SOME CONSERVATION LAWS OF FLUID MECHANICS

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STATEMENT

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

(THOMAS JOSEPH)

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Chapter 1

INTRODUCTION

The governing equations of fluid mechanics are based on a set of conservation laws. These are conservation laws of mass, momentum, angular momentum, energy etc. A general approach is developed for the derivation of conservation laws in continnum physics by the pioneers like d'Alembert, Euler, Daniel Bernoulli, and Lagrange. The kinematics of inviscid fluids bring out further conservation laws like Kelvin's circulation theorem, Helmholtz vorticity theorems, conservations of potential vorticity and helicity. These conservation laws are derived based on the dynamical equations.

A fluid flow can be considered as an infinite dimensional dynamical system with infinite degrees of freedom. In this sense the conservation laws are of great significance as integrals of equations of motion.

As is well-known there are two approaches to the study of fluid mechanics - Eulerian and Lagrangian, though both are really due to Euler. The classical study of hydrodynamics and later of real fluids and boundary layer theory are mainly in the

Eulerian frame work because of its simplicity. Especially in steady flows the Eulerian method greatly reduces the complexity of the governing equations, while such simplifications are not possible by Lagrangian method. Thus the faster developments in fluid mechanics were using Eulerian method, which is a field theoretic approach. On the other hand the Lagrangian method which is in the way of particle dynamics has found place in recent studies. Especially using a variational approach the Lagrangian method is almost straightforward. This simplicity consists in the adaptation of Hamilton's principle of least action to a mechanical system with infinite degrees of freedom. A variational formulation in the Eulerian system is not that simple. For example in the early literature, the only flow with vorticity amenable to Eulerian treatment was found to be isentropic.

One of the assumptions made in the early studies of hydrodynamics, especially in water waves, is that the flow is irrotational. This has its root in the well-known theorem that all motions started from rest by natural forces are irrotational. It is to be noted that the only exact solution for a rotational wave is trochoidal waves of Gerstner which has not gained much attention in the past more than a century. In fact the simplifying assumption of irrotational flow drastically alters the problem. For example, in an incompressible fluid if the flow is irrotational everywhere the fluid really ceases to be a fluid in the sense that it looses its infinite number of degrees of freedom which make possible the infinite variety of fluid motions and becomes a flexible extension of the bodies whose movements generate the flow. The irrelevance of classical hydrodynamics to the real world is summarised by the d'Alembert's paradox of zero drag in steady flow. At least for the motion of homogeneous incompressible fluids vorticity is the property of the flow field of crucial importance and it is not an exaggeration to say that all the problems of such flows can be posed as questions about the strength and location of the vorticity.

The conservation laws studied in this thesis are of voticity, potential vorticity and helicity. It is an attempt to give a formal presentation of derivation of conservation laws. There are two approaches: one is the classical method based on governing differential equations, the second method is based on variational principles on a formal footing using Lie group theory.

1.1 Vorticity:

The credit for the creation and unification of the discipline of vorticity transport goes to Truesdell (1953). The great significance of vorticity is aptly and beautifully recorded by Truesdell (1953) in the following words: "Before our eyes opens forth now the splendid prospect of three dimensional kinematics, the mother tongue for man's perception of the changing world about him. Its peculiar and characteristic glory is the vorticity vector $\bar{\omega}$, for whose existence it is both requisite and sufficient that the number of dimensions be three".

The earliest concept of vorticity can be traced back at least to Leonardo da Vinci and Descartes, though the first treatment of vorticity occurs in the work of d'Alembert and Euler; Lagrange and Cauchy were the first to introduce single letters to stand for the vorticity components. The kinematical significance of vorticity was recognised only when Mac Cullagh and Cauchy proved that the components of the curl (of the velocity vector) satisfy the vectorial law of transformation (see Truesdell (1953), pp 59 -.).

The mathematical understanding of vortex motions begins with the renowned work of Helmholtz (1858). The name "vorticity" was introduced by Lamb (1916). The original physical problem that motivated Helmholtz's great study, was nonlinear - the study of motions of an ideal incompressible fluid governed by Euler's equations. He found interesting invariance properties of a dual analytic and topological nature for the vorticity vector. The purely mathematical aspects of Helmholtz's ideas have been developed into the modern Hodge-Kodaira decomposition theorem for differential forms on Riemannian manifolds (Berger (1982)).

The name "irrotational", for flows with vanishing vorticity was introduced by Kelvin (1869). Both Euler and Lagrange repeatedly emphasized that irrotational motions constituted only a special case, while d'Alembert contented in effect that all motions of inviscid incompressible fluids are irrotational. Again in the words of Truesdell (1953), vorticity "generates those beautiful, intricate and perplexing phenomena which makes the challenge of the theory of the motion of fluids, whether perfect, viscous, of more complicated in their dynamical response - a challenge for the most part declined by classical hydrodynamics - and that analysis of the basic kinematical properties of vorticity initiates a frontal attack upon the citadel of the nonlinear convective acceleration". Truesdell's work, though published fourty years back points out to many hitherto unresolved problems of the kinematics of voticity.

Vorticity has provided a powerful qualitative description for many of the important phenomena of fluid mechanics. The formation and separation of boundary layers have been described in terms of the production, convection and diffusion of vorticity. In turbulent flows the dissipation of energy at a rate independent of viscosity is explained by the amplification of vorticity by the stretching of vortex lines. The lift on an air wing is explained by the bound vorticity and trailing vortex structure. The concept of coherent structures in turbulent shear flows has led to the picture of such flows as a superposition of organised deterministic vortices whose evolution and interaction is the turbulence. The strong nonlinearity of the equations of vortex motion has made quantitative use of the concept difficult for the great scientist who founded and developed the subject. But the advent of high speed computers has made it possible to attack many of the difficult problems, leading to the realization that difficult problems like almost irrotational motions such as surface waves on uniform irrotational fluid and air bubbles in water may be successfully treated as problems of vortex motion, the free surface of water waves or the boundary of air bubble being a vortex sheet whose position satisfies an integro-differential equation.

1.2 Potential Vorticity:

It is well-known that Kelvin's circulation theorem, Helmholtz vorticity theorems and Euler's equations of motion are equivalent (Von Mises and Friedrichs (1971)). But Kelvin's theorem is an integral theorem and requires a knowledge of detailed evolution of material surfaces in the fluid. The vorticity equation, though deals directly with the vector character of vorticity, is more a description of how vorticity is changed than a useful constraint on that change. Consideration of potential vorticity due to Ertel (1942) provides the way of translating the informations in the Kelvin's circulation theorem into local conservation laws. Thus it was shown that the quantity

$$\Pi = \frac{\bar{\omega}}{\rho} \nabla S,$$

where S is the specific entropy and ρ density, is conserved under the following conditions.

a) S is conserved for each fluid element, i.e., the flow is isentropic

b) The fluid is inviscid.

c) The flow is barotropic.

In fact S need not be entropy, but any conserved property of fluid element (Mobbs (1981), Pedlosky (1979), Gaffet (1985), Abarbanel (1987), Katz (1984),Katz and Lynden-Bell (1982)). It can be shown that potential vorticity is really Kelvin's circulation theorem for a very special but useful contour - a contour lying on a S-surface. The term potential vorticity may seem to be a misnomer since this entity does not have even the dimensions of vorticity. But if the density does not vary very much, as the distance between two adjacent S-surfaces increases, ∇S must decrease and the component of the vector $\bar{\omega}$ parallel to ∇S must increase proportionally if the potential voticity must remain a constant. This will be manifested as an increase in $\bar{\omega}$ and we may consider that there is a reservoir of vorticity associated with the packing together of S-surfaces which can be released as the surfaces are stretched apart by the mechanism of vortex tube stretching (Pedlosky (1979)).

Ertel's theorem associates a conserved quantity with any scalar field quantity that is conserved along fluid particles. Since potential vorticity itself is such a conserved quantity, we can use it again and again to obtain an infinite number of conserved quantities, a fact which sheds light on the integrability of the Euler equations. But not all of them are of physical significance (Abarbanel (1987)).

1.3 Helicity:

In particle dynamics the term helicity is used for the scalar product of momentum and spin of a particle. This concept has been adapted to fluid dynamics and in particular to magnetohydrodynamics by Moffatt (1978).

Let V_∞ denote the whole three-dimensional space. For any vector field $\bar{A}(x)$ the quantity $\bar{A}.(\nabla x\bar{A})$ is called the helicity density of the field $\bar{A}$. Its integral

$$I_\infty = \int_{V_\infty} \bar{A}.(\nabla x\bar{A})dV,$$

is called the helicity of $\bar{A}$. Let $V_m \subset V_\infty$ be any volume with surface S on which $\hat{n}.(\nabla x\bar{A}) = 0$, $\hat{n}$ being the unit normal to the surface. Then

$$I_m = \int_{V_m} \bar{A}.(\nabla x\bar{A})dV,$$

is called a partial helicity.

The helicity density is a pseudo-scalar quantity, its sign changing from a right-handed to a left-handed frame of reference. Thus a reflectionally symmetric vector field must have zero helicity density, though the converse is not true.

Setting $\bar{B} = \nabla x\bar{A}$ the helicity density becomes $\bar{A}.\bar{B}$. When $\bar{B}$ is magnetic field intensity, $\bar{A}$ can be the vector potential for $\bar{B}$. Since vorticity $\bar{\omega} = \nabla x\bar{u}$, $\bar{u}$ the velocity field of a flow, $\bar{u}.\bar{\omega}$ is the helicity density of the velocity field.

Suppose $\bar{A}.\bar{B} = 0$. This is the necessary and sufficient condition for the existence of scalar functions $\phi(x)$ and $\psi(x)$ such that

$$\bar{A} = \psi\nabla\phi \quad \text{and} \quad \bar{B} \quad \nabla\psi x\nabla\phi$$

It is clear from these that $\bar{B}$-lines are the intersections of the surfaces ϕ = constant and ψ = constant, and the $\bar{A}$-lines are every where orthogonal to the surfaces ϕ = constant. Thus $\bar{B}$-fields having linked or knotted $\bar{B}$-lines cannot admit such a representation. The same arguments can be adapted to fluid mechanics when the velocity field has a Clebsch's representations,

$$\bar{u} = \psi\nabla\phi + \nabla\chi$$

The term "helicity" for helicity of velocity field was first introduced by Moffatt in 1969. A detailed discussion on the invariance and topological interpretation of helicity and its significance in the dynamo theory of celestial magnetic fields and in turbulent flows with and without magnetic fields is given in Moffatt (1976, 1978).

Helicity is a measure of the degree of knottedness of a vector field (Moffatt (1969), Bretherton (1970), Holm et al. (1991)). This can be given a kinematical interpretation as follows: the fluid particles in any small volume element dV undergo a superposition of three motions - a uniform translation, an irrotational uniform strain $\nabla\phi$ and a rigid body rotation. The streamlines of flow $\bar{u} - \nabla\phi$ ($\bar{u}$ - velocity) passing near a point P in dV are helices about the streamline through P. Thus the contribution $\bar{u}.\bar{\omega}\, dV \cong \bar{u}_o \bar{\omega}_o\, dV$ ($\bar{u}_o$ and $\bar{\omega}_o$, velocity and vorticity respectively at P) to the helicity from the volume element dV is positive or negative according as the sense of rotation of these helices is right-handed or left-handed. Arnold (1965a, 1965b) has discussed the importance of helicity (measure of knottedness) as a topological property invariant under volume preserving diffeomorphism.

Moffatt (1969) has derived helicity conservation under the following conditions:

a) The flow is inviscid and barotropic.

b) The body forces are conservative.

c) The volume V is a material volume.

d) $\bar{\omega}.\hat{n} = 0$ on ∂V, the boundary of V.

To make the best use of vortex dynamics, it is important to use as effectively as possible the basic results connecting the vorticity $\bar{\omega}$ and the velocity $\bar{u}$ of fluid element which are provided by helicity conservation. In a number of recent papers the distribution of helicity in turbulent flows has been discussed and computed (Levich and Tsinober (1983), Rogers and Moin (1987), Vallis et al (1989), Holm and Kimura (1991)). Numerical computations such as for viscous flows at high Reynold's numbers are usually interpreted in terms of concepts from vortex dynamics developed for inviscid flows. The helicity density is not a conserved property and in general varies in space and time in turbulent flows. There have been many works reported on computations and measurements of helicity density in various flows (Kida and Murakami (1989)). Since there can be no production of vorticity if $\bar{u}\times\bar{\omega} = 0$; it follows that the production of vorticity and the cascade of energy to smaller scales in turbulence must be very weak if $\bar{u}$ is parallel to $\bar{\omega}$ and helicity density becomes comparable with $(u^2\omega^2)^{1/2}$ (André and Lesieur (1977)).

