SOME PROBLEMS IN ALGEBRA AND TOPOLOGY

A STUDY OF FUZZY CONVEXITY WITH SPECIAL REFERENCE TO SEPARATION PROPERTIES

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CERTIFICATE

This is to certify that this thesis is a bona fide research work carried out by Smt. M.V. Rosa under my supervision and guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology, for the award of the Degree of Doctor of Philosophy of the Cochin University of Science and Technology.and no part of it has previously formed the basis for the award of any other degree or diploma in any other University.

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DECLARATION

I hereby declare that this thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person except where due reference is made in the text of the thesis.

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Chapter 0

INTRODUCTION

This thesis is a study of abstract fuzzy convexity spaces and fuzzy topology fuzzy convexity spaces.

0.1 Fuzzy Sets and Fuzzy Topology

The formation of fuzzy mathematics rests on the notion of fuzzy sets. The basic concept of a fuzzy set was introduced by L.A. Zadeh in 1965 [40]. A fuzzy set 'A' in a set X is characterized by a membership function $\mu_A(x)$ from X to the unit interval [0,1], $x \in X$. Fuzzy set theory is a generalization of abstract (non fuzzy) set theory. If A is an ordinary subset of X, its characteristic function is a fuzzy set.

The theory of fuzzy sets deals with a subset A of a set X, where the transition between full membership and no membership is gradual. The grade of membership 'one' is assigned to those objects that fully and completely belong to A, while zero is assigned to objects that do not belong to A at all. The more an object x belongs to A, the closer to 'one' is its grade of membership $\mu_A(x)$. The fuzzy set A' defined by $\mu_{A'}(x) = 1-\mu_A(x)$ is called the complement of the fuzzy set A.

Several mathematicians have applied the theory of fuzzy sets to various branches of pure mathematics also, resulting in the development of new areas like, fuzzy topology, fuzzy groups and fuzzy topological semi-groups. Among these, fuzzy topology is well developed. It was C.L. Chang [1] who defined fuzzy topology for the first time in 1968.

According to Chang, a family T of fuzzy sets in X is called a fuzzy topology for X, if

(i) $\emptyset, X \in T$

(ii) if $A, B \in T$ then $A \cap B \in T$

(iii) if $A_i \in T$ for each $i \in I$, then $\bigcup A_i \in T$.

Then the pair (X,T) is called a fuzzy topological space or fts in short. The elements of T are called open sets and their complements are called closed sets.

In 1976 R. Lowen [20] has given another definition for a fuzzy topology by taking the set of constant functions instead of $\emptyset$ and X in axiom (i) of Chang's definition. In this thesis we are following Chang's definition rather than Lowen's definition.

For other details of fuzzy topological spaces like product and quotient spaces, we refer to C.K. Wong [39].

0.2 Convex Sets

The study of convex sets is a branch of geometry, analysis and linear algebra that has numerous connections with other areas of mathematics. Though convex sets are defined in various settings, the most useful definition is based on a notion of betweenness. When X is a space in which such a notion is defined, a subset C of X is called convex provided that for any two points x and y of C, C includes all the points between x and y. i.e., C is said to be convex if $\lambda x + (1-\lambda)y \in C$, for every $x,y \in C$ and $\lambda \in [0,1]$.

The theory of convexity can be sorted into two kinds. One deals with concrete convexity and the other that deals with abstract convexity. In concrete situations it was considered by R.T. Rockafellar [26], Kelly and Weiss [16], S.R. Lay [18] and many others.

In abstract convexity theory a convexity space was introduced by F.W. Levi in 1951 [19]. He defined a convexity space as a pair $(X, \mathcal{L})$ consisting of a set X and a family '$\mathcal{L}$' of subsets of X called convex sets satisfying the conditions

(i) $\emptyset, X \in \mathcal{L}$

(ii) If $A_i \in \mathcal{L}$, for each $i \in I$, then $\bigcap_{i \in I} A_i \in \mathcal{L}$.

The convexity space introduced by Levi was further developed by many authors like D.C. Kay and Womble E.W [13], R.E. Jamison-Waldner [9], G. Sierksma [29], M. Van de Vel [33] etc. In addition to the above conditions (i) and (ii) if $\bigcup A_i \in \mathcal{L}$ whenever $A_i \in \mathcal{L}$ and A_i's are totally ordered by inclusion, then $(X, \mathcal{L})$ is called an aligned space which was introduced by R.E. Jamison-Waldner.

In abstract situations the notion of a topological convexity structure ~~has been introduced~~ by R.E. Jamison Waldner in 1974. A triple $(X, \mathcal{L}, T)$ consisting of a set X, a topology T and convexity $\mathcal{L}$ on X is called a topological convexity structure, provided the Topology T is compatible with the convexity $\mathcal{L}$. Now a topology T is compatible with a convexity $\mathcal{L}$, if all polytopes of $\mathcal{L}$ are closed in (X,T). R.E. Jamison-Waldner also introduced the concept of local convexity.

0.3 Convex Fuzzy Sets

The notion of convexity can be generalized to fuzzy subsets of a set X. L.A. Zadeh introduced the concept of a convex fuzzy set in 1965. A fuzzy subset 'A' of X is convex if and only if for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$.

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geqslant \min\left\{\mu_A(x_1), \mu_A(x_2)\right\}$$

or equivalently a fuzzy set A is convex if and only if the ordinary set

$$A_d = \left\{x \in X \mid \mu_A(x) \geqslant d\right\} \text{ is convex for each } d > 0$$

and $d \in [0,1]$.

In concrete situations the concept of a convex fuzzy set was initiated by M.D. Weiss [35]. A.K. Katsaras and D.B. Liu [11], R. Lowen [21], Zhou Feiyue [5] etc. M.D. Weiss considered a convex fuzzy set in a vector space over real or complex numbers in 1975. In 1977 Katsaras and Liu applied the concept of a fuzzy set to the elementary theory of vector spaces and topological vector spaces. They have also considered convex fuzzy sets. In 1980, R. Lowen applied the theory of fuzzy sets to some elementary known results of convex sets.

For the definition of convex fuzzy sets in vector spaces, we refer to A.K. Katsaras and D.B. Liu [11].

No attempt seems to have been made to develop a fuzzy convexity theory in abstract situations. The purpose of this thesis is to introduce fuzzy convexity theory in abstract situations.

0.4 Summary of the Thesis

Chapter 1.

This chapter is a study of abstract fuzzy convexity spaces (fcs). In Section 1, we quote some results on fuzzy convexity in vector spaces from [11]. Also it is proved that the union of any family of convex fuzzy sets in a vector space E, totally ordered by inclusion is a convex fuzzy set in E. This motivates the introduction of abstract fuzzy convexity spaces.

