vacation. Otherwise the server continues the vacation. If the vacation is completed before service completion the service is restarted at normal rate.

The duration of vacations and services follow mutually independent general distributions. The distribution functions that we bring in here and the corresponding density functions are as defined below: Further

Table 8.1:

Operation	Distribution function	PDF	LST	Mean
Normal service	$S(t)$	$s(t)$	$S^*(s)$	$1/\mu$
Vacation service	$S_v(t)$	$s_v(t)$	$S_v^*(s)$	$1/\mu_v$
type I vacation	$V_1(t)$	$v_1(t)$	$V_1^*(s)$	$1/\gamma_1$
type II vacation	$V_2(t)$	$v_2(t)$	$V_2^*(s)$	$1/\gamma_2$

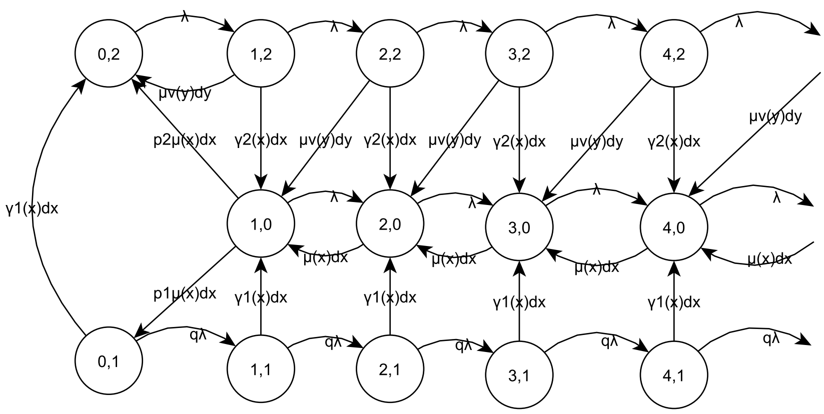

Figure 8.1: Model description

define: $N(t)$ - Number of customers in the system;

$C(t)$ - Status of server;

$S'(t)$ - Elapsed service time of the customer in normal service mode at time t;

$S_v'(t)$ - Elapsed service time in working vacation;

$V_1'(t)$ - Elapsed vacation time duration of type I vacation;

$V_2'(t)$ - Elapsed vacation time duration of type II vacation.

$$C(t) = \begin{cases} 0, & \text{if server is busy with normal service;} \\ 1, & \text{if the server is in type I vacation;} \\ 2, & \text{if the server is in type II vacation;} \end{cases}$$ Now let $P_{n,0}(x,t)dx =$

$P\{N(t) = n, C(t) = 0, x \leq S'(t) < x + dx\}$,

for $t \geq 0, x \geq 0, n \geq 1$

$P_{n,1}(x,t)dx = P\{N(t) = n, C(t) = 1, x \leq V_1'(t) < x + dx\}$,

for $t \geq 0, x \geq 0, n \geq 0$, $P_{n,2}(x,y,t)dxdy = P\{N(t) = n, C(t) = 2, x \leq V_2'(t) < x + dx,$ $y \leq S_v'(t) < y + dy\}$, for $t \geq 0, x > 0, y \geq 0, n \geq 0$

Let $\mu(x), \mu_v(x), \gamma_1(x)$, and $\gamma_2(x)$ be the conditional completion rates of

normal service, vacation service, type I vacation and type II vacation respectively. Then $\mu(x)dx = \frac{dS(x)}{1-S(x)}, \mu_v(x)dx = \frac{dS_v(x)}{1-S_v(x)}, \gamma_1(x)dx = \frac{dV_1(x)}{1-V_1(x)}, \gamma_2(x)dx = \frac{dV_2(x)}{1-V_2(x)}$.
$P_{n,0}(x) = \lim_{t\to\infty} P_{n,0}(x,t), P_{n,1}(x) = \lim_{t\to\infty} P_{n,1}(x,t)$,
$P_{n,2}(x,y) = \lim_{t\to\infty} P_{n,2}(x,y,t)$
Figure.8.1 provides a pictorial representation of the system evolution.
Define $a_k = \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} dS(x)$, $b_k = \int_0^\infty \frac{(q\lambda x)^k}{k!} e^{-q\lambda x} dV_1(x)$,
$c_k = \int_0^\infty \int_0^x \frac{(\lambda x)^k}{k!} e^{-\lambda x} dV_2(y) dS_v(y)$ and $d_k = \int_0^\infty \int_0^y \frac{(\lambda y)^k}{k!} e^{-\lambda y} dS_v(y) dV_2(x)$
where a_k, b_k, c_k and d_k are the probability for k arrivals during normal service, type I vacation, type II vacation and vacation service respectively. The corresponding probability generating functions are
$A(z) = S^*(\lambda(1-z))$, $B(z) = V_1^*(q\lambda(1-z))$,
$C(z) = \int_0^\infty e^{-\lambda(1-z)x} S_v(x) dV_2(x)$ and $D(z) = \int_0^\infty e^{-\lambda(1-z)y} V_2(x) dS_v(y)$.