In two dimensional flow, the vortex lines are normal to the plane of flow and therefore the question of knottedness of vortex lines does not arise. Thus helicity is a property possessed by three dimensional flows only indicating the complexity of the flows and may also indicate something about its mixing property.

Non-zero helicity is a necessary condition for non-integrable particle motion, which includes chaotic particle path. Holm and Kimura (1991) have investigated integrable and non-integrable particle motion for three dimensional incompressible flows with zero helicity. Computational techniques that preserve volume and helicity are developed and used to visualise the Lagrangian particle trajectories of three dimensional motion in a periodic domain. A new class of steady solution of the Euler equation for the axisymmetric flow with non-zero helicity of an incompressible inviscid fluid has been obtained by Turkington (1989). Hunt and Hussain (1991) have shown that a net contribution to the partial helicity integral is generated outside fluid volumes as they move through regions of fluid with background vorticity. A calculation is given for the helicity generated outside a spherical volume as it moves through a region of weakly rotating flow.

The concept of helicity and its conservation has been extended to non-barotropic flows by Mobbs (1981), Gaffet (1985), and Joseph and Mathew (1990).

1.4 Conservation Laws:

Conservation laws are of great significance in continuum mechanics. They have been applied to study stability of fluid flows. In elasticity, conservation laws are of key importance in the study of cracks and dislocations. They constitute a basic tool in the analysis of systems of partial differential equations also. Lax (1968) has used conservation laws to prove global existence theorems and determined realistic conditions for shock wave solutions to hyperbolic systems.

It is well-known that the mathematical structure for ideal hydrodynamics was developed a long time ago by Clebsch and a complete treatment of the then existing literature can be found in the works of Lamb (1932). Since then these results have found applications in all branches of engineering. In fact the field of applications is expanding with the result that even branches of science like medicine is being benefited from the developments in fluid dynamics. Even then the basic results need further investigations and much work in this direction is remaining. For example the Clebsch's potential finds only a touching reference in the works of Lamb (1932) and much of its applications can be found in the literature of the last two decades. Studying the connection between Eulerian and Lagrangian treatments, Bretherton (1970) has found that Kelvin's circulation theorem is a consequence of particle identity during a flow. This has been extended to non-barotropic flows by Mathew and Vedan (1988). Vorticity and potential vorticity has found applications in geophysical and meteorological studies. Helicity has great significance in the studies of turbulence. The conservation laws of vorticity, potential vorticity and helicity have found applications in developing algorithms for numerical studies in almost all branches of fluid mechanics.

The study of conservation laws is closely related to a variational formulation for fluid mechanics. Attempts for variational formulations of hydrodynamics can be traced back to Bateman (1929), Lichtenstein (1929) and Lamb (1932). Eckart (1938) and Taub (1949) tried to extend the variational principle to adiabatic compressible flows. Using the field variables velocity, density and entropy expressed in space time co-ordinates, Herivel (1955) presented both Lagrangian and Eulerian variational formulation for ideal fluid flows. However his Eulerian variational principle was incomplete in the sense that for isentropic flows this prnciple led to irrotational motion of the fluid only. Numerical procedure for irrotational flow past circular cylinder using variational method can be found in Lush and Chery (1956).

Later Lin (1963) showed that variational principle for rotational flows in Eulerian treatment could be obtained by introducing the requirement that the end points of the particle trajectories should be preassigned and not subjected to variations. The modified version (Herivel-Lin variational formulations) appeared first in an article by Serrin (1959). A generalization of Herivel-Lin variational principle can be found in Lundgren (1963). In this formulation the constraints are taken by means of Lagrangian multipliers (Monge potentials) which lead to a Clebsch's representation of the velocity field involving eight potentials. It is difficult to assign physical meanings to these potentials.

Clebsch's potentials have been used by Ito (1953) for a Hamiltonian formulation of hydrodynamics. He has shown that, for an isentropic flow, vortex motion is closely connected with the entropy. Eulerian hydrodynamical equations emerged factorized into four fundamental equations which were discussed from the thermodynamical point of view. A discussion on the Landau's theory of quantum hydrodynamics also can be found in this paper.

There have been attempts during the last three decades to find new variational techniques in the Eulerian descriptions avoiding the difficulties due to the redundancies and indeterminacies of the Clebsch's potentials.

There is reference to Lin's constraints in Drobot and Rybarski (1959). But this work has not received due attention. In fact their hydromechanical variation replaces the Lin's constraints in a variational formulation for a truly Eulerian approach.

Eckart (1960) used the energy-momentum tensor to derive equations of motion and some conservation laws in Lagrangian description. In this paper he introduced the concept of thermasy and extended Kelvin's circulation theorem to non-barotropic flows. Eckart (1963) studied transformation of Lagrangian equations of hydrodynamics to general coordinates which are useful in stability studies.

Hamilton's principle does not exist for the flows of viscous fluids except in some very restricted cases. A variational principle for steady laminar motion of simple non-newtonian fluids for which the viscosity is a function of the second invariant of the rate of deformation tensor was given by Bird (1960). This principle simplifies to Helmholtz principle for newtonian fluids. In this case the equation of continuity and motion are equivalent to the statement that the rate of entropy production is a minimum. But for more general fluids the variational principle does not admit this simple interpretation. Variational formulation for viscous fluids has been given by Johnson (1960) and Becker (1987). Discussions regarding the non-existence of variational principle in viscous fluid flows can be found in Finlayson (1972a, 1972b) and Mobbs (1982).

A discussion on the drawbacks of many variational principles usuing Eulerian coordinates can be found in Zaslavskii and Perfilev (1969). A four dimensional treatment as in the case of Drobot and Rybarski (1959) can be found in Penfield (1966).

Seliger and Whitham (1968) have studied Eulerian and Lagrangian variational priciples in continuum mechanics and shown that the number of Clebsch's potentials in Eulerian variational formulation of ideal fluid flows can be reduced to four from eight. But Bretherton (1970) has pointed out that though Seliger-Whitham representation is locally valid, in isentropic case the flows determined by such a representation do not include those with non-zero helicity. He has given a detailed discussion on Lin constraints and shown that the relation between the Eulerian and Lagrangian variations of the field variables can be used to derive the equation of motion in Eulerian form from fundamental Lagrangian without using them. The relation between Eulerian and Lagrangian variational principle has been investigated by Bampi and Morro (1982, 1984). The use of Lagrangian multipliers can be seen in Kalikstein (1981) also. Guderly and Bhutani (1973) have suggested a method to derive the variational principle for three dimensional steady flows of compressible fluids from the Herivel-Lin variational formulation for unsteady flows. Wilhelm (1979) has obtained conservation equations for particle density, momentum density and energy density for compressible fluids from a variational principle.

According to Mobbs (1982) the most general variational formulation for invscid fluids is that due to Serrin (1959) in which the Clebsch's potentials are identified with initial coordinates and initial velocity. He has attempted to extend the variational principle to thermally conducting viscous fluids using local potentials. Capriz (1984) has shown that Lin's constraints can be replaced by Eulerian expansion formulas in an Eulerian formulation of ideal fluid flows. Moreau (1981, 1982, 1985) has introduced a new variational technique - method of horizontal variations - to derive Eulerian equations of motion of inviscid non-barotropic flows. Some other variational principles are due to Gouin (1981), Benjamin (1984), Schutz and Sorkin (1977), Loffredo (1989) and Loffredo and Morato (1989).

One of the most important applications of variational formulations is in the study of stability of stationary flows (Lynden-Bell and Katz (1981)). Arnold (1965a, 1965b) has shown that the existence of a variational priciple and an invariant of the flow can be used to study stability. This method has been used for studying stability of barotropic and non-barotropic flows (Grinfeld (1981, 1982, 1984) and Abarbanel and Holm (1987)).

Variational methods have been used in the study of water waves (Luke (1967), Whitham (1967, 1974), Miles (1977) and Milder (1977)).

It is to be noted that the classical studies of kinematics of fluid flows are not based on a variational formulation. The conservation laws are derived from the governing equations. This method has been used by Mobbs (1981) in his generalization of conservation laws to non-barotropic flows. Another method, using Helmholtz fields has been employed by Thyagaraja (1975), Joseph and Mathew (1990) and Mathew (1991). A Jacobian interpretation of helicity using differential forms has been given by Nigam (1988).

1.5 Variational symmetries and conservation laws:

Influenced by the works of Lie (1912) and Klein (1918) on the transformation properties of differential equations under continuous groups of transformations, Noether (1918) proved two fundamental results now known as Noether theorems. This marked the beginning of the study of invariance properties of the action integrals in the calculus of variations. Noether theorems related symmetry groups of a variational integral to properties of its associated Euler-Lagrange equations. But the potentials of Noether theorem were unnoticed for thirty years until Hill (1951) popularised a limited version of these results among the physics community. Since then a number of papers have appeared in the literature either modifying these theorems or applying these theorems to particular dynamical systems by relating familiar conservation laws to transformation groups. In the case of hydrodynamics only a few attempts have been made in these directions. It is to be noted that even in the works of Eckart (1963) and Bretherton (1970), though reference has been made to the relations between transformation groups and conservation laws, nothing is mentioned about Noether theorems. The first reference to Noether's theorem can be found in Drobot and Rybarski (1959). Gouin (1976) has shown that Kelvin's circulation theorem is related to the invariance of the action integrals under certain transformation groups. Moreau (1977) has pointed out that helicity can be obtained as a consequence of Noether's theorem. Mathew and Vedan (1989, 1991) have used Noether's theorems in the study of non-barotropic flows.

1.6 Scope of the thesis:

The present thesis is a study of conservation laws of fluid mechanics in the barotropic and non-barotropic flows.

Chapter 2 deals with non-barotropic generalization of conservation laws of Kelvin's circulation theorem, Helmholtz vorticity theorems, conservations of potential vorticity and helicity. The earlier generalization due to Eckart (1960), Bretherton (1970) and Mobbs (1981) are valid for isentropic flows only, which is a stringent condition in many cases. We relax this condition. The results for isetropic flows are obtained as special cases. In this case it is to be noted that a true generalization of potential vorticity is the one presented here. Two different methods are used: one in the line of Mobbs and second using the concept of Helmholtz fields.

Chapter 3 and 4 present a variational approach to inviscid flows. In chapter 3 the method of Drobot and Rybarski (1959) is developed using Lie group theory. Basic results of Lie group theory necessary for the variational formulation of capter 3 and 4 are presented. Much of the complexities in application of the theory and actual computations are absent due to the functional simplicity of the Lagrangian. As an example we derive the conservation of helicity using Noether's theorems. This chapter concern with barotropic flows.

In chapter 4 the theory is developed for non-barotropic flows. Since the system is underdetermined the investigation of generalized symmetries is required. Noether theorem is used to derive conservation law of potential vorticity.

Chapter 5 presents a general discussion of the results in this thesis and pointed out the direction for further research.

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Chapter 2

GENERALIZED NON-BAROTROPIC FLOWS

2.1 Introduction:

The generalizations of kinematics of fluid flows to non-barotropic flows is brought in by the dependence of the internal energy on specific entropy in addition to specific volume. Eckart (1960) has found a generalized form of Kelvin's circulation theorem which holds for a non-barotropic perfect fluid. Later Mobbs (1981) has shown that this is a particular case of a generalization which can be applied to several vorticity theorems. Joseph and Mathew (1990) and Mathew (1991) have obtained similar conservation laws by identifying the generalized vorticity field with a Helmholtz field. All these results hold for isentropic flows only.

Now we relax the isentroy condition. Following Mobbs (1981), let be an equilibrium thermodynamic function

$$\lambda = \lambda(S,T), \tag{2.1}$$

such that

$$\frac{D\lambda}{Dt} = 0, \tag{2.2}$$

where D/Dt stands for material differentiation.

Now we introduce an enthalpy function I^* such that

$$dI^* = T\,d\lambda + V\,dP. \tag{2.3}$$

This amounts to replacing the state variable entropy S in terms of λ and T and assuming that

$$\left[\frac{\partial T}{\partial P}\right]_{\lambda} = \left[\frac{\partial V}{\partial \lambda}\right]_{P} \tag{2.4}$$

where T the absolute temperature, V the specific volume and P the pressure. From equation (2.3) we get

$$\frac{\nabla P}{\rho} = \nabla I^* - T\nabla\lambda \tag{2.5}$$

Also we have the relation

$$\nabla\lambda = \frac{\partial\lambda}{\partial S}\nabla S + \frac{\partial\lambda}{\partial T}\nabla T. \tag{2.6}$$

The law of conservation of momentum now takes the form,

$$\frac{D\overline{u}}{Dt} + \nabla(I^{*} + \phi) - T\nabla\lambda = 0. \tag{2.7}$$

The mass conservation gives the continuity equation

$$\frac{D\rho}{Dt} + \rho\nabla.\overline{u} = 0. \tag{2.8}$$

In the above equations $\overline{u}$ is the velocity field, P the pressure, ρ the density, S the specific entropy and ϕ the potential energy due to any conservative body forces.