In Section 2, we define abstract fuzzy convexity spaces- introduce and study the concept of a fuzzy convex hull operator. Also we define a fuzzy convex to convex (FCC) map and a fuzzy convexity preserving (FCP) map in such spaces and study some related topics under these maps.

In Section 3, we introduce ~~the~~ subspace, product and quotient of fuzzy convexity spaces.

Chapter 2.

In this chapter we introduce the notion of fuzzy topology fuzzy convexity spaces (ftfcs). In Section 1, we have considered a fuzzy topology together with a fuzzy convexity on the same underlying set and introduced fuzzy topology fuzzy convexity spaces. Also we introduce the subspace, product and quotient of an ftfcs.

A fuzzy topology is compatible with a fuzzy convexity in a set X, if fuzzy convex hulls of finite fuzzy sets are fuzzy closed in X. Using this concept we introduce fuzzy topological convexity spaces (ftcs) in Section 2.

Chapter 3.

An attempt has been made to study fuzzy local convexity in this chapter. It is a continuation of the study of ftfcs introduced in Chapter 2. We also study subspace, product and quotient of such spaces.

Chapter 4.

This chapter is a study of separation axioms in fuzzy convexity spaces.

In Section 1, we introduce fuzzy hemispaces and study certain related results.

In Sections 2 through 6, we introduce and study concepts FS_o, FS_1, FS_2, FS_3 and FS_4 spaces analogous to S_o, S_1, S_2, S_3 and S_4 spaces in (crisp) convexity theory introduced by R.E. Jamison-Waldner. We also study the invariance or otherwise of these separation properties under subspace, product and quotient operations.

Chapter 5.

In a topological convexity structure (crisp) a number of separation properties were considered by M. Van de vel [33]. The separation axioms in fuzzy topology fuzzy convexity spaces are introduced in this chapter. The separation involves closed convex fuzzy neighbourhoods.

In Section 1, we introduce and study concepts FNS_o, FNS_1 and FNS_2 spaces where 'FNS' stands for 'Fuzzy Neighbourhood separation'.

In Section 2, we introduce pseudo FNS_3 and FNS_3 spaces in an ftfcs and in Section 3, we introduce semi FNS_4 and FNS_4 spaces in an ftfcs.

Also we study concepts like subspace, product and quotient in all these cases.

0.5 Basic Definitions used in the Thesis

Definition 0.5.1.

Let A and B be fuzzy sets in a set X. Then

(i) $A = B \Longleftrightarrow \mu_A(x) = \mu_B(x)$ for all $x \in X$

(ii) $A \subset B \Longleftrightarrow \mu_A(x) \leqslant \mu_B(x)$ for all $x \in X$

(iii) $C = A \cup B \Longleftrightarrow \mu_C(x) = \max\left\{\mu_A(x), \mu_B(x)\right\}$ for all $x \in X$.

(iv) $D = A \cap B \Longleftrightarrow \mu_D(x) = \min\left\{\mu_A(x), \mu_B(x)\right\}$ for all $x \in X$.

(v) $E = A' \Longleftrightarrow \mu_E(x) = 1 - \mu_A(x)$ for all $x \in X$.

For any family $\{A_i\}_{i \in I}$ of fuzzy sets in X, we define intersection $\bigcap_{i \in I} A_i$ and the union $\bigcup_{i \in I} A_i$ respectively by

$$\mu_{\bigcap_{i \in I} A_i}(x) = \inf_{i \in I} \mu_{A_i}(x) \quad \text{and}$$

$$\mu_{\bigcup_{i \in I} A_i}(x) = \sup_{i \in I} \mu_{A_i}(x) \quad \text{for } x \in X$$

The symbol ϕ will be used to denote the empty set such that $\mu_\phi(x) = 0$ for all $x \in X$. For X, we have by definition $\mu_X(x)=1$ for all x in X.

Definition 0.5.2.

Let f be a mapping from a set X to a set Y. If A is a fuzzy set in X, then the fuzzy set f(A) in Y is defined by

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

where $f^{-1}(y) = \{x \in X \mid f(x) = y\}$.

If B is a fuzzy set in Y, then the fuzzy set $f^{-1}(B)$ in X is defined by

$$\mu_{f^{-1}(B)}(x) = \mu_B(f(x)).$$

Definition 0.5.3.

A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions.

(i) $\emptyset, X \in T$

(ii) If $A, B \in T$, then $A \cap B \in T$

(iii) If $A_i \in T$ for each $i \in I$, then $\bigcup_{i \in I} A_i \in T$.

The pair (X,T) is called a fuzzy topological space or fts in short. Every member of T is called a T-open fuzzy set (or simply an open fuzzy set). A fuzzy set is T-closed (or simply closed) if and only if its complement is T-open.

As in general topology, the indiscrete fuzzy topology contains ϕ and X while the discrete fuzzy topology contains all fuzzy sets.

Definition 0.5.4.

A function f from a fuzzy topological space (X,T) to a fuzzy topological space (Y,U) is fuzzy continuous if and only if the inverse of each U-open fuzzy set is a T-open fuzzy set.

Definition 0.5.5.

A function from an fts (X,T) to an fts (Y,U) is said to be F-open (F-closed) if and only if it maps a fuzzy open (closed) set in (X,T) onto a fuzzy open (closed) set in (Y,U).

Definition 0.5.6.

A fuzzy point p in X is a fuzzy set with membership function

$$\mu_p(x) = \begin{cases} \lambda & \text{for } x = x_o \\ 0 & \text{otherwise} \end{cases}$$

where $0 < \lambda \leqslant 1$. P is said to have support x_o and value λ and we write $P = x_{o\lambda}$.

Two fuzzy points are said to be distinct if their supports are distinct. When $\lambda = 1$, P is called a fuzzy singleton.

Chapter 1

FUZZY CONVEXITY SPACES

1.1 Fuzzy Convexity in Vector Spaces

In this section, we quote some results on fuzzy convexity in vector spaces from [11]. Also it is proved that the union of any family of convex fuzzy sets, in a vector space E, totally ordered by inclusion is a convex fuzzy set in E. This motivates the introduction of abstract fuzzy convexity spaces.