8.2 Stability of the system

Theorem :*The inequality $\rho = \frac{\lambda}{\mu} < 1$ is necessary and sufficient condition for the system to be stable.*
Proof: Let t_n be the departure time of n^{th} customer from the system after service completion or at the end of a vacation. X_n be the number of customers in the system just after the n^{th} departure, or just at the end of a vacation.

$$X_{n+1} = \begin{cases} X_n - 1 + M_{n+1}, & \text{for } X_n \geq 1 \\ M_n, & \text{for } X_n = 0 \end{cases}$$

where M_{n+1} is the number of arrivals during the service of a customer or during vacation. The arrivals are independent. Then $\{X_n, n \geq 1\}$ is a Markov chain with state space $Z^+ \cup \{0\}$. This Markov chain is irreducible

and aperiodic. Now we have to prove the positive recurrence. For that we use Foster's Criterion.
Foster's Criterion (see Pakes [40]): An irreducible and aperiodic Markov chain is positive recurrent if there exists a non negative function $f(i)$ $i \in Z^+ \cup \{0\}$ and $\epsilon > 0$ such that the mean drift $\psi(i) = E[f(X_n + 1) - f(X_n)/X_n = i]$ is finite for all $i \in Z^+ \cup \{0\}$ and $\psi(i) \leq -\epsilon$ $\forall i$ except for a finite number. Here let us consider $f(s) = s$, $s \in Z^+ \cup \{0\}$
Then the mean drift when $i > 0$ is given by
$\psi(i) = E[f(X_n + 1) - f(X_n)/X_n = i] = \sum_{j=0}^{\infty}(i + j - 1 - i)a_j$

$$= \sum_{j=0}^{\infty}(j-1)a_j = (\rho - 1).$$

When $i = 0$, $\psi(i) = p_1\rho_1 + p_2(\rho_2 + \rho')$ where $\rho = \frac{\lambda}{\mu}, \rho_1 = \frac{q\lambda}{\gamma_1}, \rho_2 = \frac{\lambda}{\gamma_2}, \rho' = \frac{\lambda}{\mu_b}$.
Obviously $\psi(i) \leq -\epsilon$, except for $i = 0$, which is the sufficient condition for ergodicity. The necessary condition follows from Kaplan's condition which states that $\psi(i) < \infty$ $\forall i \in Z^+ \cup \{0\}$ and there exists $j \in Z^+ \cup \{0\}$ such that $\psi(i) \geq 0$ for $i \geq j$.

8.3 Steady state distribution

In the long run when the system stabilizes, let we get the following system of equations satisfied by the probabilities of the system state.

$$\frac{dP_{n,0}(x)}{dx} = -(\mu(x) + \lambda)P_{n,0}(x) + \lambda P_{n-1,0}(x)(1 - \delta_{1n}), n \geq 1. \qquad (8.1)$$

$$\frac{dP_{n,1}(x)}{dx} = -(\gamma_1(x) + q\lambda)P_{n,1}(x) + q\lambda P_{n-1,1}(x), n \geq 1. \quad (8.2)$$

$$\frac{\partial P_{n,2}(x,y)}{\partial x} = -(\gamma_2(x)+\lambda)P_{n,2}(x,y)+\lambda P_{n-1,2}(x,y)(1-\delta_{1n}), n \geq 1. \quad (8.3)$$

$$\frac{\partial P_{n,2}(x,y)}{\partial y} = -(\mu_v(y)+\lambda)P_{n,2}(x,y)+\lambda P_{n-1,2}(x,y)(1-\delta_{1n}), n \geq 1. \quad (8.4)$$

The steady state boundary conditions at $x = 0$ and $y = 0$ are

$$P_{n,0}(0) = \int_0^\infty P_{n,1}(x)\gamma_1(x)dx + \int_0^\infty P_{n+1,0}(x)\mu(x)dx$$

$$+ \int_0^\infty P_{n+1,2}(x,y)\mu_v(y)dy + \int_0^\infty P_{n,2}(x,y)\gamma_2(x)dx, n \geq 1. \quad (8.5)$$

$$P_{0,1}(0) = p_1 \int_0^\infty P_{10}(x)\mu(x)dx. \quad (8.6)$$

$$P_{0,2}(0) = p_2 \int_0^\infty P_{1,0}(x)\mu(x)dx + \int_0^\infty P_{0,1}(x)\gamma_1(x)dx. \quad (8.7)$$