We can now define a barotropic flow as the one in which $\nabla\eta x\nabla\lambda = 0$, where η is thermasy, the time integral of temperature (Schutz and Sorkin 1977) defined by

$$\frac{D\eta}{Dt} = T, \quad \eta = 0 \text{ at } t = 0. \tag{2.9}$$

Because of equation (2.6) this will include the two different usual requirements $\nabla\eta x\nabla S = 0$ and $\nabla S x\nabla T = 0$.

Definition 2.1

A flow in which $\nabla\eta x\nabla\lambda \neq 0$ is called a generalized non-barotropic flow.

2.2 Generalized vorticity equation:

Theorem 2.1

For a generalized non-barotropic flow, the vorticity $\bar{\omega}$ satisfies the equation

$$\frac{D}{Dt}\left[\frac{\bar{\omega} - \nabla\eta x \nabla\lambda}{\rho}\right] = \left[\frac{\bar{\omega} - \nabla\eta x \nabla\lambda}{\rho} .\nabla\right] \bar{u}. \qquad (2.10)$$

Proof:

From equation of continuity (2.8) and equations of motion (2.7) we have the Vazsonyi's vorticity equation

$$\frac{D}{Dt}\left[\frac{\bar{\omega}}{\rho}\right] = \left[\frac{\bar{\omega}}{\rho} .\nabla\right]\bar{u} + \frac{\nabla T x \nabla\lambda}{\rho} \qquad (2.11)$$

for a generalized non-barotropic flow. Also from equation of continuity (2.8), equation (2.9) and the vector identity

$$\nabla\lambda x(\nabla\eta.\nabla)\bar{u} - \nabla\eta x(\nabla\lambda.\nabla)\bar{u} + \nabla\lambda x(\nabla\eta x\bar{\omega}) - \nabla\eta x(\nabla\lambda x\bar{\omega})$$
$$(\nabla\lambda x\nabla\eta)\nabla.\bar{u} - \left[(\nabla\lambda x\nabla\eta).\nabla\right]\bar{u}, \qquad (2.12)$$

we get

$$\frac{D}{Dt}\left[\frac{\nabla\eta\times\nabla\lambda}{\rho}\right] = \left[\frac{\nabla\eta\times\nabla\lambda}{\rho}.\nabla\right]\bar{u} + \frac{\nabla T\times\nabla\lambda}{\rho} \tag{2.13}$$

Subtracting (2.13) from (2.11) we get (2.10).

This completes the proof.

Integrating (2.10), we get

$$\frac{\bar{\omega} - \nabla\eta\times\nabla\lambda}{\rho} = \frac{\bar{\omega}_o}{\rho_o}\,\text{Grad}\,\bar{x}, \tag{2.14}$$

where $\bar{\omega}_o$ and ρ_o are the initial vorticity and density of a fluid particle respectively, $\bar{x}$ is the current position and

$$\text{Grad} = \frac{\partial}{\partial\alpha^i}, \quad \alpha^i \ (i = 1,2,3) \text{ being Lagrangian co-ordinates.}$$

2.3 Generalization of Kelvin's circulation theorem:

Theorem 2.2

In the case of an inviscid generalized non-barotropic fluid flow

$$\frac{D}{Dt}\oint_C (\bar{u} - \eta\nabla\lambda).d\bar{l} = 0, \tag{2.15}$$

where C is a closed curve moving with the fluid.

Proof:

From equation (2.2) we have the identity

$$\frac{D}{Dt}(\nabla\lambda) = -(\nabla\lambda.\nabla)\bar{u} - \nabla\lambda x\bar{\omega}. \tag{2.16}$$

Since C is a material curve

$$\frac{D}{Dt}\oint_C (\bar{u}-\eta\nabla\lambda).d\bar{l} = \oint_C\left[\frac{D\bar{u}}{Dt} - \frac{D\eta}{Dt}\nabla\lambda - \eta\frac{D}{Dt}\left(\nabla\lambda\right)\right].d\bar{l}$$

$$+ \oint_C (\bar{u}-\eta\nabla\lambda).(d\bar{l}.\nabla)\bar{u}. \tag{2.17}$$

From equation (2.17), and on using the identity

$$\bar{u}.(d\bar{l}.\nabla)\bar{u} = \nabla\left[\frac{1}{2}|\bar{u}|^2\right].d\bar{l} \tag{2.18}$$

we get

$$\frac{D}{Dt}\oint_C (\bar{u} - \eta\nabla\lambda).d\bar{l} = -\oint_C \left[\nabla(I^* + \phi) - \eta(\nabla\lambda.\nabla)\bar{u} - \eta\,\nabla\lambda\, x\bar{\omega}\right.$$

$$\left. - \nabla\left[\tfrac{1}{2}|\bar{u}|^2\right]\right].d\bar{l} - \oint_C \eta\,\nabla\lambda.(d\bar{l}.\nabla)\bar{u}. \qquad (2.19)$$

Then using the vector identity

$$-\nabla\lambda.(d\bar{l}.\nabla)\bar{u} + (\nabla\lambda.\nabla)\bar{u}.d\bar{l} = -(\nabla\lambda\, x\bar{\omega}).d\bar{l}, \qquad (2.20)$$

and applying Stoke's theorem we get,

$$\frac{D}{Dt}\oint_C (\bar{u} - \eta\nabla\lambda).d\bar{l} = -\oint_C \nabla\left[I^* + \phi - \tfrac{1}{2}|\bar{u}|^2\right].d\bar{l}$$

$$= 0.$$

This completes the proof.

We call the quantity $\oint_C (\bar{u} - \eta\nabla\lambda).d\bar{l}$ the generalized circulation around C.

Corollary 2.1

$$\frac{D}{Dt}\oint_C \bar{u}.d\bar{l} = \oint_C T d\lambda, \qquad (2.21)$$

which is the generalized form of Bjerknes' theorem.

Proof:

From theorem (2.2) we have,

$$\frac{D}{Dt}\oint_C \bar{u}.d\bar{l} = \frac{D}{Dt}\oint_C (\eta\nabla\lambda).d\bar{l},$$

$$\oint_C \left[\frac{D\eta}{Dt}\nabla\lambda + \eta\frac{D}{Dt}\left[\nabla\lambda\right]\right].d\bar{l} + \oint_C (\eta\nabla\lambda).(d\bar{l}.\nabla)\bar{u},$$

$$= \oint_C T\,\nabla\lambda.d\bar{l},$$

(By equations (2.2), (2.9), (2.16), (2.20))

$$= \oint_C Td\lambda.$$

This completes the proof.

Note:

The particular case $\lambda = S$ gives,

$$\frac{D}{Dt}\oint_C \bar{u}.d\bar{l} = \oint_C TdS,$$

Which is the Bjerknes' theorem in the usual case.

2.4 Generalization of potential vorticity:

Theorem 2.3

In generalized non-barotropic flow of an inviscid fluid the quantity

$$\frac{\bar{\omega} - \nabla\eta \times \nabla\lambda}{\rho} \cdot \nabla\mu,$$

is constant in time for each fluid element where μ is any fluid property satisfying equation (2.2).

Proof:

$$\frac{D}{Dt}\left[\frac{\bar{\omega} - \nabla\eta \times \nabla\lambda}{\rho} \cdot \nabla\mu\right] = \frac{D}{Dt}\left[\frac{\bar{\omega} - \nabla\eta \times \nabla\lambda}{\rho}\right] \cdot \nabla\mu$$

$$+ \frac{\bar{\omega} - \nabla\eta \times \nabla\lambda}{\rho} \cdot \frac{D}{Dt}\left[\nabla\mu\right]. \qquad (2.22)$$

Using equation (2.10) and the identity

$$\frac{D}{Dt}\left[\nabla\mu\right] = \nabla\left[\frac{D\mu}{Dt}\right] - (\nabla\mu.\nabla)\bar{u} - \nabla\mu x\bar{\omega}$$

by straight forward simplification we can show that the right hand side of equation (2.22) vanishes identically if $D\mu/Dt = 0$.

We can generalize Ertel's potential vorticity theorem to the case of non-barotropic isentropic inviscid fluid flows.

Theorem 2.4

In the class of non-barotropic inviscid fluid flows in which the entropy S is conserved during motion,

$$\frac{D}{Dt}\left[\frac{\bar{\omega} - \nabla\eta x\nabla\lambda}{\rho}\nabla S\right] \quad 0. \qquad (2.23)$$

Proof:

The proof follows trivially from theorem (2.3), by identifying μ with the specific entropy S.

We remark that this generalization is not possible in Mobbs(1981).

2.5 Generalization of helicity conservation:

Mobbs helicity conservation law for non-barotropic flows is now generalized as follows:

Theorem 2.5

In the class of generalized non-barotropic flows of an inviscid fluid,

$$\frac{D}{Dt}\int_V (\bar{\omega} - \nabla\eta \times \nabla\lambda)\cdot(\bar{u} - \eta\nabla\lambda)dV = 0, \tag{2.24}$$

where V is a material volume, if $\bar{\omega} - \nabla\eta \times \nabla\lambda$ is everywhere parallel to the boundaries of the volume.

Proof:

$$\frac{D}{Dt}\int_V (\bar{\omega} - \nabla\eta \times \nabla\lambda)\cdot(\bar{u} - \eta\nabla\lambda)dV = \int_V \frac{D}{Dt}\left[\frac{\bar{\omega} - \nabla\eta \times \nabla\lambda}{\rho}\right]\cdot\left[\bar{u} - \eta\nabla\lambda\right]\rho dV$$

$$+ \int_V \left[\frac{\bar{\omega} - \nabla\eta \times \nabla\lambda}{\rho}\right]\cdot\left[\frac{D\bar{u}}{Dt} - \frac{D}{Dt}\left[\eta\nabla\lambda\right]\right]\rho dV,$$

(By Reynold's transport theorem)

$$= \int_V \left[\left[\frac{\bar{\omega} - \nabla\eta \times \nabla\lambda}{\rho} . \nabla\right] \bar{u}\right] . \left[\bar{u} - \eta\nabla\lambda\right] \rho dV$$

$$+ \int_V \left[\frac{\bar{\omega} - \nabla\eta \times \nabla\lambda}{\rho}\right] . \left[- \nabla\left[I^* + \phi\right] + T\nabla\lambda - T\nabla\lambda - \eta \frac{D}{Dt}\left[\nabla\lambda\right]\right] \rho dV,$$

(By equations (2.7), (2.9) and (2.10))

$$- \int_V \left[\left[\left[(\bar{\omega} - \nabla\eta \times \nabla\lambda) . \nabla\right] \bar{u}\right] . \bar{u} - (\bar{\omega} - \nabla\eta \times \nabla\lambda) . \nabla(I^* + \phi)\right] dV$$

$$- \int_V \left[\frac{D}{Dt}\left[\frac{\bar{\omega} - \nabla\eta \times \nabla\lambda}{\rho}\right] . \nabla\lambda + \frac{\bar{\omega} - \nabla\eta \times \nabla\lambda}{\rho} . \frac{D}{Dt}\left[\nabla\lambda\right]\right] \eta\rho dV,$$

$$= \int_V \nabla . \left[\left(\frac{1}{2}|\bar{u}|^2 - I^* - \phi\right)\left(\bar{\omega} - \nabla\eta \times \nabla\lambda\right)\right] dV$$

$$- \int_V \frac{D}{Dt}\left[\frac{\bar{\omega} - \nabla\eta \times \nabla\lambda}{\rho} \nabla\lambda\right] \eta\rho dV. \qquad (2.25)$$

The first integral on the right hand side of equation (2.25) vanishes by applying Gauss's divergence theorem and since $(\bar{\omega} - \nabla\eta \times \nabla\lambda) . \hat{n} = 0$ on the boundaries of the volume V. Since λ

satisfies equation (2.2), the second integral on the right hand side of equation (2.25) also vanishes.

This completes the proof of the theorem.

2.6 Helmholtz theorem for generalized non-barotropic flows:

In the case of non-barotropic flows the Helmholtz theorems are proved using Kelvin's circulation theorem. The vector lines of $\bar{\omega} - \nabla\eta \times \nabla\lambda$ of a generalized non-barotropic flow have all the properties of vortex lines of a barotropic flow.

Definition 2.2

$\bar{\omega} - \nabla\eta \times \nabla\lambda$ is called the vorticity vector of a generalized non-barotropic flow.

Vortex lines and vortex tubes are defined using this definition of vorticity. Then we have the following generalizations of Helmholtz theorems.