Definition 1.1.1 [11]

Let $A_1, A_2, A_3, \ldots, A_n$ be fuzzy sets in a vector space E over K, where K is the space of real or complex numbers. Then define $A_1 \times A_2 \times A_3 \times \ldots \times A_n$ to be the fuzzy set A in E^n whose membership function is given by

$$\mu_A(x_1, x_2, x_3, \ldots, x_n) = \min \left\{ \mu_{A_1}(x_1),\ \mu_{A_2}(x_2), \ldots \mu_{A_n}(x_n) \right\}.$$

Let $f: E^n \longrightarrow E$,

$$f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n.$$

Now define $A_1 + A_2 + A_3 + \ldots + A_n = f(A)$. For λ scalar, and B a fuzzy set in E, define $\lambda B = g(B)$ where $g: E \longrightarrow E$, $g(x) = \lambda x$.

Definition 1.1.2 [11]

A fuzzy set A in a vector space E over K is said to be convex if $kA + (1-k)A \subset A \;\forall\; k \in [0,1]$.

Proposition 1.1.3 [11]

Let A be a fuzzy set in a vector space E over K. The following assertions are equivalent.

(i) A is convex

(ii) $\mu_A(kx + (1-k)y) \geqslant \min\{\mu_A(x), \mu_A(y)\}$ for every $x, y \in E$ and for every $k \in [0,1]$.

(iii) For each $d \in [0,1]$, the crisp set $A_d = \{x \in E \mid \mu_A(x) \geqslant d\}$ is convex.

Note:

$\emptyset$ and E are convex fuzzy sets.

Proposition 1.1.4 [11]

If $\{A_i\}_{i \in I}$ is a family of convex fuzzy sets in E, then $A = \bigcap_{i \in I} A_i$ is a convex fuzzy set in E.

Proposition 1.1.5.

The union of any family of convex fuzzy sets in E totally ordered by inclusion is a convex fuzzy set in E.

Proof:

Let $\{A_\alpha\}_{\alpha \in I}$ be a family of convex fuzzy sets totally ordered by inclusion in E. By proposition 1.1.3 it is enough to show that the crisp set

$$(\cup A_\alpha)_d = \left\{ x \in E \mid \mu_{\cup A_\alpha}(x) \geqslant d \right\} \text{ is convex.}$$

Let $x, y \in (\cup A_\alpha)_d$

i.e. $\mu_{\cup A_\alpha}(x) \geqslant d$ and $\mu_{\cup A_\alpha}(y) \geqslant d$,

i.e. $\vee \mu_{A_\alpha}(x) \geqslant d$ and $\vee \mu_{A_\alpha}(y) \geqslant d$.

Therefore given $\varepsilon > 0$ there exist α, β such that

$$\mu_{A_\alpha}(x) > d-\varepsilon \text{ and } \mu_{A_\beta}(y) > d-\varepsilon$$

Now (A_α) is totally ordered by inclusion,

$\therefore A_\alpha \subseteq A_\beta$ or $A_\beta \subseteq A_\alpha$. w.l.o.g assume that $A_\alpha \subseteq A_\beta$.

$\therefore \mu_{A_\beta}(x) \geqslant \mu_{A_\alpha}(x) > d-\varepsilon$ and $\mu_{A_\beta}(y) \geqslant \mu_{A_\alpha}(y) > d-\varepsilon$.

Now A_β is a convex fuzzy set.

$$\therefore \mu_{A_\beta}(kx + (1-k)y) \geqslant \min\left\{\mu_A(x),\ \mu_A(y)\right\}$$
$$> d - \varepsilon$$

$$\therefore \bigvee_{\alpha<1} \mu_{A_\alpha}(kx + (1-k)y) > d - \varepsilon$$

This is true for all $\varepsilon > 0$.

$$\therefore \quad \mu_{\cup A_\alpha}(kx + (1-k)y) \geqslant d$$

i.e. $\quad kx + (1-k)y \in (\bigcup A_\alpha)_d$

$\therefore \quad (\bigcup A_\alpha)_d$ is a convex crisp set.

$\therefore \quad (\bigcup A_\alpha)$ is a convex fuzzy set

1.2 Fuzzy Convexity Spaces

In this section we define abstract fuzzy convexity spaces- introduce and study the concept of a fuzzy convex hull operator. Also we define a fuzzy convex to convex (FCC) map and a fuzzy convexity preserving (FCP) map in such spaces.

Definition 1.2.1.

Let X be any set. A fuzzy alignment on X is a family '$\mathcal{L}$' of convex fuzzy sets in X which satisfies ?
the following conditions:

(i) $\emptyset, X \in \mathcal{L}$

(ii) If $A_i \in \mathcal{L}$ for each $i \in I$, then $\bigcap_{i \in I} A_i \in \mathcal{L}$.

(iii) If $A_i \in \mathcal{L}$ for each $i \in I$, and if A_i's are totally ordered by inclusion then $\bigcup_{i \in I} A_i \in \mathcal{L}$.

The pair $(X, \mathcal{L})$ is called a fuzzy aligned space or a fuzzy convexity space or fcs in short. Every member of '$\mathcal{L}$' is called an $\mathcal{L}$-convex fuzzy set or just a convex fuzzy set. As in ordinary convexity, the indiscrete fcs contains only $\emptyset$ and X, while the discrete fcs contains all fuzzy sets.

Example:

Let X=N, the set of natural numbers.

$$\mathcal{L} = \{\emptyset\} \cup \{Y \subset I^X \mid A \subseteq Y\}$$

where the fuzzy set A is given by

$$A: \quad \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow \frac{1}{2} \\ x \longrightarrow 0 \quad \text{for every } x \geqslant 3 \end{array}$$

Then $(X, \mathcal{L})$ is a fuzzy convexity space.

Example:

Let $X = [0,1]$

$\mathcal{L} = \{\emptyset, X, A, B\}$ where A and B are fuzzy sets,

$$A: \begin{array}{ll} x \longrightarrow 0, & 0 \leqslant x \leqslant \frac{1}{2} \\ x \longrightarrow \frac{1}{2}, & \frac{1}{2} < x \leqslant 1 \end{array}, \quad B: \begin{array}{ll} x \longrightarrow 0, & 0 \leqslant x \leqslant \frac{1}{2} \\ x \longrightarrow 1, & \frac{1}{2} < x \leqslant 1 \end{array}$$

Then $(X, \mathcal{L})$ is a fuzzy convexity space.

Example:

Let $X = \{a, b, c\}$

$\mathcal{L} = \{\emptyset, X, \{a\}, A\}$ where A is the fuzzy set

$$A: \quad \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow \frac{1}{3} \end{array}$$

Now $(X, \mathcal{L})$ is a fuzzy convexity space.