$$P_{1,2}(x,0) = \lambda P_{0,2}(x) \quad (8.8)$$

To solve the system of equations (8.1) – (8.4) , define the following probability generating functions (for $|z| < 1$):
$P_0(x,z) = \sum_{n=1}^\infty P_{n,0}(x)z^n$, $P_1(x,z) = \sum_{n=0}^\infty P_{n,1}(x)z^n$, $P_2(x,y,z) = \sum_{n=1}^\infty P_{n,2}(x,y)z^n$
Multiplying equations (8.1) – (8.4) byz^n and summing over n we get

$$\frac{\partial P_0(x,z)}{\partial x} = -[\lambda(1-z) + \mu(x)]P_0(x,z) \quad (8.9)$$

$$\frac{\partial P_1(x,z)}{\partial x} = -[q\lambda(1-z) + \gamma_1(x)]P_1(x,z) \quad (8.10)$$

$$\frac{\partial P_2(x,y,z)}{\partial x} = -[\lambda(1-z) + \gamma_2(x)]P_2(x,y,z) \quad (8.11)$$

$$\frac{\partial P_2(x,y,z)}{\partial y} = -[\lambda(1-z)+\mu_v(y)]P_2(x,y,z). \tag{8.12}$$

Solving (8.9) and (8.10) we get

$$P_0(x,z) = P_0(0,z)(1-S(x))e^{-\lambda(1-z)x}. \tag{8.13}$$

$$P_1(x,z) = P_1(0,z)(1-V_1(x))e^{-q\lambda(1-z)x} = P_{0,1}(0)(1-V_1(x))e^{-q\lambda(1-z)x}. \tag{8.14}$$

Solving (8.11) and (8.12) we obtain

$$P_2(x,y,z) = P_{0,2}(0)(1-V_2(x))(1-S_v(y))e^{-\lambda(1-z)(x+y)}. \tag{8.15}$$

Now

$$P_{n,0}(x) = \sum_{i=1}^{n} P_{i,0}(0)\frac{(\lambda x)^{n-i}e^{-\lambda x}}{(n-i)!}[1-S(x)]. \tag{8.16}$$

$$P_{n,1}(x) = P_{0,1}(0)\frac{(q\lambda x)^{n}e^{-q\lambda x}}{n!}[1-V_1(x)]. \tag{8.17}$$

$$P_{n,2}(x,y) = P_{0,2}(0)\frac{(\lambda y)^{n-1}e^{-\lambda y}}{(n-1)!}[1-S_v(y)][1-V_2(x)]. \tag{8.18}$$

Solving (8.6) and (8.7) using (8.16) we get

$$P_{0,1}(0) = p_1 P_{1,0}(0)S^*(\lambda) \tag{8.19}$$

$$P_{0,2}(0) = [p_2 + p_1 V_1^*(\lambda)]P_{1,0}(0)S^*(\lambda) \tag{8.20}$$

Using the boundary condition we can write $P\Delta = P$
where $P = (P_{0,1}(0), P_{0,2}(0), P_{1,0}(0), P_{2,0}(0), \ldots)$ and

$$\Delta = \begin{bmatrix} 0 & b_0 & b_1 & b_2 & b_3 & \cdots \\ 0 & c_0+d_0 & c_1+d_1 & c_2+d_2 & c_3+d_3 & \cdots \\ p_1a_0 & p_2a_0 & a_1 & a_2 & a_3 & \cdots \\ & & a_0 & a_1 & a_2 & \ddots \\ & & & a_0 & a_1 & \ddots \\ & & & & a_0 & \ddots \\ & & & & & \ddots \\ & & & & & \end{bmatrix}$$

.

It is clear that the matrix Δ is irreducible. It is stochastic since $\sum_{k=0}^{\infty} b_k = B(1) = 1$, $\sum_{k=0}^{\infty} c_k + d_k = C(1) + D(1) = 1$ and $\sum_{k=1}^{\infty} a_k + (p_1 + p_2)a_0 = A(1) = 1$.

Now we have to prove that Δ is positive recurrent when $\rho < 1$. Δ is positive recurrent when $\sum_{k=1}^{\infty} ka_k < 1$, and this condition is satisfied when $\rho < 1$.