Theorem 2.6

In the case of a generalized non-barotropic flow:

(i) If C_1 and C_2 are any two circuits encircling a vortex tube in the same direction, then the circulation of $\bar{u} - \eta\nabla\lambda$ around C_1 is equal to the circulation around C_2.

(ii) The vortex lines are material lines.

(iii) The strength of a vortex tube defined as the circulation around any circuit encircling the tube remains constant, as the tube moves with the fluid.

2.7 On Helmholtz fields:

We have seen that, provided $D\lambda/Dt = 0$, $\bar{\omega} - \nabla\eta x\nabla\lambda$ satisfies the equation (2.10). We also have

$$\nabla.(\bar{\omega} - \nabla\eta x\nabla\lambda) = 0. \qquad (2.26)$$

Thyagaraja (1975) has defined a Helmholtz field $\bar{g}(x,t)$ as follows:

Definition 2.3

A Helmholtz field $\bar{g}(x,t)$ is a solenoidal vector field which satisfies the equation

$$\frac{D}{Dt}\left[\frac{\bar{g}}{\rho}\right] = \left[\frac{\bar{g}}{\rho}.\nabla\right]\bar{u}, \qquad (2.27)$$

where $\bar{u}$ is the velocity of the fluid flow.

The vorticity vector of a barotropic flow is an example. The equation also implies that $\bar{g}(x,t)$ may be calculated if $\bar{g}(x,0)$, the velocity and the density fields are given.

In fact this definition of Helmholtz field is based on an extensive discussion in Truesdell (1953) where $\bar{g}$ is the vorticity. This discussion is concerning the circulation preserving motions and the deductions of Kelvin's circulation theorem in Thyagaraja (1975), Joseph and Mathew (1990) and Mathew (1991) are in fact redundant. But the fact remains that these studies help to shed light on fundamental ideas about the vorticity. It may be recalled that vorticity, though not a measurable quantity, has gained the greatest importance in almost all studies in fluid mechanics.

In spite of what is mentioned above, the deductions of helicity conservation law in the aforementioned works have something novel. Helicity conservation as derived by Moffatt (1969) is not directly following from Kelvin's circulation theorem and is not equivalent to it also. It is related to the knottedness of vortex lines. The above mentioned works show that conservation of helicity is also a property of circulation preserving motions.

The author together with Mathew (1990) has shown that in the case of non-barotropic flows helicity conservation follows as a property of Helmholtz field. Here we show that the result is true for a generalized non-barotropic flow also.

Definition 2.4

A $\bar{g}$-tube in a fluid is a material structure formed by the closed field lines of a Helmholtz field $\bar{g}$.

Definition 2.5

A $\bar{g}$-tube of infinitesimal cross-section is called a $\bar{g}$-filament.

Theorem 2.7

In a generalized non-barotropic flow of an inviscid fluid,

$$\frac{D}{Dt}\left[\frac{\bar{g}}{\rho} \quad \nabla\mu\right] = 0, \qquad (2.28)$$

where μ is any fluid property satisfying $D\mu/Dt = 0$.

Proof:

Proof follows from Mathew (1991).

Corollary 2.2

In the class of non-barotropic inviscid fluid flows in which the specific entropy S is conserved during motion:

$$\frac{D}{Dt}\left[\frac{\bar{g}}{\rho}\cdot\nabla S\right] = 0. \tag{2.29}$$

This is in fact Ertel's potential vorticity theorem.

2.7.1 <u>A generalization of helicity which is conserved:</u>

Theorem 2.8

For any $\bar{g}$-tube in a generalized non-barotropic inviscid fluid flow,

$$\frac{D}{Dt}\int_V \bar{g}\cdot(\bar{u} - \eta\nabla\lambda)dV = 0, \tag{2.30}$$

where V is a material volume bounded by $\bar{g}$-surfaces.

Proof:

$$\frac{D}{Dt}\int_V \bar{g}\cdot(\bar{u}-\eta\nabla\lambda)dV = \int_V \frac{D}{Dt}\left[\frac{\bar{g}}{\rho}\right]\cdot(\bar{u}-\eta\nabla\lambda)\rho dV + \int_V \frac{\bar{g}}{\rho}\cdot\left[\frac{D\bar{u}}{Dt} - \frac{D}{Dt}\left[\eta\nabla\lambda\right]\right]\rho dV,$$

(By Reynolds transport theorem)

$$\int_V \left[\left[\frac{\bar{g}}{\rho}\quad\nabla\right]\bar{u}\right]\quad\left[\bar{u} - \eta\nabla\lambda\right]\rho dV$$

$$+\int_V \left[\frac{\bar{g}}{\rho}\right]\cdot\left[-\nabla(\overset{*}{I}+\phi) + T\nabla\lambda - T\nabla\lambda - \eta\frac{D}{Dt}\left[\nabla\lambda\right]\right]\rho dV,$$

(By equations (2.7), (2.9) and (2.27))

$$= \int_V \left[\left[\left[\bar{g}\cdot\nabla\right]\bar{u}\right]\cdot\bar{u} - \bar{g}\cdot\nabla\left[\overset{*}{I} + \phi\right]\right]dV$$

$$- \int_V \left[\frac{D}{Dt}\left[\frac{\bar{g}}{\rho}\right]\cdot\nabla\lambda + \frac{\bar{g}}{\rho}\ \frac{D}{Dt}\left[\nabla\lambda\right]\right]\eta\rho dV,$$

$$= \int_V \nabla . \left[\left[\frac{1}{2}|\bar{u}|^2 - I^* - \phi \right] \bar{g} \right] dV - \int_V \frac{D}{Dt} \left[\frac{\bar{g}}{\rho} . \nabla\lambda \right] \eta\rho dV.$$

...............(2.31)

The first integral on the right hand side of equation (2.31) vanishes by applying Guass' divergence theorem and since $\bar{g}.\hat{n} = 0$ on the boundaries of the volume V. Since λ satisfies equation (2.2), the second integral on the right hand side of equation (2.31) also vanishes by equation (2.28).

This completes the proof of the theorem.

Theorem 2.9

In the class of generalized non-barotropic inviscid fluid flows in which $\bar{g}.\nabla\lambda$ vanishes identically,

$$\frac{D}{Dt}\int_V \bar{g}.\bar{u}dV = 0, \qquad (2.32)$$

where V is any material volume bounded by $\bar{g}$-surface.

Proof:

From theorem (2.8) we have,

$$\frac{D}{Dt}\int_V \bar{g}.\bar{u}dV = \frac{D}{Dt}\int_V \bar{g}.(\eta\nabla\lambda)dV$$

$$= 0.$$

Note that this conservation law is analogous to that given by Gaffet (1985), and Mathew (1991). It is to be noted that contrary to the claims of these authors isentropy condition DS/Dt = 0 is not relaxed in their proof, as conservation of potential vorticity is applied in the proof.

2.7.2 A generalization of Kelvin's circulation theorem:

Lemma 2.1

Let C be a closed curve defined by a closed $\bar{g}$-filament and let the volume of filament be V_C Then

$$\int_{V_C} \bar{g}.(\bar{u} - \eta\nabla\lambda)dV = |\bar{g}|\delta\sigma \oint_C (\bar{u} - \eta\nabla\lambda) \quad d\bar{l}, \qquad (2.33)$$

where $\delta\sigma$ is the infinitesimal cross-sectional area of the filament.

Proof:

The proof follows from the solenoidal nature of $\bar{g}$ which makes the strength of the $\bar{g}$-filament "$|\bar{g}|\delta\sigma$" constant along the $\bar{g}$-tube.

The above lemma is a particular case of the generalized circulation theorem and is analogous to that given by Joseph and Mathew (1990).

Theorem 2.10

Let C be a closed curve moving with the fluid, formed by the closed $\bar{g}$-filaments of a $\bar{g}$-tube,

$$\frac{D}{Dt} \oint_C (\bar{u} - \eta\nabla\lambda) \, d\bar{l} = 0. \tag{2.34}$$

Proof:

The proof follows from lemma (2.1) and theorem (2.8).

2.8 Discussion:

As pointed out earlier the conservation of circulation, potential vorticity and helicity follows from the property that $\bar{\omega} - \nabla\eta \times \nabla\lambda$ is a Helmholtz field. This depends on the condition that $D\lambda/Dt = 0$. Thus the possible relaxation of isentropy conditions is obtained not by replacing $\bar{u}$ by $\bar{u} - \eta\nabla S$ but by a term $\bar{u} - \eta\nabla\lambda$, λ having the above property.

It is interesting to note that in a paper dedicated to Truesdell, Serrin (1979) has questioned the entropy form of second law of thermodynamics. He has pointed out that classical thermodynamics is not a closed subject as most of the physicists and chemists think. He has also shown that the different statements of second law of thermodynamics are not equivalent. To quote Serrin "the general belief in entropy is based largely on arguments by analogy. This argument however may be inappropriate not only because entropy, even at the simplest level is a subtle idea far from direct experience, but also because in complicated cases, we can scarcely understand, let alone calculate, it". We simply add here that the generalization of kinematics of barotropic flows to non-barotropic one by the addition of entropy which is conserved does not give much deeper insight into the subject. The basic property of entropy relevant to all previous studies is DS/Dt = 0 which can be relaxed only by considering another quantity which has the same property.

The condition DS/Dt = 0 is satisfied in general for flows with very large Reynolds numbers, ie. large scale atmospheric and oceanic flows. Such high Reynolds numbers are difficult to occur in laboratory scale simulations.

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Chapter 3

VARIATIONAL SYMMETRIES AND CONSERVATION LAWS FOR BAROTROPIC FLOWS

3.1 Introduction:

In this chapter and the following one we develop a variational formulation for fluid mechanics. The variational principle is not a new one but the approach here is to develop it using Lie group theory. The identification of transformation groups leading to different conservation laws are significant in their application to studies of stability and turbulence.

At the outset we give the basic mathematical tools necessary for the studies of these two chapters. Here we follow Olver (1986a, 1986b).

We consider a system $\mathcal{S}$ of differential equations involving p independent variables $x = (x^1, x^2, \ldots\ldots, x^p) \in X \approx R^p$ and q dependent variables $u = (u^1, u^2, \ldots\ldots, u^q) \in U \approx R^q$ defined over an open subset $M \in X \times U$. All our cosiderations are local, justifying restrictions to Euclidean space but extensions to vector bundles and smooth manifolds follow easily. Clearly M is a manifold.

3.1.1 Lie group:

Definition 3.1

An r-parameter Lie group is a group G which also carries the structure of an r - dimensional smooth manifold in such a way that the group operations of multiplication and inversion are smooth maps between manifolds.

The lie group is connected if G is a connected manifold. We consider only connected Lie groups.

We are often not interested in the full Lie group, but only in group elements close to the identity element. In this case we can define a local Lie group solely in terms of local co-ordinate expressions for the group operations avoiding the abstract manifold theory.

3.1.2 Local Lie group

Definition 3.2

An r - parameter local Lie group consists of connected open subsets $V_o \subset V \subset R^r$ containing the origin O and smooth maps

$$m \quad V \times V \longrightarrow R^r$$

defining the group operation, and

$$i \quad V_o \longrightarrow V$$

defining the group inversion with the following properties:

a) Associativity

If $x, y, z \in V$, and also $m(x,y)$ and $m(y,z)$ are in V, then

$$m(x,m(y,z)) = m(m(x,y),z).$$

b) Identity element

$\forall x \in V$, $m(0,x) = x = m(x,0)$;

c) Inverses:

For each x in V_o $\quad m(x,i(x)) = 0 = m(i(x),x)$.

The identity element of the group is the origin O and the inverse is defined only for x sufficiently near O. It can be shown that every local Lie group is locally isomorphic to a neighbourhood of the identity element of some global Lie group G.

In our studies local Lie groups arise as groups of transformation on some manifold M.

3.1.3 Local transformation groups:

Definition 3.3

Let M be a smooth manifold. A local group of transformations acting on M is given by a (local) Lie group G, an open subset $\mathcal{U}$, with

$$\{e\}xM \subset \mathcal{U} \subset GxM,$$

which is the domain of definition of the group action, and a smooth map ψ $\mathcal{U} \longrightarrow M$ with the following properties:

a) If $(h,x) \in \mathcal{U}$, $(g,\psi(h,x)) \in \mathcal{U}$, and also $(g.h,\ x) \in \mathcal{U}$, then

$$\psi(g,\psi(h,x)) = \psi(g.h,\ x).$$

b) For all $x \in M$,

$$\psi(e,x) = x.$$

c) If $(g,x) \in \mathcal{U}$, then $(g^{-1},\ \psi(g,x)) \in \mathcal{U}$ and

$$\psi(g^{-1},\ \psi(g,x)) = x.$$

3.1.4 Connected group of transformations:

Definition 3.4

A group of transformations G acting on M is called connected if the following requirements hold:

a) G is a connected Lie group and M is a connected manifold;

b) $\mathcal{U} \subset G \times M$ is a connected open set; and

c) for each $x \in M$, the local Lie group $G_x = \left\{ g \in G : (g,x) \in \mathcal{U} \right\}$, is connected.