Note:

Given two fuzzy convexities $\mathcal{L}_1$ and $\mathcal{L}_2$ on the same set X, we say that $\mathcal{L}_1$ is smaller (weaker or coarser) than $\mathcal{L}_2$ or $\mathcal{L}_2$ is larger (stronger or finer) than $\mathcal{L}_1$ if and only if $\mathcal{L}_1 \subset \mathcal{L}_2$.

Note:

From axioms (i) and (ii) in definition 1.2.1 we have that for any subset S of X there is a smallest convex fuzzy set $\mathcal{L}(S)$ containing S and is called the convex hull of the fuzzy set S.

i.e. $\mathcal{L}(S) = \bigcap \{K \in \mathcal{L} \mid S \subseteq K\}$.

Proposition 1.2.2.

The convex hull operator $S \longrightarrow \mathcal{L}(S)$ satisfies the following conditions

(i) $\mathcal{L}(\emptyset) = \emptyset$

(ii) $S \subseteq \mathcal{L}(S)$

(iii) If $S \subseteq T$ then $\mathcal{L}(S) \subseteq \mathcal{L}(T)$

(iv) $\mathcal{L}(\mathcal{L}(S)) = \mathcal{L}(S)$

(v) $\mathcal{L}(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} \mathcal{L}(S_i)$ for any family

of fuzzy sets $\{S_i \mid i \in I\}$, which is totally ordered by inclusion.

Proof:

(i), (ii), (iii) and (iv) can be proved very easily. To prove (v), for $i \in I$

$$s_i \subset \cup s_i$$

$$\therefore \quad \mathcal{L}(s_i) \subset \mathcal{L}(\cup s_i) \text{ for every } i$$

$$\therefore \quad \cup \mathcal{L}(s_i) \subset \mathcal{L}(\cup s_i) \tag{1}$$

Now $s_i \subset \mathcal{L}(s_i)$

$$\therefore \quad \cup s_i \subset \cup \mathcal{L}(s_i)$$

Since $\mathcal{L}(s_i)$'s are convex fuzzy sets totally ordered by inclusion, $\cup \mathcal{L}(s_i)$ is a convex fuzzy set containing $\cup s_i$. Now $\mathcal{L}(\cup s_i)$ is the smallest convex fuzzy set containing $\cup s_i$ and hence

$$\cup s_i \subset \mathcal{L}(\cup s_i) \subset \cup \mathcal{L}(s_i) \tag{2}$$

From (1) and (2) we have

$$\mathcal{L}\left(\bigcup_{i \in I} s_i\right) = \bigcup_{i \in I} \mathcal{L}(s_i).$$

Note:

Conversely given an operator $\mathcal{L}$, on the set of fuzzy subsets of a set X, satisfying conditions (i) — (v) we can find the fuzzy alignment from that, namely, $\mathcal{L} = \{S \subset X \mid \mathcal{L}(S) = S\}$, with respect to which $\mathcal{L}(S)$ becomes the convex hull of S. Thus a fuzzy alignment and its hull operator uniquely determine each other.

Definition 1.2.3.

Let $(X, \mathcal{L}_1)$ and $(Y, \mathcal{L}_2)$ be fuzzy convexity spaces and let $f: X \longrightarrow Y$. Then f is said to be

(i) a fuzzy convexity preserving function (FCP function) if for each convex fuzzy set K in Y, $f^{-1}(K)$ is a convex fuzzy set in X.

(ii) a fuzzy convex to convex function (FCC function) if for each convex fuzzy set K in X, f(K) is a convex fuzzy set in Y.

Proposition 1.2.4.

Let $(X, \mathcal{L}_1)$ and $(Y, \mathcal{L}_2)$ be fuzzy convexity spaces. Then $f: X \longrightarrow Y$ is an FCP function if and only if $f(\mathcal{L}_1(S)) \subseteq \mathcal{L}_2(f(S))$, for every fuzzy subset S of X.

Proof: (Necessity)

Let S be any fuzzy subset of X. Let K be a convex fuzzy set containing f(S) in Y. Now $f(S) \subseteq K$.

$\therefore \quad S \subseteq f^{-1}(K)$

$\therefore \quad \mathcal{L}_1(S) \subseteq \mathcal{L}_1(f^{-1}(K)) = f^{-1}(K)$,

since f is FCP and K is convex.

$\therefore \quad f(\mathcal{L}_1(S)) \subseteq K.$

In particular $f(\mathcal{L}_1(S)) \subseteq \mathcal{L}_2(f(S))$.

Thus necessity follows.

Sufficiency

Let K be a convex fuzzy set in Y. Then $f^{-1}(K)$ is a fuzzy set in X. Therefore we have

$$f(\mathcal{L}_1(f^{-1}(K))) \subseteq \mathcal{L}_2(f(f^{-1}(K))) \subseteq \mathcal{L}_2(K)$$

$$= K, \text{ K being convex.}$$

$$\therefore \quad \mathcal{L}_1(f^{-1}(K)) \subseteq f^{-1}(K) \tag{1}$$

$$\text{Now} \quad f^{-1}(K) \subseteq \mathcal{L}_1(f^{-1}(K)) \tag{2}$$

From (1) and (2) we have

$$\mathcal{L}_1(f^{-1}(K)) = f^{-1}(K)$$

$\therefore \quad f^{-1}(K)$ is a convex fuzzy set in X

$\therefore \quad$ f is an FCP function.

Proposition 1.2.5.

Let $(X, \mathcal{L}_1)$ and $(Y, \mathcal{L}_2)$ be fuzzy convexity spaces. Then $f: X \longrightarrow Y$ is an FCC function if and only if $\mathcal{L}_2(f(S)) \subseteq f(\mathcal{L}_1(S))$ for each fuzzy subset S of X.

Proof:

Suppose $f: X \longrightarrow Y$ is an FCC function. Let S be a fuzzy subset of X. Now $\mathcal{L}_1(S)$ is a convex fuzzy set and $S \subseteq \mathcal{L}_1(S)$. Since f is an FCC map, $f(\mathcal{L}_1(S))$ is a convex fuzzy set in Y, and $f(S) \subseteq f(\mathcal{L}_1(S))$

$$\therefore \quad \mathcal{L}_2(f(S)) \subseteq f(\mathcal{L}_1(S)).$$

Conversely assume that

$\mathcal{L}_2(f(S)) \subseteq f(\mathcal{L}_1(S))$, for every fuzzy subset S of X; and let K be a convex fuzzy set in X.

Then $\mathcal{L}_1(K) = K$ and hence

$$f(\mathcal{L}_1(K)) = f(K).$$

Now,

$$\mathcal{L}_2(f(K)) \subseteq f(\mathcal{L}_1(K)) = f(K)$$

$$\therefore \quad \mathcal{L}_2(f(K)) \subseteq f(K) \tag{3}$$

and

$$f(K) \subseteq \mathcal{L}_2(f(K)) \tag{4}$$

From (3) and (4) we have

$$\mathcal{L}_2\,(f(K)) \;=\; f(K)$$

$\therefore$ f(K) is a convex fuzzy set in Y.