From the matrix Δ,

$$P_{0,1}(0) = P_{1,0}(0)p_1a_0. \tag{8.21}$$

$$P_{0,2}(0) = P_{0,1}(0)b_0 + P_{0,2}(0)(c_0 + d_0) + P_{1,0}(0)p_2a_0. \tag{8.22}$$

$$P_{j,0}(0) = P_{0,1}(0)b_j + P_{0,2}(0)(c_j + d_j) + \sum_{i=0}^{j} P_{i+1,0}(0)a_{j-i}. \tag{8.23}$$

From (8.23),

$$P_0(0, z) = z\frac{[P_{0,1}(0)(B(z) - 1) + P_{0,2}(0)(C(z) + D(z) - 1)]}{z - A(z)}. \tag{8.24}$$

From (8.13),

$$P_0(z) = P_0(0,z)\frac{1 - S^*(\lambda(1-z))}{(\lambda(1-z))}. \tag{8.25}$$

From (8.14),

$$P_1(z) = p_1 P_{1,0}(0) S^*(\lambda)\frac{1 - V_1^*(q\lambda(1-z))}{(q\lambda(1-z))}. \tag{8.26}$$

From (8.15),

$$P_2(z) = [p_2 + p_1 V_1^*(\lambda)] P_{1,0}(0) S^*(\lambda)\Omega(z), \tag{8.27}$$

where $\Omega(z) = \int_0^\infty \int_0^x (1 - V_2(x))(1 - S_V(y)) e^{-\lambda(1-z)(x+y)} dxdy.$
Let, $P(z) = P_0(z) + P_1(z) + P_2(z)$ be the PGF of the stationary queue size distribution irrespective of the server's state.
Then,
$P(z) = P_0(0,z)\frac{1-S^*(\lambda(1-z))}{(\lambda(1-z))} + p_1 P_{1,0}(0) S^*(\lambda)\frac{1-V_1^*(q\lambda(1-z))}{(q\lambda(1-z))} +$
$[p_2 + p_1 V_1^*(\lambda)] P_{1,0}(0) S^*(\lambda)\Omega(z)$. Using the condition P(1)=1, we get
$P_{1,0}(0) = \left[\left(\frac{p_1\rho_1 + (p_2 + p_1 V_1^*(q\lambda))(\rho_2+\rho_3)}{\mu-\lambda} + \frac{p_1}{\gamma_1} + (p_2 + p_1 V_1^*(q\lambda))\Omega(1)\right) S^*(\lambda)\right]^{-1}$
The expected queue length $E(L) = P'(z)|z = 1$
$= \frac{1}{2}(2P_0'(0,1)S^{*'}(0) + \lambda P_0(0,1)S^{*''}(0) - P_0''(0,1)S^{*'}(0)) +$
$\frac{q}{2}\lambda P_{0,1}(0)V_1^{*''}(0) + P_{0,2}(0)\Omega'(1)$

8.4 Waiting time Analysis

To find the waiting time of a customer who joins for service at time t, we have to consider different possibilities depending on the status of server at that time . The server may be in general busy period, vacation I or in vacation II. Let $W(t)$ be the waiting time of a customer who arrives at time t and $W^*(s)$ be the corresponding LST.

Case1. The customer arrives to the system when the number of customers is 0 and the server is in vacation. It may be either in vacation I or vacation II. If it is in vacation II the customer starts getting service immediately and the waiting time is zero. Let $W^*_{0,2}(s)$ be the corresponding LST. Then

$$W^*_{0,2}(s) = 1.$$

If the server is on vacation I, the customer has to wait till the completion of vacation. Let x be the elapsed vacation time until the arrival of the customer and $W^*_{01}(s)$ be the LST of the waiting time of the customer who arrives when the system is empty and the server in vacation I. Then

$$W^*_{0,1}(s) = \int_0^\infty e^{-st} \frac{dV_1(x+t)}{1 - V_1(x)}.$$

Case 2. The waiting time of the customer who arrives to the system when there are n customers in the system and the server is providing normal service to customer is the sum of the remaining service time of the customer in service and the service time of the remaining $n-1$ customers. Let x be the elapsed service time of the customer in service and $W^*_{n,0}(s)$ be the LST of the waiting time of the customer who arrives to the system when there are n customers and the server is busy. Then

$$W^*_{n,0}(s) = S^{*(n-1)}(s) \int_0^\infty e^{-st} \frac{dS(x+t)}{1 - S(x)}.$$

Case 3. The waiting time of the customer who arrives to the system when there are n customers in the system and the server is is in vacation I is the sum of the remaining vacation time and the service time of the remaining n customers. Let x be the elapsed vacation time and $W^*_{n,1}(s)$ be the LST of the waiting time of the customer who arrives to the system when there are n customers and the server is in vacation I. Then