3.1.5 Vector field:

A vector field $\bar{v}$ on M assigns a tangent vector $\bar{v}|_x$ to each point $x \in M$ varying smoothly from point to point. In local coordinates $(x^1, x^2, \ldots\ldots, x^m)$, $\bar{v}|_x$ has components $\xi^1(x)$, $\xi^2(x), \ldots\ldots, \xi^m(x)$. We write

$$\bar{v}|_x = \xi^1(x) \frac{\partial}{\partial x^1} + \ldots\ldots\ldots + \xi^m(x) \frac{\partial}{\partial x^m}.$$

An integral curve of a vector field $\bar{v}$ is a smooth parametrized curve $x = \phi(\varepsilon)$, whose tangent vector at any point coincides with the value of v at the same point. Thus

$$x = (\phi^1(\varepsilon), \ldots\ldots, \phi^m(\varepsilon)),$$

must be a solution of the autonomous system of the ordinary differential equations

$$\frac{dx^i}{d\varepsilon} \quad \xi^i(x), \qquad i = 1, 2, \quad m.$$

where the $\xi^i(x)$ are the coefficients of $\bar{v}$ at x.

An integral curve which is not contained in any larger integral curve is called a maximal integral curve.

3.1.6 <u>Flow generated by a vector field:</u>

Definition 3.5

If $\bar{v}$ is a vector field, the parametrized maximal integral curve passing through a point $x \in M$ is called the flow generated by $\bar{v}$.

3.1.7 One-parameter group of transformations:

Definition 3.6

The flow generated by a vector field $\bar{v}$ is called a one-parameter group of transformations, and $\bar{v}$ is called its infinitesimal generator.

3.1.8 Action of a vector field on functions:

Definition 3.7

Let $v = \xi^1 \frac{\partial}{\partial x^1} + \ldots\ldots\ldots + \xi^m \frac{\partial}{\partial x^m}$ be a vector field on a manifold M and $f : M \longrightarrow R$ a smooth fuction, then the action of $\bar{v}$ on f is defined as

$$\bar{v}(f(x)) = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial x^i} (f(x)).$$

$\bar{v}(f)$ gives the infinitesimal change in the function "f" under the flow generated by $\bar{v}$.

3.1.9 Lie brackets:

Definition 3.8

Let $\bar{v}$ and $\bar{w}$ be vector fields on a manifold M, then their Lie bracket or commutator $[\bar{v}\ \ \bar{w}]$ is the unique vector field satisfying

$$[\bar{v}\ \ \bar{w}](f) = \bar{v}(\bar{w}(f)) - \bar{w}(\bar{v}(f)),$$

for all smooth functions $f \quad M \longrightarrow R$.
It can be shown that Lie bracket is bilinear, skew-symmetric and satisfies the Jacobi identity,

$$[\bar{u},[\bar{v},\bar{w}]] + [\bar{v},[\bar{w},\bar{u}]] + [\bar{w},[\bar{u},\bar{v}]] = 0.$$

3.1.10 Lie algebra:

Definition 3.9

A Lie algebra is a vector space g together with a bilinear operation

$$[\cdot,\cdot] \quad g x g \longrightarrow g,$$

called the Lie bracket for g, satisfying the axioms of bilinearity, skew-symmetry and Jacobi identity for all $u,v,w \in g$.

It can be shown that associated with a Lie group G, there is a Lie algebra g which is the tangent space to G at the identity element. Moreover there is a one to one correspondence between one dimensional subspaces of g and one-parameter connected subgroups of G.

As is mentioned at the beginnig we are concerned with systems of differential equations. These equations turn out to be Euler-Lagrange equations of a variational problem. The symmetry group of a system of differential equations is the largest local group of transformations acting on the independent and dependent variables of the system with the property that it transforms solutions of the system to other solutions. Now we give the necessary definitions and results.

Definition 3.10

A symmetry group of the system $\mathcal{S}$ of differential equations is a local group of transformations G acting on M with the property that whenever $u = f(x)$ is a solution of $\mathcal{S}$ and whenever $g.f$ is defined for $g \in G$ then $u = g.f(x)$ is also a solution of the system.

In order to use infinitesimal method for finding symmetries of differential equations; it is necessary to develop a concrete geometric structure for the system. This structure is determined by the vanishing of certain functions. To do this we "prolong" the basic space $X \times U$ of independent and dependent variables to a space which also represents the various partial derivatives occurring in the system.

Let $f: X \longrightarrow U$ be a smooth function, then it is possible to take partial derivatives of $u = f(x)$ with respect to different x^i which amounts to taking partial derivatives of different components of u. Let U_k be the Euclidean space endowed with co-ordinates which are partial derivatives of components of u of order k with all possible combinations of independent variables.

Definition 3.11

$U^{(n)} = U_1 \times \ldots\ldots \times U_n$ is a catesian product space whose co-ordinates represents all the derivatives of u of all orders from 0, 1, n. A typical element of $U^{(n)}$ will be denoted by $u^{(n)}$

3.1.11 Jet space:

Definition 3.12

The space $X \times U^{(n)}$ is called the n-th order jet space of the underlying space $X \times U$.

3.1.12 Prolongation:

Definition 3.13

The function $u^{(n)} = pr^{(n)} f(x)$, is called the n-th prolongation of $u = f(x)$.

Let $\mathcal{S}$ be the system of n-th order differential equations

$$\Delta_{\nu}(x, u^{(n)}) = 0, \qquad \nu = 1, 2, \ldots\ldots, l, \tag{3.1}$$

involving $x = (x^1, x^2, \ldots\ldots, x^p)$, $u = (u^1, u^2, \ldots\ldots, u^q)$ and the derivatives of u with respect to x upto order n. If we write

$$\Delta(x, u^{(n)}) = \left[\Delta_1(x, u^{(n)}), \ldots\ldots\ldots, \Delta_l(x, u^{(n)})\right], \tag{3.2}$$

Δ can be viewed as a smooth map from the jet space

$$\Delta : X \times U^{(n)} \longrightarrow R^{l}.$$

Then the differential equations determines a subvariety

$$\mathcal{S}_\Delta = \left\{ (x, u^{(n)}) : \Delta(x, u^{(n)}) = 0 \right\} \subset X \times U^{(n)}. \qquad (3.3)$$

3.1.13 Prolongation of group actions:

Suppose G is a local group of transformation acting on M. There is an induced local action of G acting on its n-jet space. This is called the n-th order prolongation of G and is denoted by $pr^{(n)}(G)$.

Given a smooth real valued function $f(x) = f(x^1, \ldots, x^p)$ of p independent variables, there are $\begin{pmatrix} p+k-1 \\ k \end{pmatrix}$ different k-th order partial derivatives of f. These derivatives can be denoted using the multi-index notation

$$\partial_J f(x) = \frac{\partial^k f(x)}{\partial x^{j_1} \partial x^{j_2} \ldots \partial x^{j_k}},$$

where $1 \le j_k \le p$ and $J = (j_1, j_2, \ldots, j_k)$

3.1.14 Prolongation of vector fields:

Definition 3.14

Let

$$\bar{v} = \sum_{i=1}^{p} \xi^{i} \frac{\partial}{\partial x^{i}} + \sum_{\alpha=1}^{q} \phi_{\alpha} \frac{\partial}{\partial u^{\alpha}} \qquad (3.4)$$

be a vector field in an open subset $M \subset X \times U$, the n-th prolongation of $\bar{v}$ is a vector field on the n - jet space

$M^{(n)} = M \times U_1 \times \ldots\ldots \times U_n$ defined by

$$pr^{(n)} v \quad \sum_{i=1}^{p} \xi^{i} \frac{\partial}{\partial x^{i}} + \sum_{\alpha=1}^{q} \sum_{J} \phi_{\alpha}^{J} \frac{\partial}{\partial u_{J}^{\alpha}}$$

where

$$\phi_{\alpha}^{J}(x, u^{(n)}) \quad D_J\left[\phi_{\alpha} - \sum_{i=1}^{p} \xi^{i} u_{i}^{\alpha} \right] + \sum_{i=1}^{p} \xi^{i} u_{J,i}^{\alpha} \qquad (3.5)$$

$u_{i}^{\alpha} = \dfrac{\partial u^{\alpha}}{\partial x^{i}}$ $\quad u_{J,i}^{\alpha} = \dfrac{\partial u_{J}^{\alpha}}{\partial x^{i}}$ and D_J stands for total derivatives.

3.1.15 <u>Maximal rank:</u>

Definition 3.15

The system of equations

$$\Delta_{\nu}(x,u^{(n)}) = 0 \qquad \nu = 1, 2, \ldots\ldots, l,$$

is of maximal rank if the Jacobian matrix of Δ with respect to all the variables in the n-jet space is of rank l whenever $\Delta(x,u^{(n)}) = 0$.

3.1.16 <u>Characteristic of vector field:</u>

Given the vector fields (3.4); its characteristic is the q-tuple

$$Q(x,u^{(n)}) \qquad (Q_1, Q_2, \ldots\ldots, Q_q),$$

where

$$Q_{\alpha}(x,u^{(n)}) \qquad \phi_{\alpha}(x,u) - \sum_{i=1}^{p} \xi^{i}(x,u)u_{i}^{\alpha}, \qquad \alpha = 1, 2, \ldots\ldots, q.$$

3.1.17 Non-degenerate system:

Definition 3.16

A system of differential equations is non-degenerate, if at every point $(x_0, u_0) \in \mathscr{S}_\Delta$ it is both locally solvable and of maximal rank.

Given a system of equations we can find its k-th prolongation by differentiating the equations in all possible ways upto order k.

3.1.18 Totally non-degenerate system:

Definition 3.17

A system of differential equations is called totally non-degenerate if it and all its prolongations are non-degenerate.

3.1.19 Characteristic direction:

Definition 3.18

Let Δ be an n-th order systems of differential equations having the same number of equations as unknowns. A non-zero p-tuple ω is said to define a characteristic direction to Δ at $(x_0, u_0^{(n)}) \in \mathcal{S}_\Delta$ if the qxq matrix of polynomials $M(\omega)$ with elements

$$M_{\alpha\nu}(\omega) = \sum_J \frac{\partial \Delta_\nu}{\partial u_J^\alpha}(x_0, u_0^{(n)}).\omega_J \qquad \alpha, \nu = 1, 2, \ldots, q$$

is non-singular. Otherwise it is called a non-characteristic direction.

3.1.20 Normal systems:

Definition 3.19

A system of equations $\Delta_\nu(x, u^{(n)}) = 0$ is said to be normal at a point $(x_0, u_0^{(n)}) \in \mathcal{S}_\Delta$ if there is at least one non-characteristic direction for the system at that point. The system is normal if it is normal at every point of $\mathcal{S}_\Delta$.

3.1.21 Over-determined and Under-determined systems:

Definition 3.20

Let Δ be an n-th order system of differential equations. Let $(x_0, u_0^{(n)})$ be initial values satisfying the system.

a) Δ is over-determined at $(x_0, u_0^{(n)})$ if there exist homogeneous k-th order differential operators $D_1, D_2, \ldots, D_q$ for some $k \geq 0$, not all zero, such that the linear combination $\sum D_\nu \Delta_\nu$ Q of equations in $\Delta^{(k)}$, at the point $(x_0, u_0^{(n)})$, depends only on derivatives of u of order at most $n+k-1$, and the linear combination Q does not vanish as an algebraic consequence of $\Delta^{(k-1)}$

b) Δ is under-determined at $(x_0, u_0^{(n)})$ if

(i) there exist at least one set of homogeneous k-th order operators $D_1, D_2, \ldots\ldots, D_q$, not all zero, with $\sum D_\nu \Delta_\nu$ Q depending on at most $(n+k-1)$-st order derivatives at the point x_0, and

(ii) whenever $D_1, D_2, \ldots, D_q$ satisfy the conditions in part (i), the resulting Q vanishes as an algebraic consequence of the previous prolongation $\Delta^{(k-1)}$

Over-determined systems are characterized by their lack of existence and under-determined systems are characterized by their lack of uniqueness of solutions for Cauchy problem.