$\therefore$ f is an FCC function.

Definition 1.2.6.

Let $(X, \mathcal{L})$ be a fuzzy convexity space. A collection $\mathcal{C}$ of convex fuzzy subsets of X generates $\mathcal{L}$ (or is a subbase for $\mathcal{L}$) if $\mathcal{C} \subset \mathcal{L}$ and $\mathcal{L}$ is the smallest fuzzy alignment containing $\mathcal{C}$. A subbase $\mathcal{C}$ will be called a base for $\mathcal{L}$ if it is closed for arbitrary intersections.

Note that every family $\mathcal{C}$ of fuzzy subsets of X generates a fuzzy alignment on X by taking the union of collections of sets totally ordered by inclusion of arbitrary intersections of members of $\mathcal{C}$.

Proposition 1.2.7.

Let $(X, \mathcal{L}_1)$ and $(Y, \mathcal{L}_2)$ be fuzzy convexity spaces. If $\mathcal{C}$ is a subbase for $\mathcal{L}_2$, then $f: X \longrightarrow Y$ is an FCP function if and only if the inverse images of members of $\mathcal{C}$ is in $\mathcal{L}_1$.

Proof:

Let $f: X \longrightarrow Y$ be an FCP function. Then for any convex fuzzy set K in Y $f^{-1}(K)$ is a convex fuzzy set in X. This is true for every convex fuzzy set K. Therefore inverse images of members of $\mathcal{C}$ is a convex fuzzy set in X. Conversely, since $\mathcal{C}$ is a subbase for $\mathcal{L}_2$, each convex fuzzy set in Y is the union of a totally ordered family of intersections of members of $\mathcal{C}$ and f^{-1} preserves unions and intersections; hence it follows that f is an FCP function.

1.3 Subspaces, Products and Quotients of Fuzzy Convexity Spaces

Definition 1.3.1.

Let $(X, \mathcal{L})$ be an fcs and M a crisp subset of X. Then a fuzzy alignment on M is given by the fuzzy sets of the form $\{L \cap M \mid L \in \mathcal{L}\}$. Then the pair $(M, \mathcal{L}_M)$ is a fuzzy subspace of $(X, \mathcal{L})$.

Note:

The convex hull operator on M is given by

$\mathcal{L}_M(S) = \mathcal{L}(S) \cap M$ for fuzzy subsets S of M.

Definition 1.3.2.

Let $(X_\alpha, \mathcal{L}_\alpha)_{\alpha \in I}$ be a family of fuzzy convexity spaces. Let $X = \prod_{\alpha \in I} X_\alpha$ be the product space and let $\pi_\alpha : X \longrightarrow X_\alpha$ be the projection map. Then X can be equipped with the fuzzy alignment $\mathcal{L}$ generated by the convex fuzzy sets of the form $\left\{\pi_\alpha^{-1}(C_\alpha) \mid C_\alpha \in \mathcal{L}_\alpha,\ \alpha \in I\right\}$. Then $\mathcal{L}$ is called the product fuzzy alignment for X and $(X, \mathcal{L})$ is called the product fuzzy convexity space.

Remark:

The product fuzzy alignment is the alignment which has for a subbase the collection

$$\left\{\pi_\alpha^{-1}(u_\alpha) \mid u_\alpha \in \mathcal{L}_\alpha,\ \alpha \in I\right\}.$$

Definition 1.3.3.

Let X be any set. Let R be an equivalence relation defined on X. Let X/R be the usual quotient set and let π be the projection map from X to X/R. If $(X, \mathcal{L})$ is an fcs one can define a fuzzy alignment on X/R as follows:

Let $\mathcal{Y}$ be the family of fuzzy sets in X/R defined by

$\mathcal{Y} = \left\{u \mid \pi^{-1}(u) \in \mathcal{L}\right\}$. Then $\mathcal{Y}$ is a fuzzy alignment on X/R and $(X/R, \mathcal{Y})$ is called the quotient fcs.

Example:

Let $X = N$, the set of natural numbers.

$$\mathcal{L} = \{\emptyset\} \cup \{Y \subset I^X \mid A \subseteq Y\}$$

where A is the fuzzy set given by

$$A: \quad \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow \frac{1}{2} \\ x \longrightarrow 0 \text{ for every } x \geqslant 3 \end{array}$$

Then $(X, \mathcal{L})$ is an fcs.

Define $\pi: N \longrightarrow \tilde{N}$ where $\tilde{N} = \{0,1\}$ as follows:

$$\pi(x) = \begin{cases} 0 \text{ if } x \text{ is an odd number} \\ 1 \text{ if } x \text{ is an even number} \end{cases}$$

Consider

$$\tilde{\mathcal{L}} = \{\emptyset\} \cup \{Y \subset I^X \mid \tilde{A} \subseteq Y\}$$

where $\tilde{A}$ is the fuzzy set given by

$$\tilde{A}: \quad \begin{array}{l} 0 \longrightarrow \frac{1}{2} \\ 1 \longrightarrow \frac{1}{2} \end{array}$$

. Then $(\tilde{N}, \tilde{\mathcal{L}})$ is a quotient fcs.

Proposition 1.3.4.

Every quotient fcs of a discrete fcs is a discrete fcs.

Proof:

Let $f : (X, \mathcal{L}) \longrightarrow (Y, \mathcal{Y})$ be a quotient map where $(X, \mathcal{L})$ is a discrete fcs and $\mathcal{Y}$ is the quotient fuzzy alignment on Y. Since $u \in \mathcal{Y}$ if and only if $f^{-1}(u) \in \mathcal{L}$, where $\mathcal{L}$ is discrete, it follows that $(Y, \mathcal{Y})$ is a discrete fuzzy convexity space.

Chapter 2

FUZZY TOPOLOGY FUZZY CONVEXITY SPACES *

2.1. Fuzzy Topology Fuzzy Convexity Spaces

Definition 2.1.1.

A triple $(X,\mathcal{L},T)$ consisting of a set X, a fuzzy alignment $\mathcal{L}$, and a fuzzy topology T is called a fuzzy topology fuzzy convexity space or ftfcs in short.