$$W^*_{n,1}(s) = S^{*(n)}(s)\int_0^\infty e^{-st}\frac{dV_1(x+t)}{1-V_1(x)}.$$

Case 4. The waiting time of the customer who arrives to the system when there are n customers in the system and the server is in vacation II is the sum of the remaining vacation time and the service time of the remaining n customers, if the vacation completes before the service given while in vacation. If the service is completed before vacation completion then the waiting time is the sum of the remaining vacation service time and the service time of the remaining $n-1$ customers. Let x be the elapsed vacation time, y be the elapsed vacation service time and $W^*_{n,2}(s)$ be the LST of the waiting time when vacation is completed before service and $W^{*'}_{n,2}(s)$ be the LST of the waiting time of the customer when service is completed before vacation of the customer who arrives to the system when there are n customers and the server is on vacation II. Then

$$W^*_{n,2}(s) = S^{*(n)}(s)\int_0^\infty e^{-st}\frac{dV_2(x+t)}{1-V_2(x)}.$$

$$W^{*'}_{n,2}(s) = S^{*(n-1)}(s)\int_0^\infty e^{-st}\frac{dS_v(y+t)}{1-S_v(y)}.$$

$$W^*(s) = p_1\int_0^\infty P_{0,1}(x)dx\int_0^\infty e^{-st}\frac{dV_1(x+t)}{1-V_1(x)} + p_2\int_0^\infty P_{0,2}(x,0)dx$$
$$+\sum_{n=1}^{\infty} S^{*(n-1)}(s)\int_0^\infty P_{n,0}(x)dx\int_0^\infty e^{-st}\frac{dS(x+t)}{1-S(x)}+$$

$$+p_1 \sum_{n=1}^{\infty} S^{*(n)}(s) \int_0^{\infty} P_{n,1}(x)dx \int_0^{\infty} e^{-st} \frac{dV_1(x+t)}{1-V_1(x)}$$

$$+p_2 \sum_{n=1}^{\infty} S^{*(n)}(s) \int_0^{\infty} \int_0^{x} P_{n,2}(x,y)dxdy \int_0^{\infty} e^{-st} \frac{dV_2(x+t)}{1-V_2(x)}$$

$$+p_2 \textstyle\sum_{n=1}^{\infty} S^{*(n-1)}(s) \int_0^{\infty} \int_0^{y} P_{n,2}(x,y)dxdy \int_0^{\infty} e^{-st} \frac{dS_v(y+t)}{1-S_v(y)}.$$

8.5 Numerical results

In this section we provide some numerical examples for this model. Assume normal service time is exponentially distributed with parameter μ, vacation service time is exponentially distributed with parameter μ_v, duration of type I vacation is exponentially distributed with parameter γ_1 and that of type II vacation is exponentially distributed with parameter γ_2.

The conclusion drawn below are purely based on input parameters.

The variation in queue length due to the variation in vacation service rate and arrival rate

Let $\mu = 5$, $\gamma_1 = 0.4$, $\gamma_2 = 0.3$, $p_1 = 0.7$, $p_2 = 0.3$ $q = 0.5$. Fig 8.2 and Fig 8.3 represent the variation in queue length due to the variation in vacation service rate and arrival rate when $p_1 = 0.7$, $p_2 = 0.3$ and $p_1 = 0.3$, $p_2 = 0.7$, respectively. As the value of vacation service rate increases the expected queue length decreases and as the arrival rate increases the queue length also increases which are on expected lines. From Figure 8.2 we not that when the probability of opting for type I vacation decreases the expected queue length decreases.

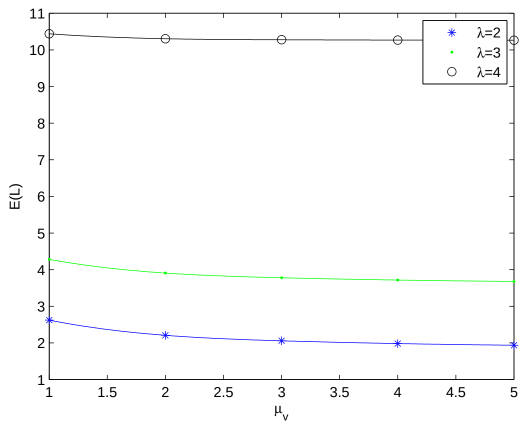

Figure 8.2: The variation in queue length due to the variation in vacation service rate and arrival rate $p_1 = 0.7$, $p_2 = 0.3$

The variation in queue length due to the variation in vacation service rate and duration of vacation

Let $\lambda = 4$, $\gamma_2 = 0.05$, $p_1 = 0.3$, $p_2 = 0.7$. Fig 8.4 and Fig 8.5 represent the variation in queue length due to the variation in vacation service rate and vacation duration when $q = 0.5$ and $q = 0.2$ respectively. As the duration of vacation decreases the queue length decreases. This is due to the early return of server from vacation. When the server returns early from vacation the customer starts getting service earlier and the length of the queue reduces. When the probability of a customer joining the queue