3.1.22 Calculus of variations:

Let $X = R^p$ with co-ordinates $x = (x^1, x^2, \ldots\ldots, x^p)$ representing the independent variables and $U = R^q$ with co-ordinates $u = (u^1, u^2, \ldots\ldots, u^q)$ representing the dependent variables. Let $\Omega_0 \subset X$ be an open connected subset with smooth boundary $\partial\Omega_0$. A variational problem consists of finding the extrema of a functional

$$\mathcal{L}[u] = \int_{\Omega_0} L(x, u^{(n)})dx \tag{3.6}$$

in some class of functions $u = f(x)$. The integrand $L(x, u^{(n)})$, called the Lagrangian of the variational problem $\mathcal{L}$, is a smooth function of x, u and various derivatives of u.

3.1.23 Euler operator:

Definition 3.21

For $1 \leq \alpha \leq q$, the α-th Euler operator is defined by

$$E_\alpha = \sum_J (-D)_J \frac{\partial}{\partial u_J^\alpha} \tag{3.7}$$

the sum extending over all multi-indices over $J = (j_1, j_2, ..., j_k)$ with $1 \leq j_k \leq p$, $k \geq 0$.

Theorem 3.1

If $u = f(x)$ is a smooth extremal of the variational problem (3.6), then it must be a solution of the Euler-Lagrange equations

$$E_\alpha(L) = 0, \qquad \alpha = 1, 2, \quad ,q.$$

Definition 3.22

The total divergence of a function

$$P(x, u^{(n)}) = \left[P_1(x, u^{(n)}), \; ...., P_p(x, u^{(n)}) \right]$$

is defined as

$$\text{Div } P = D_1 P_1 + ... + D_p P_p \qquad (3.8)$$

where each D_j is the total derivative with respect to x^j.

It can be easily shown that if $L = \text{Div } P$, for some functions P, then the Euler-Lagrange equations are satisfied identically. In this case L is called a null-Lagrangian. This condition can be shown to be necessary also.

3.1.24 Variational symmetry:

Definition 3.23

A local group of transformations G acting on $M \subset \Omega_o \times U$ is a variational symmetry group of the functional (3.6), if whenever Ω is a subdomain with closure $\bar{\Omega} \subset \Omega_o$, $u = f(x)$ is a smooth function defined over Ω whose graph lies in M, and $g \in G$ is such that $\tilde{u} = \tilde{f}(\tilde{x}) = g.f(\tilde{x})$ is a single-valued function defined over $\tilde{\Omega}$, then

$$\int_{\tilde{\Omega}} L\left[\tilde{x},\ pr^{(n)}\tilde{f}(\tilde{x}) \right]d\tilde{x} = \int_{\Omega} L\left[x,\ pr^{(n)}f(x) \right]dx.$$

The infinitesimal criterion for a transformation group to be a variational symmetry is given by the following theorem.

Theorem 3.2

A connected group of transformations G acting on $M \subset \Omega_o \times U$ is a variational symmetry group of the functional (3.6) if and only if

$$Pr^{(n)}\bar{v}(L) + L\,\mathrm{Div}\,\xi = 0 \tag{3.9}$$

for all $(x, u^{(n)}) \in M^{(n)}$ and every infinitesimal generator

$$\bar{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha} \tag{3.10}$$

of G.

3.1.25 <u>Conservation laws:</u>

Definition 3.24

Given a system of differential equations $\Delta_\nu(x, u^{(n)}) = 0$ A conservation law is a divergence expression Div P which vanishes for all solutions $u = f(x)$.

In a dynamical problem, one of the independent variables is distinguished as "t" and the remaining as spatial variables. In this case a conservation law takes the form

$$\frac{\partial T}{\partial t} + \mathrm{div}\, X = 0, \tag{3.11}$$

where div X is the divergence of X with respect to the spatial co-ordinates. Here T is called the conserved density and X the associated flux.

3.1.26 Generalized vector fields:

Consider a vector field (3.4), defined on some open subset of XxU. Provided the coefficient function ξ^i and ϕ_α depend only on x and u, $\bar{v}$ is the infinitesimal generator of a transformation group acting pointwise on the underlying space.

Definition 3.25

$\bar{v}$ is called a generalized vector field if ξ^i and ϕ_α are functions of derivatives of u also.

The theorem (3.2) applies in the case of generalized vector fields also.

Noether's theorem relates the symmetries of a variational problem to conservation laws. This method is used for finding conservation laws for dynamical systems which can be formulated as variational problems.

3.2 Statement of problem:

The variational symmetry of an action integral for barotropic inviscid fluid flows is considered. By imposing two conditions on the infinitesimal generators of the transformation based on physical arguments appropiate to fluid flows, we get the hydromechanical variations of Drobot and Rybarski (1959). The invariance of the action integral leads to conservation laws of fluid mechanics as an application of Noether's theorem. Conservation of helicity is obtained as an example.

3.3 Hydromechanical transformations:

Following Drobot and Rybarski (1959), we consider the Euclidean four-dimensional space X. A point x in X has coordinates x^α, $\alpha=0,1,2,3$, where x^0 is the time t and x^i, $i=1,2,3$ are space-like coordinates. $\tilde{p}(x)$ is a four-dimensional vector field with components $p^\alpha(x)$, $\alpha=0,1,2,3$. Here p^0 is the density ρ, and p^i, $i=1,2,3$ are the impulses $p^0 u^i$ $(i=1,2,3)$ where u^i denotes the components of the velocity u. Tensor notation is used through out.

Let H denote the three dimemsional hypersurface in X and dH_α denote the oriented element on H,

$$dH_\alpha = \varepsilon_{\alpha\beta\gamma\delta}\, dl^\beta dl^\gamma dl^\delta \tag{3.12}$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ is the "Levi-Civita tensor" and $dl^\beta, dl^\gamma, dl^\delta$ are three linearly independent vectors lying on H so that dH_α is normal to the hypersurface H (Einstein's summation convention is used through out). The mass contained on H will be represented by the integral $\int_H dH_\alpha p^\alpha$, called the complete matter flow. In particular, when the hypersurface H is the space-like three dimensional volume V, we have,

$$dH_0 = dV; \quad dH_1 = dH_2 = dH_3 = 0 \quad \text{and}$$

$$\int_H dH_\alpha p^\alpha \quad \int_V p^0 dV,$$

reduces to the usual mass. If V is any four dimensional region contained in X and ∂V is its boundary, by Gauss theorem applied to $\oint_V dH_\alpha\, p^\alpha$, we obtain

$$\partial_\alpha p^\alpha = \frac{\partial p^0}{\partial t} + \operatorname{div}(p^0 \bar{u}), \tag{3.13}$$

representing the density of the source of matter existing in V.

Let F be a function space of vector valued functions $p^\alpha(x)$, $\alpha = 0,1,2,3$, supposed to be sufficiently regular in X. We consider the one-parameter group of transformations of XxF into itself generated by the infinitesimal generator

$$\bar{v} = \xi^\alpha \frac{\partial}{\partial x^\alpha} + \phi^\alpha \frac{\partial}{\partial p^\alpha} \tag{3.14}$$

The flow generated by $\bar{v}$ is subject to the conservation laws of momentum and mass. Accordingly we restrict the functions ϕ^{α} as follows:

(a) $\xi^{\alpha} = 0 \quad \Rightarrow \quad \bar{v}\,(p^{\alpha}) + p^{\alpha}\partial_{\beta}\xi^{\beta} = 0,$

and

(b) $\partial_{\alpha}\phi^{\alpha} = 0.$

Then for arbitrary ξ^{α},

$$\bar{v} = \xi^{\alpha}\frac{\partial}{\partial x^{\alpha}} + \phi^{\alpha}\frac{\partial}{\partial p^{\alpha}}$$

$$\text{where} \quad \phi^{\alpha} = \partial_{\beta}(p^{\beta}\xi^{\alpha} - p^{\alpha}\xi^{\beta}) \qquad \text{****} \qquad (3.15)$$

is the generator of a flow subject to the above constraints. While (a) corresponds to conservation of mass (b) makes it possible to consider points with variable masses, and in hydromechanics flows with sources.

The one-parameter family of transformations generated by

$\bar{v} \quad \xi^{\alpha}\frac{\partial}{\partial x^{\alpha}} + \partial_{\beta}(p^{\beta}\xi^{\alpha} - p^{\alpha}\xi^{\beta})\frac{\partial}{\partial p_{\alpha}}$ is called a hydromechanical transformation. The conditions (a) and (b) replace the Lin constraints used in other variational formulations.

3.4 Hydromechanical variational principle:

Given a physical problem; it can be formulated as a variational problem in more than one way. In the case of classical problem of dynamics in which there is no dissipation, the formulations corresponding to Hamilton's principle have a very special status since they lead to the equations of motions. The usual Lagrangian for barotropic flow which corresponds to Hamilton's principle is given by,

$$L = \frac{1}{2p^{o}} \left\{ (p^{1})^{2} + (p^{2})^{2} + (p^{3})^{2} \right\} - E(p^{o}) - p^{o}U(x), \qquad (3.16)$$

where E is the internal energy which depends only on $p^{o} = \rho$ and U is the potential of the external forces.

Coming back to (3.15) let us consider a standard variational problem. Let D denote an open subset of R^{n}. For any $y \in \mathscr{C}^{1}(cl(D), R)$ satisfying some prescribed boundary conditions, we consider the functional,

$$I(y) = \int_{D} L(x, y(x), \mathrm{grad}\ y(x))dx, \qquad (3.17)$$

where $x=(x_1, x_2, \ldots, x_n)$, dx denotes the n-dimensional Lebesgue measure and L denotes the given element of $\mathscr{C}^2(cl(D)xRxR^n, R)$. The conventional calculus of variation derives necessary condition for y to make I an extremum by studying the real function $\varepsilon \to I(y+\varepsilon\eta)$; the variable ε ranges over a neighbourhood of 0 in R and η denotes an arbitrary continuously differentiable real function whose support related to D is compact. This amounts to make the surface $S(0) \subset R^{n+1}$ which constitutes the graph of y to compete with a family of surface $S(\varepsilon)$; every point $(x, y+\varepsilon\eta)$ of $S(\varepsilon)$ results from the corresponding point $(x, y(x))$ of $S(0)$ by the displacement $\varepsilon\eta$.

A more general way of inserting S(0) into a chain of nearby surfaces would be to define, on some neighbourhood of S(0) in R^{n+1} a vector field $\bar{v}$, and to call $S(\varepsilon)$ the image of S(0) under the geometric transform $\exp(\varepsilon\bar{v})$ generated by this vector field. The classical calculus of variation amounts to choose the vector field $\bar{v}$ to have the special form,

$$\bar{v}(x,y) = (0, \delta y(x)) \tag{3.18}$$

In the case of (3.15) since ϕ^α is defined in terms of ξ^α the independent variation is given by ξ^α only. Thus (3.15) correspods to the case

$$\bar{v} = (\delta x, \delta y). \tag{3.19}$$

While (3.18) is called vertical variation (3.19) is called horizontal variation, since independent variations are in the direction of x^{i} only (Moreau (1981)).

Consider the action integral

$$W = \int_V dV\, L(x, p(x)), \tag{3.20}$$

where L is defined by equation (3.16) and the transformation

$$(x,\ p) \longrightarrow (\bar{x},\ \bar{p})$$

generated by equation (3.14).

Then the action W is transformed to

$$\tilde{W} = W + \Delta W = \int_{\bar{V}} dV\, L(\bar{x}, \bar{p}),$$

where $\bar{V}$ is the transformed region V and

$$\Delta W = \int_V dV \left\{ \frac{\partial L}{\partial p^{\alpha}} \phi^{\alpha} + \partial_{\alpha}(L\xi^{\alpha}) \right\}$$

is called the total variation of the action (3.20).

When $\phi^{\alpha} = \partial_{\beta}(p^{\beta}\xi^{\alpha} - p^{\alpha}\xi^{\beta})$,we write

$$\Delta W = \Delta_{o} W.$$

Now we state the variational principle:

For all hydromechanical transformations generated by

$$\bar{v} = \xi^{\alpha}\frac{\partial}{\partial x^{\alpha}} + \partial_{\beta}(p^{\beta}\xi^{\alpha} - p^{\alpha}\xi^{\beta})\frac{\partial}{\partial p^{\alpha}}$$

we have,

$$\Delta_{o} W = \int_{V} dV\left\{ \frac{\partial L}{\partial p^{\alpha}}\, \partial_{\beta}\left[p^{\beta}\xi^{\alpha} - p^{\alpha}\xi^{\beta}\right] + \partial_{\alpha}\left[L\xi^{\alpha} \right]\right\} = 0, \quad (3.21)$$

provided $\xi^{\alpha} = 0$ on ∂V.

The integral on the right hand side of equation (3.21) corresponds to that given by Courant and Hilbert (1953). Note that $\partial L/\partial p^{\alpha}$ is the usual Euler-Lagrange expression corresponding to our action integral.