Example:

Let X = N, the set of natural numbers

$T = \{\emptyset, X, A, B\}$, where A and B are the fuzzy sets

$$A: \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow \frac{1}{2} \\ x \longrightarrow 0 \ \forall\ x \geq 3 \end{array} \qquad \text{and} \qquad B : x \longrightarrow \frac{1}{2}\ \forall\ x.$$

Choose $\mathcal{L} = \{\emptyset\} \cup \{Y \subset I^X \mid A \subseteq Y\}$

Then $(X,\mathcal{L},T)$ is an ftfcs.

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* some of the results of this chapter appeared in Fuzzy Sets and Systems 1994 [28].

Definition 2.1.2.

Let a_λ , be a fuzzy point in an ftfcs $(X, \mathcal{L}, T)$ Then a fuzzy set N is called a fuzzy neighbourhood of a_λ if there exists $A \in T$ such that $a_\lambda \in A \subseteq N$.

Definition 2.1.3.

Let $(X, \mathcal{L}, T)$ be an ftfcs. Let M be an ordinary subset of X. Then a fuzzy topology on M is defined as $T_M = \{M \cap Y | Y \in T\}$ and a fuzzy alignment on M is given by

$$\mathcal{L}_M = \{M \cap L \mid L \in \mathcal{L}\}.$$

Then the corresponding triple $(M, \mathcal{L}_M, T_M)$ is a subspace of $(X, \mathcal{L}, T)$.

Definition 2.1.4

Let $(X_\alpha, \mathcal{L}_\alpha, T_\alpha)_{\alpha \in I}$ be a family of fuzzy topology fuzzy convexity spaces. Let $X = \prod_{\alpha \in I} X_\alpha$ be the product set. Let π_α be the projection map from X to X_α. Then the family of fuzzy sets $\{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \in T_\alpha, \alpha \in I\}$ will generate a fuzzy topology T on X. Also $X = \prod_{\alpha \in I} X_\alpha$ can be equipped with the fuzzy alignment $\mathcal{L}$ generated by the fuzzy sets of the form $\{\pi_\alpha^{-1}(C_\alpha) | C_\alpha \in \mathcal{L}_\alpha, \alpha \in I\}$.

Then T is called the product fuzzy topology on X and $\mathcal{L}$ is called the product fuzzy alignment on X and $(X, \mathcal{L}, T)$ is called the product ftfcs.

Definition 2.1.5.

Let X be any set. Let R be an equivalence relation defined on X. Let X/R be the usual quotient set and let π be the projection map from X to X/R. If X is an ftfcs, one can define a fuzzy topology and a fuzzy alignment on X/R in such a way that a set U in X/R is F-open if and only if $\pi^{-1}(U)$ is F-open in X and a set V in X/R is fuzzy convex if and only if $\pi^{-1}(V)$ is fuzzy convex in X. Then the topology on X/R is called the quotient fuzzy topology T_q and the alignment on X/R is called the quotient fuzzy alignment $\mathcal{L}_q$ and X/R with topology T_q and alignment $\mathcal{L}_q$ is called the quotient ftfcs denoted by $(X/R, \mathcal{L}_q, T_q)$.

Example:

Consider the ftfcs $(X, \mathcal{L}, T)$ given below (Definition 2.1.1), where X = N, the set of natural numbers.

$T = \{\emptyset, X, A, B\}$ where A and B are the fuzzy sets

$$A : \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow \frac{1}{2} \\ x \longrightarrow 0 \ \forall \ x \geq 3 \end{array} \qquad \text{and } B : x \longrightarrow \tfrac{1}{2} \ \forall \ x.$$

and $\mathcal{L} = \{\emptyset\} \cup \{Y \subset I^X \mid A \subseteq Y\}$

(i) Define $\pi : N \longrightarrow \tilde{N}_1$, where

$\tilde{N}_1 = N \setminus \{1\}$ as follows:

$\pi(1) = 2, \quad \pi(x) = x$ for every $x \geqslant 2$.

Define a fuzzy topology and a fuzzy alignment on $\tilde{N}_1$ as follows:

$\tilde{T}_1 = \{\emptyset, \tilde{N}_1, \tilde{A}\}$ where $\tilde{A}$ is the fuzzy set given by

$$\tilde{A} : \begin{array}{l} 2 \longrightarrow \frac{1}{2} \\ x \longrightarrow 0 \quad \text{for every } x \geqslant 3. \end{array}$$

$$\tilde{\mathcal{L}}_1 = \{\emptyset\} \cup \{Y \subset I^X \mid \tilde{A} \subseteq Y\}$$

Then $(\tilde{N}_1, \tilde{\mathcal{L}}_1, \tilde{T})$ is a quotient ftfcs of $(X, \mathcal{L}, T)$

(ii) Define $\pi : N \longrightarrow \tilde{N}_2$ where

$\tilde{N}_2 = \{0, 1\}$ as follows:

$$\pi(x) = \begin{cases} 0 & \text{if } x \text{ is an odd number} \\ 1 & \text{if } x \text{ is an even number} \end{cases}$$

Consider $T_2 = \{\emptyset, \tilde{N}_2, B\}$ where

B is the fuzzy set given by

$$\tilde{B} : \quad \begin{array}{l} 0 \longrightarrow \frac{1}{2} \\ 1 \longrightarrow \frac{1}{2} \end{array}$$

$$\tilde{\mathcal{L}}_2 = \{\emptyset\} \cup \{Y \subset I^X \mid \tilde{B} \subseteq Y\}$$

Then $(\tilde{N}_2, \tilde{\mathcal{L}}_2, \tilde{T}_2)$ is a quotient ftfcs.

2.2 Fuzzy Topological Convexity Spaces

Definition 2.2.1.

Let X be a set with a fuzzy topology T and a fuzzy convexity $\mathcal{L}$. Then T is said to be compatible with $\mathcal{L}$, if the fuzzy convex hulls of finite fuzzy sets are fuzzy closed in (X,T). Then $(X,\mathcal{L},T)$ is called a fuzzy topological convexity space or ftcs in short.

Note:

We say that a fuzzy set A in X is finite if its support is a finite subset of X.

Example:

Let $X = \{a, b, c\}$

$T = \{\emptyset, X, A_1, A_2, A_3, A_4, A_5, A_6\}$

$\mathcal{L} = \{\emptyset, X, A_1, A_3, 1-A_2\}$ where

$$A_1: \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{array}, \quad A_2: \begin{array}{l} a \longrightarrow \frac{3}{4} \\ b \longrightarrow \frac{3}{4} \\ c \longrightarrow 1 \end{array} \quad A_3 = 1-A_1: \begin{array}{l} a \longrightarrow 0 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{array}$$

$$A_4: \begin{array}{l} a \longrightarrow \frac{1}{2} \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow 0 \end{array}, \quad A_5: \begin{array}{l} a \longrightarrow \frac{3}{4} \\ b \longrightarrow \frac{3}{4} \\ c \longrightarrow 0 \end{array} \quad A_6: \begin{array}{l} a \longrightarrow \frac{1}{2} \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow 1 \end{array}$$

Then $(X, \mathcal{L}, T)$ is an ftcs.