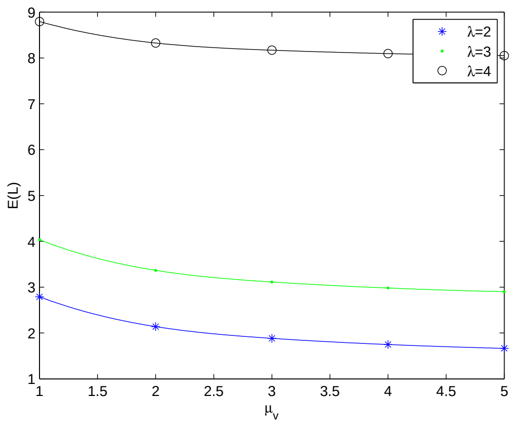

Figure 8.3: Expected queue length $E(L)$ against vacation service rate μ_v

during vacation I reduces the queue length also decreases which are on expected lines.

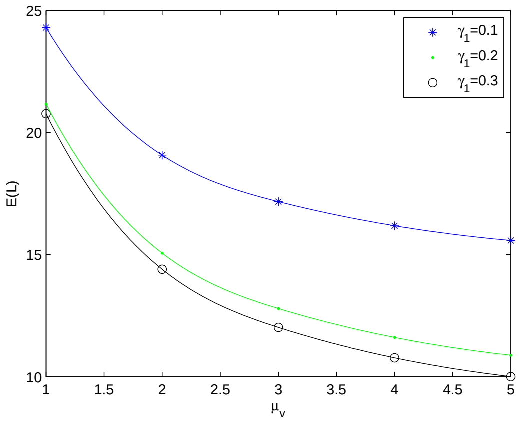

Figure 8.4: Expected queue length $E(L)$ against vacation service rate μ_v, $q = 0.5$

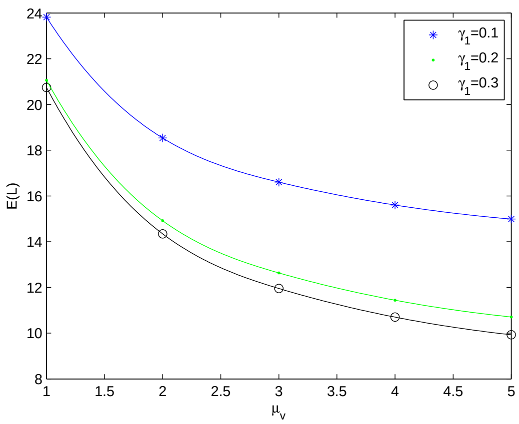

Figure 8.5: Expected queue length $E(L)$ against vacation service rate μ_v, $q = 0.2$

Concluding remarks and suggestions for further study

In this thesis we have introduced and studied the notion of environment dependent server interruption and server vacation in queueing systems. Both random environment and Markovian environment are considered. Queueing systems with customer and server induced interruption have been extensively discussed in literature. In all cases the interruption is induced by some factors. These factors are called environmental factors. Sometimes these factors are interrelated. In this work we study different queueing models with environment dependent interruption and vacation. In the second chapter we analyzed a queueing model with interruption due to a finite number of environmental factors in which the interruption remains unidentified until a random amount of time elapses. The interruption is controlled by two clocks. In chapter 3 all the assumptions are same as in the second chapter except the interruption is inducing environmental factors are the states of a Markovian chain. In chapter 4 we have studied a queueing model with partially ignored interruption in Markovian environment. We introduced two clocks in the model to determine whether to resume or restart the service. Then we proceeded to a queueing model with totally ignored interruption (Chapter 5). We introduced the notion of self correction in this chapter.

Chapter 6 - chapter 8 discuss queueing models with environment dependent vacation. In chapter 6 we considered a queueing model with $n+1$ types of environment dependent vacations. The vacations are taken at the end of a nonzero busy period. We derived a formula to calculate the expected queue length and expected waiting time. In the 7^{th} chapter we obtained stochastic decomposition of expected queue length and expected waiting time of an $M/M/1$ queue with environment dependent working vacation. In the last chapter (chapter 8) we considered an $M/G/1$ queue with two types of vacation - normal vacation and working vacation. The type of vacation the server selects after service, is based on the environment. Since the models are not analytically tractable, a large number of numerical illustrations were given in each chapter to illustrate the working of the systems.

Extensions of the work reported in the thesis to the case of arbitrary distribution especially those in chapters 2-7 is being taken up. As a first step we replace exponential distribution by phase type distribution for service and vacation and go for Markovian arrival process.

Bibliography

[1] Awi Federgruen and Linda Green(1986): Queueing Systems with Service Interruptions, Operations Research, 34(5), 752-768.