Now

$$\Delta_o W = \int_V dV\, \partial_\beta \left[T^\beta_\alpha \xi^\alpha \right] - \int_V dV\, \psi_\alpha \xi^\alpha, \tag{3.22}$$

where

$$T^\beta_\alpha = p^\beta \frac{\partial L}{\partial p^\alpha} + \delta^\beta_\alpha \left[L - p^\gamma \frac{\partial L}{\partial p^\gamma} \right], \tag{3.23}$$

and

$$\psi_\alpha = p^\beta \left[\partial_\beta \left[\frac{\partial L}{\partial p^\alpha} \right] - \partial_\alpha \left[\frac{\partial L}{\partial p^\beta} \right] \right]. \tag{3.24}$$

The expressions ψ_α are called the hydromechanical Euler expressions.

Since $\xi^\alpha = 0$ on ∂V, the first integral on the right-hand side in (3.22) vanishes. Thus we obtain from (3.21) & (3.22),

$$\int_V dV\, \psi_\alpha \xi^\alpha = 0$$

Since ξ^α are arbitrary in the interior of ∂V, we get the equations of motion,

$$\psi_\alpha = 0. \quad (3.25)$$

Since $p^\alpha \psi_\alpha = 0$ the four equations of motion are linearly dependent and we get only three linearly independent solutions. In the case the Lagrangian takes the usual form (3.16), we get the following equations:

$\alpha = 0$:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\bar{u}|^2\right) + \bar{u}.\nabla\left(\frac{1}{2}|\bar{u}|^2\right) + \bar{u}.\left(\frac{1}{p^o}\nabla p + \nabla U\right) = 0, \quad (3.26)$$

$\alpha = 1, 2, 3$:

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u}.\nabla)\,\bar{u} = -\frac{1}{p^o}\nabla p - \nabla U, \quad (3.27)$$

where

$$p = p^o \frac{\partial E}{\partial p^o} - E, \quad (3.28)$$

is the pressure.

These are the conservation laws of energy, and angular momentum.

Definition 3.26

The vector field $\tilde{v}$ is called an infinitesimal divergence symmetry of W if there exists a vector C^{α}, $\alpha=0,1,2,3$ such that

$$\Delta W = \int_V dV\, \partial_{\alpha} C^{\alpha}, \tag{3.29}$$

identically in V. If $C^{\alpha}=0$ so that $\Delta W = 0$, W is said to be absolutely invariant.

Theorem 3.3

If the action (3.20) is absolutely invariant under the infinitesimal transformations (3.15) for arbitrary volume V, then a linear combination of the hydromechanical Euler expressions $\xi^{\alpha}\psi_{\alpha}$ is a divergence.

Proof:

The proof follows from equation (3.22).

3.5 Variational symmetries and Noether's theorem:

Now we consider the symmetry group of the action defined by (3.20). The group considered here will be local group of transformations G, acting on an open subset $M \subset V \times F \subset X \times F$. Unlike Drobot and Rybarski (1959) and Mathew and Vedan (1989) we can find variational symmetries using theorem (3.3).

The infinitesimal criterion for invariance is given by the following theorem (analogous to theorem (3.2)). Here we note that in our case we do not have to consider prolongation.

Theorem 3.4

A connected group of transformations G acting on $M \subset V \times F$ is a variational symmetry group of the (action) functional (3.20) if and only if

$$\bar{v}(L) + L \, \mathrm{Div}\xi = 0, \qquad (3.30)$$

for all $(x, p^{\alpha}) \in M$ and every infinitesimal generator

$$\bar{v} = \xi^{\alpha} \frac{\partial}{\partial x^{\alpha}} + \phi^{\alpha} \frac{\partial}{\partial p^{\alpha}}$$

of G.

Proof:

This is a particular case of theorem (3.2).

Example 3.1

Let $\bar{v}$, L be as defined in equations (3.15) and (3.16), and ξ^{α} be any arbitrary vector of the Galilean group of transformations G. When L is not an explicit function of x^{α}, choosing a Lie algebra g of G defined by the vanishing of the Lie bracket

$$[v^i, v^j] = v^i v^j - v^j v^i = 0, \tag{3.31}$$

we have

$\bar{v}(L) = 0$ and $Div\xi = 0$ so that

$\bar{v}(L) + L\, Div\xi = 0.$

Thus $\bar{v}$ is a variational symmetry of the action defined by (3.20).

Here we note that the above equation (3.31) is nothing but the condition for the absolute invariance of the action as stated by Drobot and Rybarski (1959). The corresponding conservation laws are those of mass, energy, impulse, and angular momentum.

Remark:

If G is a variational symmetry group of the action defined by (3.20), then from the above theorem (3.4)

$$\bar{v}(L) + L\, \mathrm{Div}\xi = 0$$

$$\text{i.e.}\quad \left[\xi^{\alpha} \frac{\partial}{\partial x^{\alpha}} + \phi^{\alpha} \frac{\partial}{\partial p^{\alpha}} \right](L) + L\, \partial_{\alpha}\xi^{\alpha} = 0$$

$$\text{i.e.}\quad \xi^{\alpha} \frac{\partial L}{\partial x^{\alpha}} + \phi^{\alpha} \frac{\partial L}{\partial p^{\alpha}} + L\, \partial_{\alpha}\xi^{\alpha} = 0$$

$$\text{i.e.}\quad \mathrm{Div}(L\xi) + \left[\phi^{\alpha} - \xi^{\alpha} \frac{\partial p^{\beta}}{\partial x^{\alpha}} \right] \frac{\partial L}{\partial p^{\alpha}} = 0$$

This can be written as a conservation law,

$$\mathrm{Div}(P) = \left[\phi^{\alpha} - \xi^{\alpha} \frac{\partial p^{\beta}}{\partial x^{\alpha}} \right] \frac{\partial L}{\partial p^{\alpha}}$$

where $P = -L\xi$. This leads to the following theorem, since the Euler expressions in this case are $\frac{\partial L}{\partial p^{\alpha}}$.

Theorem 3.5

Suppose G is a (local) one-parameter group of symmetries of the variational problem (3.20). Let

$$\bar{v} = \xi^{\alpha} \frac{\partial}{\partial x^{\alpha}} + \phi^{\alpha} \frac{\partial}{\partial p^{\alpha}}$$

be the infinitesimal generator of G, and

$$Q_{\alpha} = \phi^{\alpha} - \xi^{\beta}\partial_{\beta}p^{\alpha}, \tag{3.32}$$

the corresponding characteristic of $\bar{v}$. Then there is a P such that

$$\text{Div } P = \sum Q_{\alpha}E_{\alpha}(L) = Q.E(L), \tag{3.33}$$

is a conservation law in characteristic form for the usual Euler-Lagrange equations $E_{\alpha}(L) = 0$. This is a particular case of Noether theorem.

Theorem (3.3) is related to our hydromechanical variational principle and Theorem (3.5) is related to the usual (vertical) variational principle. Choosing a variational symmetry for the usual variational principle amounts to finding functions ϕ^{α} and ξ^{α} satisfying equation (3.30). This leads to the corresponding conservation laws in the usual variational formulation. But from Theorem (3.3) we find that the problem is reduced to finding functions ξ^α only. As shown by Drobot and Rybarski, when ξ^α defines a Galilean transformation these conservation laws are of energy, impulse, and angular momentum.

Now let us consider the connection between Theorems (3.3) & (3.5). The infinitesimal criterion for invariance leads to

$$\partial_\beta\left(T^\beta_\alpha\, \xi^\alpha\right) - \psi_\alpha \xi^\alpha - \xi^\alpha \partial_\alpha p^\beta \frac{\partial L}{\partial p^\beta} = 0. \tag{3.34}$$

Thus a linear combination of the usual Euler-Lagrange expressions and the hydromechanical Euler expressions is a divergence. Further discussion of this will be given in the next chapter. As a simple example we consider the case when the flow determined by $\bar{v}$ is isochoric $(\partial_\alpha \xi^\alpha \quad 0)$.

Example 3.2 (Conservation of helicity)

Suppose

$$\bar{v}(L) + L\, \mathrm{Div}\xi = 0$$

Now

$$\psi_\alpha \xi^\alpha = \partial_\beta \left[T^\beta_\alpha \xi^\alpha \right] - \xi^\alpha \partial_\alpha p^\beta \frac{\partial L}{\partial p^\beta}$$

$$= \partial_\beta \left\{ \left[p^\beta \frac{\partial L}{\partial p^\alpha} + \delta^\beta_\alpha \left[L - p^\gamma \frac{\partial L}{\partial p^\gamma} \right] \right] \xi^\alpha \right\} - \xi^\alpha \partial_\alpha p^\beta \frac{\partial L}{\partial p^\beta}$$

$$= \partial_\beta \left[p^\beta \xi^\alpha \frac{\partial L}{\partial p^\alpha} + \left[L - p^\gamma \frac{\partial L}{\partial p^\gamma} \right] \xi^\beta \right] - \xi^\alpha \partial_\alpha p^\beta \frac{\partial L}{\partial p^\beta}$$

$$= \partial_\beta \left[p^\beta \xi^\alpha \frac{\partial L}{\partial p^\alpha} + \left[L - p^\gamma \frac{\partial L}{\partial p^\gamma} \right] \xi^\beta - L\xi^\beta \right]$$

(if L is not an explicit function of x^α and $\partial_\alpha \xi^\alpha = 0$)

$$= \partial_\beta \left[\left(p^\beta \xi^\alpha - p^\alpha \xi^\beta \right) \frac{\partial L}{\partial p^\alpha} \right].$$

For $\xi^0 = 0$, the corresponding density

$$p^0 \xi^i \frac{\partial \Gamma}{\partial p^i}, \qquad i = 1, 2, 3$$

$$= p^0 \xi^i u_i .$$

Identifying $\xi^i = \omega^i$ (ω^i, the vorticity vector) and taking a volume such that $\omega^i n_i = 0$ (n_i the unit normal to the surface), this gives the conservation of helicity.

3.6 Discussion:

In the usual variational problem we consider the "vertical variations" of the dependent variables. Instead, Moreau (1982) condider variations of the independent variables which he calls the "horizontal variations" or "transport method". In both the cases the variations are infinitesimal transformations acting either on the space of dependent variables or on the space of independent variables.

Here we have considered the transformation groups acting on the space of both independent and dependent variables in an Eulerian frame-work. Restricting the transformations to hydromechanical ones reduces the problem to the invariance of action under transformations of independent variables, i.e. the transport method. This method avoids the use of Lagrange multipliers to account for the physical constraints of fluid flows.

In the examples that we have considered in this chapter, the explicit forms of the functions ξ^{α} is immeterial, the only condition being that the flow generated by $\bar{v}$ is isochoric. Thus the symmetry we are considering is the ordinary variational symmetry. The difference between the classical Noether theorem and the Noether theorem for hydromechanical variations is brought out by considering the conditions for variational symmetry. The more general cases of generalized symmetries, div-invariants etc. will be discussed in the next chapter.

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Chapter 4

VARIATIONAL SYMMETRIES AND CONSERVATION LAWS FOR NON-BAROTROPIC FLOWS

4.1 Introduction:

The variational principle of Drobot and Rybarski (1959) has been extended to the case of non-barotropic flows by Mathew and Vedan (1989, 1991). They have used Noether's theorems in the derivation of conservation laws. These theorems describe a relationship between the invariants with respect to the given infinitesimal transformation of the action integral and some identities satisfied by the corresponding Euler expressions.

There are two kinds of Noether theorems. In one, the transformation is supposed to depend on scalar parameters and in the other, on functions. Mathew and Vedan were successful in extending the application of Noether's first theorem to the hydromechanical variational principle for non-barotropic flows. In the application of Noether's second theorem only a special type of transformations were considered and the method involved introduction of Lagrangian multipliers to incorporate side conditions.

In this chapter we consider applications of Noether's second theorem to derive conservation of potential vorticity for non-barotropic flows. The special form of the Lagrangian helps us to avoide the use of Lagrangian multipliers. The relation between the hydromechanical variational principle and the variational principle due to Katz and Lynden-Bell (private communication) leads to the conservation of potential vorticity.

4.2 Generalized hydromechanical transformations:

Following Mathew and Vedan, we extend the set of dependent variables to include the entropy flux. X is the four-dimensional Euclidean space with coordinates x^α, $\alpha=0,1,2,3$, and p^α also have the same meaning as in chapter 3. $\bar{s}(x)$ is a four-dimensional vector field with components s^α, $\alpha=0,1,2,3$, where

$$(s^0, s^1, s^2, s^3) = (p^0 S,\ p^0 u^1 S,\ p^0 u^2 S,\ p^0 u^3 S),$$

S being the specific entropy.