Example:

Let X be an infinite set with

$$\mathcal{L} = \{ A \subset I^X \mid A \text{ is finite} \}$$

$$T = \{\emptyset\} \cup \{ A \subset I^X \mid 1-A \text{ is finite} \}$$

Then $(X, \mathcal{L}, T)$ is an ftcs.

Remark:

It is almost obvious that an ftcs is always an ftfcs, and the converse is not true, i.e., an ftfcs need not be an ftcs.

Example:

Let X = N, the set of natural numbers.

$T = \{\emptyset, X, A, B, C, D, E, F, G\}$ and

$\mathcal{L} = X \cup \{Y \subset I^X \mid Y \subseteq K\}$ where

A,B,C,D,E,F,G and K are the fuzzy sets

$$A: \begin{cases} 1 \longrightarrow \frac{3}{4} \\ x \longrightarrow 1 \ \forall x \geqslant 2, \end{cases} \qquad B: \begin{cases} 1 \longrightarrow 1 \\ 2 \longrightarrow \frac{3}{4} \\ x \longrightarrow 1 \ \forall x \geqslant 3 \end{cases}$$

$$C: \begin{cases} 1 \longrightarrow \frac{3}{4} \\ 2 \longrightarrow \frac{3}{4} \\ x \longrightarrow 1 \ \forall x \geqslant 3 \end{cases} \qquad D: \ x \longrightarrow 1 \ \forall x$$

$$E: \begin{cases} 1 \longrightarrow \frac{1}{2} \\ x \longrightarrow 1 \ \forall x \geqslant 2 \end{cases} \qquad F: \begin{cases} 1 \longrightarrow \frac{1}{3} \\ x \longrightarrow 0 \ \forall x \geqslant 2 \end{cases}$$

$$G: \begin{cases} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow \frac{3}{4} \\ x \longrightarrow 0 \ \forall x \geqslant 3 \end{cases} \qquad K: \begin{cases} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow \frac{1}{2} \\ x \longrightarrow 0 \ \forall x \geqslant 3 \end{cases}$$

Then $(X, \mathcal{L}, T)$ is an ftfcs which is not an ftcs, Since K is convex but not closed.

Proposition 2.2.2.

Subspace of an ftcs is an ftcs.

Proof:

Let $(X,\mathcal{L},T)$ be an ftcs. Let $(M,\mathcal{L}_M,T_M)$ be a subspace of $(X,\mathcal{L},T)$. Clearly $(M, \mathcal{L}_M, T_M)$ is a fuzzy topology fuzzy convexity space. Now to show that it is an ftcs.

Let A be the fuzzy convex hull of a fuzzy set generated by a finite fuzzy set with support $\{a_1,a_2,a_3,\ldots,a_k\}$ in M. Since A is a convex fuzzy set in M, we have $A = M\cap L$ where $L \in \mathcal{L}$. Now we can take L to be the fuzzy convex hull of $\{a_1,a_2,\ldots,a_k\}$ in X. Since X is an ftcs, L is fuzzy closed in X and hence $A = M\cap L$ is a fuzzy closed set in M. Hence $(M,\mathcal{L}_M, T_M)$ is an ftcs.

Note 2.2.3.

An FCP, F-continuous image of an ftcs need not be an ftcs.

Example:

Let $X =\{a,b,c\}$

$$T_1 = \{\emptyset,X,A_1,A_2,A_3,A_4,A_5,A_6\}$$

$$\mathcal{L}_1 = \{\emptyset,X,A_1,A_3, 1-A_2\}$$ where

$$A_1 : \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{array} \quad , \quad A_2 : \begin{array}{l} a \longrightarrow \frac{3}{4} \\ b \longrightarrow \frac{3}{4} \\ c \longrightarrow 1 \end{array}$$

$$1 - A_1 = A_3 : \begin{array}{l} a \longrightarrow 0 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{array}$$

$$A_4 : \begin{array}{l} a \longrightarrow \frac{1}{2} \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow 0 \end{array} \qquad A_5 : \begin{array}{l} a \longrightarrow \frac{3}{4} \\ b \longrightarrow \frac{3}{4} \\ c \longrightarrow 0 \end{array} \qquad A_6 : \begin{array}{l} a \longrightarrow \frac{1}{2} \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow 1 \end{array}$$

Consider $(Y, \mathcal{L}_2, T_2)$, where $Y = \{1, 2\}$,

$\mathcal{L}_2 = \{ \emptyset, Y, C_1, C_2, C_3 \}$ where

$$C_1 : \begin{array}{l} 1 \longrightarrow 0 \\ 2 \longrightarrow 1 \end{array} \quad , \quad C_2 : \begin{array}{l} 1 \longrightarrow \frac{1}{4} \\ 2 \longrightarrow 0 \end{array} \quad , \quad C_3 : \begin{array}{l} 1 \longrightarrow 1 \\ 2 \longrightarrow 0 \end{array}$$

and

$T_2 = \{ \emptyset, Y, B_1, B_2, B_3, B_4 \}$ where

$$B_1 : \begin{array}{l} 1 \longrightarrow 1 \\ 2 \longrightarrow 0 \end{array} \qquad B_2 : \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow 0 \end{array} \qquad B_3 : \begin{array}{l} 1 \longrightarrow 0 \\ 2 \longrightarrow 1 \end{array} \qquad B_4 : \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow 1 \end{array}$$

Let $f : (X, \mathcal{L}_1, T_1) \longrightarrow (Y, \mathcal{L}_2, T_2)$ be defined as follows

$f(a) = 1$, $f(b) = 1$ and $f(c) = 2$.

Now $f^{-1}(B_i)$ $i = 1,2,3,4$ is an open fuzzy set in X for each i. Therefore f is a fuzzy continuous map. Also $f^{-1}(C_i)$, $i = 1,2,3$ is a convex fuzzy set in X for each i. Therefore f is an FCP map.

Now $(X,\mathcal{L}_1,T_1)$ is an ftcs since fuzzy convex hulls of each finite fuzzy set is a closed fuzzy set in X. But $(Y,\mathcal{L}_2,T_2)$ is not an ftcs, since the convex fuzzy set C_2 is not a closed fuzzy set in Y.

Proposition 2.2.4.