[2] Bellman, R. (1960): Introduction to Matrix Analysis, McGraw Hill book Co.,New York.

[3] Bhaskar Sengupta(1990): A Queue with Service Interruptions in an Alternating Random Environment, Operations Research,38(2), 308-318.

[4] Breuer. L. and Baum. D. (2005) : An introduction to queueing theory and matrix analytic methods, Springer, The Netherlands.

[5] Doshi.B.T(1986): Single-server queues with vacations-A survey, Queueing syst. 1 29-66.

[6] Doshi. B.T. (1990) : Single server queues with vacations. In Stochastic Analysis of Computer and Communication Systems, H. Takagi (editor), 217-265, Elsevier Science Publishers B.V. (North-Holland), Amsterdam.

[7] Dudin. A.N, Varghese Jacob, and Krishnamoorthy. A. (2013): A multiserver queueing system with customer induced interruption, partial

protection and repetition of service, Annals of Operations Research, Springer, DOI 10.1007/s10479-013-1318-3.

[8] Erlang, A.K(1909) : The theory of probabilities and telephone conversations, Nyt Tidsskrift Matematik, B.20, pp.139153, 33-39.

[9] Fiems.D, Maertens.T and Brunee.H(2008) : Queueing systems with different types of interruptions, European Journal of Operations Research, vol. 188/3, pp. 838-845.

[10] Fuhrmann.S, R.Cooper(1985): Stochastic decompositions in the $M/G/1$ queue with generalized vacations, Oper.Res.321117-1129.

[11] Gaver. D.P. (1962) : A waiting line with interrupted service including priority, Journal of Royal Statistical Society, B24, 73-90.

[12] Gaver. D.P., Jacobs, P.A. and Lathouche, G. (1984) : Finite birth and death models in randomly changing environments, Adv. in Appl.Probab., 16: 715-731.

[13] Gross. D. and Harris, C.M. (1988) : Fundamentals of Queueing Theory, John Wiley and Sons, New York.

[14] Ibe.O. C., Olubukola. A. Isijola (2014): M/M/1 Multiple Vacation Queueing Systems with Differentiated Vacations, Modelling and Simulation in Engineeing, 158247(6).

[15] Ibe.O. C., Olubukola. A. Isijola(2014): M/M/1 Multiple Vacation Queueing Systems with Differentiated Vacations and vacation interruption, IEEE Access, 10.1109/Access.2012372571.

[16] Jacob. V, Chakravarthy. S.R. and Krishnamoorthy. A(2012): On a Customer Induced Interruption in a service system, Stochastic Anal. Appl. 30, No. 6, 949-962.

[17] Karlin. S. and Taylor. H.M. (1975) : A first course in Stochastic Processes, Academic press, Newyork.

[18] Karlin. S. and Taylor. H.M. (1981) : A second course in Stochastic Processes, Academic press, Newyork.

[19] Ke.Jau-Chuan, Chia-Huang Wu, Z.G.Zhang(2010): Recent Developments in vacation Queueing Models: A Short Survey, International Journal of Operational Research, 7, 4, 3-8.

[20] Kim.J.D, D.W.Choi, K.C. Chae(2008): Analysis of queue-length distribution of the $M/G/1$ queue with working vacations, Hawaii International Conference on Statistics and Related Fields, 5-8 June.

[21] Kishor S.Trivedi: Probability & Statistics with Reliability, Queueing and Computer Science Applications, Wiley-india.

[22] Krishnamoorthy. A, Pramod. P.K, and Deepak. T.G. (2009) On a queue with interruptions and repeat/resumption of service: Nonlinear Analysis: Theory, Methods and Applications 71 (12), 1673-1683.

[23] Krishnamoorthy.A, Gopakumar. B., and Narayanan. V. C. (2009): A Queueing Model with interruption resumption/restart and reneging: Bulletin of Kerala Mathematics Association (Special Issue with Guest Editor: S.R.S. Varadhan FRS.) 29 -45.

[24] Krishnamoorthy. A, Pramod. P.K, and Chakravarthy. S.R. (2014): Queues with interruption, A Survey, TOP, 22, 290-320.

[25] Krishnamoorthy. A. Gopakumar. B, Viswanath. C.N (2012): A retrial queue with server interruptions, resume and restart of service, Operational Research 12(2):133-149.

[26] Krishnamoorthy. A. and Varghese Jacob (2012): Analysis of Customer Induced Interruption in a multi server system: Journal of Neural, Parallel and Scientific Computations, Vol. 20, Issue 2.

[27] Krishnamoorthy. A, Pramod.P.K, Chakravarthy.S.R (2013): A note on characterizing service interruptions with phase type distribution, Stochastic Analysis and Applications 31(4), 671-683.