Let F be the function space of vector valued functions $p^\alpha(x)$, $s^\alpha(x)$. We consider the one-parameter group of transformations of XxF into itself generated by the infinitesimal generator

$$\bar{v} = \xi^{\alpha} \frac{\partial}{\partial x^{\alpha}} + \phi^{\alpha} \frac{\partial}{\partial p^{\alpha}} + \eta^{\alpha} \frac{\partial}{\partial s^{\alpha}} \tag{4.1}$$

The flow generated by $\bar{v}$ is subject to the conservation laws of entropy in addition to mass and momentum. For this we choose the functions ϕ^{α} and η^{α} satisfying the following conditions:

(a) $\xi^{\alpha} = 0 \Rightarrow \bar{v}(p^{\alpha}) + p^{\alpha}\partial_{\beta}\xi^{\beta} = 0,$

$$\bar{v}(s^{\alpha}) + s^{\alpha}\partial_{\beta}\xi^{\beta} = 0,$$

and

(b) $\partial_{\alpha}\phi^{\alpha} = 0, \quad \partial_{\alpha}\eta^{\alpha} = 0.$

Then for arbitrary ξ^{α},

$$\bar{v} = \xi^{\alpha} \frac{\partial}{\partial x^{\alpha}} + \phi^{\alpha} \frac{\partial}{\partial p^{\alpha}} + \eta^{\alpha} \frac{\partial}{\partial s^{\alpha}}$$

where $\phi^{\alpha} = \partial_{\beta}(p^{\beta}\xi^{\alpha} - p^{\alpha}\xi^{\beta})$ and $\eta^{\alpha} = \partial_{\beta}(s^{\beta}\xi^{\alpha} - s^{\alpha}\xi^{\beta})$

is the infinitesimal generator of a flow subject to the above constraints.

Definition 4.1

The one-parameter family of transformations generated by

$$\bar{v} = \xi^{\alpha} \frac{\partial}{\partial x^{\alpha}} + \partial_{\beta}(p^{\beta}\xi^{\alpha} - p^{\alpha}\xi^{\beta}) \frac{\partial}{\partial p^{\alpha}} + \partial_{\beta}(s^{\beta}\xi^{\alpha} - s^{\alpha}\xi^{\beta}) \frac{\partial}{\partial s^{\alpha}} \qquad (4.2)$$

is called a generalized hydromechanical transformation.

The Lagrangian for non-barotropic flow is given by

$$L = \frac{1}{2p^{0}} \left[(p^{1})^{2} + (p^{2})^{2} + (p^{3})^{2} \right] - E(p^{0}, s^{0}) - p^{0}U(x), \qquad (4.3)$$

where E is the internal energy and U is the potential of the external forces.

Let V be any four-dimensional region contained in X with boundary ∂V.

4.3 Generalized hydromechanical variational principle:

For all generalized hydromechanical transformations (4.2), we have

$$\Delta_o W = \int_V \left[\frac{\partial L}{\partial p^\alpha} \partial_\beta \left[p^\beta \xi^\alpha - p^\alpha \xi^\beta \right] + \frac{\partial L}{\partial s^\alpha} \partial_\beta \left[s^\beta \xi^\alpha - s^\alpha \xi^\beta \right] + \partial_\alpha \left[L\xi^\alpha \right] \right] = 0 \tag{4.4}$$

provided $\xi^\alpha = 0$ on ∂V.

The variation $\Delta_o W$ can be written as

$$\Delta_o W = \int_V dV \, \partial_\beta \left[T^\beta_\alpha \, \xi^\alpha \right] - \int_V dV \, \psi_\alpha \xi^\alpha , \tag{4.5}$$

where

$$T^\beta_\alpha = p^\beta \frac{\partial L}{\partial p^\alpha} + s^\beta \frac{\partial L}{\partial s^\alpha} + \delta^\beta_\alpha \left[L - p^\gamma \frac{\partial L}{\partial p^\gamma} - s^\gamma \frac{\partial L}{\partial s^\gamma} \right] \tag{4.6}$$

and

$$\psi_\alpha = p^\beta \left[\partial_\beta \left[\frac{\partial L}{\partial p^\alpha} \right] - \partial_\alpha \left[\frac{\partial L}{\partial p^\beta} \right] \right] + s^\beta \left[\partial_\beta \left[\frac{\partial L}{\partial s^\alpha} \right] - \partial_\alpha \left[\frac{\partial L}{\partial s^\beta} \right] \right] . \tag{4.7}$$

The expressions ψ_α are called the generalized hydromechanical Euler expressions.

Since ξ^α is zero on ∂V, the first integral on the right hand side of equation (4.5) vanishes. As ξ^α are arbitrary in the interior of ∂V we get

$$\psi_\alpha = 0. \qquad (4.8)$$

Since $p^\alpha \psi_\alpha = 0$ (and $s^\alpha \psi_\alpha = 0$), (4.9)

only three of the above four equations are linearly independent.

4.4 Noether theorems and conservation laws:

As mentioned in chapter 3 Noether's theorems relate variational symmetries of an action integral to conservation laws associated with the corresponding usual Euler-Lagrange equations. In the case of hydromechanical variations the transformations of the dependent variables are related to the transformations of the independent variables as is clear from equations (3.15) and (4.2). Thus a generalized hydromechanical transformation is defined in terms of the functions ξ^α alone. It also follows that the infinitesimal generators of the hydromechanical transformations are generalized vector fields (Definition (3.25)).

In this sense the two theorems of Drobot and Rybarski and Mathew and Vedan are classical Noether's second theorem adapted to hydromechanical variational principle. Noether's first theorem of Drobot and Rybarski corresponds to transformations depending on scalar parameters and second theorem, transformations depending on scalar functions. The system corresponding to our variational principle is under-determined as is clear from equations (4.9) (see also page 77). Classical Noether's second theorem is concerned with such systems for which there may be trivial conservation laws determined by non-trivial variational symmetry groups. Mathew and Vedan (1991, theorem (3.7)) have proved the following theorem.

Theorem 4.1

If there exists a divergence symmetry for the action integral

$$W = \int_V dV\, L(x,\ p(x),\ s(x)), \tag{4.10}$$

depending on r - arbitrary functions and their derivatives upto a given order q, there exists exactly r linearly independent identities between the Euler-Lagrange expressions ψ_α and their derivatives, provided the symmetry corresponds to generalized hydromechanical transformations.

The applications of this theorem to derive conservation laws were considered only in the special cases in which $\phi^\alpha = 0$ and $\eta^\alpha = 0$ Even in this case the method involved use of Lagrange multipliers to incorporate side conditions.

4.5 Conservation of potential vorticity:

Although we have defined a new four-vector s^α ($\alpha=0,1,2,3$) only s^0 is entering into our variational principle. This allows us to consider conditions which lead to $\phi^\alpha = 0$ ($\alpha=0,1,2,3$) and $\eta^0 = 0$. We use generalized hydromechanical transformation with $\xi^0 = 0$. For simplicity we shall use three-dimensional vector notation with $\bar{\xi} = (\xi^1,\xi^2,\xi^3)$, $\bar{u} = (u^1,u^2,u^3)$ and ∇ - the spatial divergence operator

$$\nabla = \left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right].$$

Then following Mathew and Vedan we find that $\phi^\alpha = 0$ ($\alpha = 0,1,2,3$) and $\eta^0 = 0$ provided

$$\nabla.(\rho\bar{\xi}) = 0, \quad \rho\bar{\xi}.\nabla S = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\left[\rho\bar{\xi}\right] + \nabla x(\rho\bar{\xi} x\bar{u}) = 0 \tag{4.11}$$

A solution of equation (4.11) for $\bar{\xi}$ is then given by

$$\bar{\xi} = \frac{1}{\rho} \nabla f x \nabla S, \tag{4.12}$$

where f satisfies the equation

$$\nabla\left[\frac{Df}{Dt}\right] \times \nabla S = 0. \tag{4.13}$$

Equations (4.11) and (4.13) and its solutions (4.12) have appeared in Katz and Lynden-Bell and Friedman and Schutz (1978).

Katz and Lynden-Bell have considered variations which satisfy the conservation laws of mass and entropy. It is comparable to the generalized hydromechanical transformations assumed above. It has been shown by these authors that the invariance of the action under the transformations defined by equations (4.12) and (4.13) is that of potential vorticity.

4.6 Discussion:

In this chapter we have considered the hydromechanical variational principle for non-barotropic flows. This involves the set of dependent variables to include the entropy flux also. The system described by the hydromechanical variational principle is under-determined and the two theorems of Drobot and Rybarski (1959) are particular cases of classical Noether's second theorem. The variations considered in chapters 3 and 4 are generalized variations (Olver 1986b). The special form of the Lagrangian is made use of to avoide Lagrange multipliers. Then a symmetry group is identified which involves the transformations of the independent variables x^1, x^2, x^3 only. Conservation of potential vorticity is obtained by comparing hydromechanical variational principle to the variational principle of Katz and Lynden-Bell.

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Chapter 5

CONCLUSION

This thesis contains a study of conservation laws of fluid mechanics. These conservation laws though classical, have been put to extensive studies in the past many decades.

In chapter 2 we have considered the generalization of the well-known conservation laws of barotropic flows to the case of non-barotropic flows. The earlier generalizations are based on the dependence of internal energy on specific entropy in addition to the specific volume. Isentropy conditions are assumed in these studies. It is shown that it is the property which forms the basis of all these generalizations. Further the conservation of helicity which is shown to be independent of this property by Gaffet (1985) and Mathew (1991) is wrong as the result depends on conservation of potential vorticity. This leads to the possibility of considering a more general function $\lambda = \lambda(S,T)$ of specific entropy S and absolute temperature T, which is advected to form the basis of conservation laws for what we call a generalized non-barotropic flow. Compare the left hand side of equation (2.4) with the Joule-Thomson coefficient well-known in geophysical flows. The earlier studies are further restricted by their derivation based on Clebsch's potentials which do not exists when vortex lines are knotted. The general conclusion is that all the conservation laws are based on the fact that the vorticity vector for a generalized non-barotropic flow as well as for any other flows is a Helmholtz field. Our investigations also make it clear that the generalizations of all conservation laws to the non-barotropic case are based on the equation $DS/Dt = 0$.

The identification $\lambda = S$ leads to the isentropic cases discussed by earlier workers for non-barotropic flows, except in the case of potential vorticity. The identification $\lambda = T$ is not possible in general because of equation (2.4) but may be possible when there is no appreciable change in density with temperature. A physical meaning to the function λ is to be sought in the light of Serrin's remarks (page 45) of this thesis.

There are extensive discussions on conservation laws based on Clebsch's potentials (Seliger and Whitham (1968), Serrin (1959), Mobbs (1981) etc. for example). As pointed out by Bretherton (1970) these studies have limited validity because Clebsch's representation is not possible for flows with non-zero helicity. Thus the characterization of circulation preserving motions is truly based on functions like λ which are advected.

In chapter 3 we have considered variational formulations for barotropic flows using Lie group theory. This method helped us to systematically develop the hydromechanical variational formulations of Drobot and Rybarski (1959), and Mathew and Vedan (1989, 1991). The connection between variational symmetries and hydromechanical Euler-Lagrange equations has been brought out. Further this has helped us to compare the Noether theorem in the usual variational formulation and hydromechanical variational formulation. In this chapter and chapter 4 generalized symmetries have been considered by including the dependence of infinitesimal generators of transformations on derivatives of dependent variables also.

In chapter 4 we have considered the variational formulation of non-barotropic flows. Here the only case of isentropic process is considered. The two theorems of Drobot and Rybarski and Mathew and Vedan are in fact special cases of classical Noether's second theorem, since the system described by the variational principle is under-determined. The application of Noether's second theorem to derive the conservation laws is difficult and Drobot and Rybarski (1959) and Mathew and Vedan

(1991) have considered only the transformations in which p^{α} and s^{α} are invariant. Even in this case Lagrange multipliers were used to incorporate the constraints which define such transformations, and the corresponding conserved quantity is vorticity.

Though we have introduced four additional variables s^{0}, s^{1}, s^{2} s^{3} for non-barotropic flows, only one of them, s^{0}, appears in the Lagrangian. This fact has helped us to avoide the use of Lagrange multipliers. Then the variational principle is compared to that due to Katz and Lynden-Bell, who have shown that the corresponding conservation law is that of potential vorticity. But this method has reduced the scope of applications of Noether's theorem. A direct application of the theorem is still an unsolved problem.

The conservation laws are of interest to us because of their applications in different branches of studies of fluid mechanics. In particular studies of stability of flows is being undertaken in the department. The identification of variational symmetries is being found helpful in stability studies using Arnold's method. Hamiltonian formulation for fluid flows is also under study. The studies contained in this thesis is in this sense only a beginning and can be carried forward by the use of many more concepts of Lie group theory like De-Rahm complex, variational complex, Lie derivatives etc.

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