[28] Latouche. G. and Ramaswami. V. (1993) : A logarithmic reduction algorithm for quasi-birth-and-death processes, Journal of Applied Probability,30, 650-674.

[29] Latouche. G. and Ramaswami. V. (1999) : Introduction to Matrix Analytic Methods in Stochastic Modeling, SIAM., Philadelphia, PA.

[30] Y. Levy, U. Yechiali(1975): Utilization of idle time in an M/G/1 queueing system, Manag. Sci., 22, 202-211.

[31] Li.J, N.Tian(2007): Discrete time GI/Geo/1 queue with working vacation and vacation interruption, Appl.Math.Comput.1851-10.

[32] Li.W, X.Xu, N.Tian(2007): Stochastic decompositions in The $M/M/1$ queue with working vacations, Oper.Res.Lett. 35, 595-600.

[33] Li.J, N.Tian, Z.Ma(2008): Performance analysis of GI/M/1 queue with working vacations and vacation interruption, Applied Mathematical Modelling, 32, 2715-2730.

[34] Li.J, N.Tian(2008): Analysis of the discrete time Geo/Geo/1 queue with single working vacation, Quality Technology and Quantitative Management,5(1)77-89.

[35] Li.J, N. Tian, Z. Zhang, H. Luh(2009): Analysis of the M/G/1 queue with exponentially working vacations-a matrix analytic approach, Queueing Syst. 61, 139-166.

[36] Li.J, N.Tian(2014): The M/M/1 queue with working vacations and vacation interruptions, J Syst Sci Syst Eng,16(1), 121-127.

[37] Medhi. J. (1984) : Stochastic Processes, New age international, New Delhi.

[38] Medhi. J. (2003) : Stochastic models in queueing theory, Academic press, An imprint of Elsevier, USA.

[39] Neuts M.F(1981): Matrix-geometric solutions in stochastic models-an Algorithmic Approach. The Johns Hopkins University Press, Baltimore.

[40] Pakes.A.G(1969): Some conditions for ergodicity and recurrence of Markov chains, Operations Research, 17, 1058-1061.

[41] Servi.L.D, S.G. Finn (2002): M/M/1 queues with working vacations (M/M/1/WV), Perform. Eval. 50 41-52.

[42] Shanthikumar.J.G(1988): On stochastic decomposition in M/G/1 type queues with generalized server vacations, Oper.Res. 36 566-569.

[43] Sheldon.M.Ross(2007): Introduction to Probability Models, Accademic Press, Lendon.

[44] Sreenivasan.C, A. Krishnamoorthy(2012): An M/M/2 Queueing System with Heterogeneous Servers Including One with Working Vacation, International Journal of Stochastic Analysis, Volume 2012 , Article ID 145867.

[45] Sreenivasan.C, S. R. Chakravarthy, A. Krishnamoorthy(2013): MAP/PH/1 queue with working vacations, vacation interruptions and N policy, Applied Mathematical Modelling, 37, 6, 3879 - 3893.

[46] Takagi.H(1991): Queueing Analysis: A Foundation of Performance Analysis, Volume 1: Vacation and Priority Systems, Part 1, Elsevier Science Publishers B.V., Amsterdam.

[47] Tian.N, G. Zhang(2006): Vacation Queueing Models: Theory and Applications, Springer Verlag, New York, .

[48] White.H, and L. Christie (1958): Queueing with preemptive priorities or with breakdown, Operations Research, 1.6, 79-95.

[49] Wu.D.A, H. Takagi(2003): M/G/1 queues with multiple working vacation, Proceedings of the Queueing Symposium, Stochastic Models and their Applications, Kakegawa, 51-60.

[50] Wu. D.A, H. Takagi(2006): M/G/1 queue with multiple working vacations, Perform. Eval. 63, 654-681.

[51] Zhang, Z. Hou(2010): Performance analysis of $M/G/1$ queue with working vacations and vacation interruption, Journal of Computational and Applied Mathematics, 234, 2977-2985.mic Publishers, Boston.

List of Papers accepted/communicated

- *A.Krishnamoorthy, Jaya.S, B.Lakshmy.* :Queue with interruption in random environment. *Annals of Operations research, Springer,(Accepted for Publication).*

- *A.Krishnamoorthy, Jaya.S, B.Lakshmy.* :On an M/G/1 queue with vacation in random environment, *Communicated.*

Papers presented/accepted for presentation

- ***A.Krishnamoorthy, Jaya.S,B.Lakshmy.*** *: Queue with interruption in random environment, presented in the* 8^{th} *International Conference on Matrix Analytic Methods, January 6-10,2014,NIT Calicut.*