Quotient of an ftcs $(X,\mathcal{L}_1,T_1)$ is an ftcs if X is finite.

Proof:

Let $f : (X,\mathcal{L}_1,T_1) \longrightarrow (Y,\mathcal{L}_2,T_2)$ be the quotient map. Let A be a (finite) fuzzy set in Y and $K \subset Y$ be the fuzzy convex hull of A. Then $f^{-1}(K)$ is a convex fuzzy set in X and since X is finite, $f^{-1}(K)$ will be always finite and hence it can be considered as the fuzzy convex hull of a finite fuzzy set in X. Now X is an ftcs and hence $f^{-1}(K)$ is fuzzy closed. Therefore K is fuzzy closed in Y since f is the quotient map from X to Y. Therefore convex hulls of finite fuzzy sets are fuzzy closed in Y. Hence Y is an ftcs.

Chapter 3

FUZZY LOCAL CONVEXITY*

Introduction

In this chapter we introduce the concept of fuzzy local convexity in an ftfcs and define a locally ftfcs. Also it is proved that an FCC, F-continuous, F-open image of a locally ftfcs is a locally ftfcs. We also study the subspace, product and quotient of a locally ftfcs.

3.1 Locally Fuzzy Topology Fuzzy Convexity Spaces

Definition 3.1.1.

Let a_λ be a fuzzy point in an ftfcs $(X, \mathcal{L}, T)$. Then a fuzzy set N is called a fuzzy neighbourhood of a_λ if there exists $A \in T$ such that $a_\lambda \in A \subseteq N$.

Definition 3.1.2.

An ftfcs $(X, \mathcal{L}, T)$ is said to be locally fuzzy convex at a fuzzy point a_λ if for every fuzzy neighbourhood U of a_λ there is some convex fuzzy neighbourhood C of a_λ which is contained in U.

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* Almost all the results of this chapter appeared as a research paper in Fuzzy Sets and Systems 1994 [28].

$(X, \mathcal{L}, T)$ is locally fuzzy convex if it is locally fuzzy convex at each of its fuzzy points.

Example:

The Euclidean space R^n is locally convex under usual convexity and considering crisp as a special case of 'fuzzy', this means R^n is locally fuzzy convex.

Example:

Let $X = \{a, b, c\}$

$$T = \{\emptyset, X\} \cup \{a_\alpha | 0 < \alpha \leqslant \tfrac{1}{2}\}$$

$$\mathcal{L} = \{\emptyset, X, \{a\}, \{a, b\}\} \cup \{a_\alpha | 0 < \alpha < \tfrac{1}{2}\}$$

$(X, \mathcal{L}, T)$ is locally fuzzy convex.

Proposition 3.1.3.

An FCC, F-open, F-continuous image of a locally ftfcs is a locally ftfcs.

Proof:

Let $f: (X, \mathcal{L}_1, T_1) \longrightarrow (Y, \mathcal{L}_2, T_2)$ be an FCC, F-open, F-continuous onto map. Let a_λ be a fuzzy point in Y. Then we can find a point b in X such that $f(b) = a$. Then clearly $f(b_\lambda) = a_\lambda$. Let U be a fuzzy neighbourhood of a_λ in Y. Then $f^{-1}(U)$ is a fuzzy neighbourhood of b_λ in X. Since X is a locally ftfcs, there exists an $\mathcal{L}_1$-convex fuzzy neighbourhood C of b_λ in X such that

$$b_\lambda \in C \subseteq f^{-1}(U)$$

$$\therefore \quad f(b_\lambda) \in f(C) \subseteq U$$

i.e. $\quad a_\lambda \in f(C) \subseteq U$

Since f is an FCC, F-open onto map, f(C) is an $\mathcal{L}_2$-convex fuzzy neighbourhood of a_λ in Y. Hence Y is a locally ftfcs.

Proposition 3.1.4.

Any subspace of a locally ftfcs is a locally ftfcs.

Proof:

Let $(X, \mathcal{L}, T)$ be a locally ftfcs. Let $M \subset X$ and $(M, \mathcal{L}_M, T_M)$ be the corresponding subspace of $(X, \mathcal{L}, T)$. Let a_λ be a fuzzy point in M and let U be an F-open neighbourhood of a_λ in M. i.e. $a_\lambda \in U \in T_M$. Since U is F-open in M, we have $U = V \cap M$ where $V \in T$. Since X is locally fuzzy convex, there exists a convex fuzzy neighbourhood C of a_λ such that $a_\lambda \in C \subset V$. Then

$a_\lambda \in C \cap M \subset V \cap M$. Now $C \cap M$ is a convex fuzzy neighbourhood of a_λ in $(M, \mathcal{L}_M, T_M)$ and so M is locally fuzzy convex.

Remark: <u>Fuzzy Topology Fuzzy Convexity Fuzzy Subspace</u>

Let $(X, \mathcal{L}, T)$ be an ftfcs and M a fuzzy subset of X. Then define

$$\mathcal{L}_M = \{L \cap M \mid L \in \mathcal{L}\}$$

and $$T_M = \{A \cap M \mid A \in T\}.$$

We can say that $(M, \mathcal{L}_M, T_M)$ is a fuzzy topology fuzzy convexity fuzzy subspace of $(X, \mathcal{L}, T)$ in the following sense:

(i) $\emptyset, M \in \mathcal{L}_M$

(ii) If $A_i \in \mathcal{L}_M$ for each $i \in I$, then $\bigcap A_i \in \mathcal{L}_M$.

(iii) If $A_i \in \mathcal{L}_M$ for each $i \in I$ and if A_i's are totally ordered by inclusion,

then $\bigcup A_i \in \mathcal{L}_M$.

Again (i) $\emptyset, M \in T_M$

(ii) $A, B \in T_M \Rightarrow A \cap B \in T_M$

(iii) If $A_i \in T_M$ then $\bigcup A_i \in T_M$.

Imitating the proof of the above Proposition 3.1.4, we can show that anysuch fuzzy subspace of a locally ftfcs is a locally ftfcs with obvious definition for locally ftfcs in the case of the subspace.

In the following chapters also wherever we prove results for crisp subspaces, we could obtain analogous results for fuzzy subspaces; however we would be restricting ourselves to crisp subspaces only.

Remark:

We proved in Proposition 1.2.5 that a map $f : (X, \mathcal{L}_1) \longrightarrow (Y, \mathcal{L}_2)$ is an FCC function if and only if $\mathcal{L}_2(f(S)) \subseteq f(\mathcal{L}_1(S))$, for each fuzzy subset S of X. However, as in the crisp case (cf. Van de Vel [33]) we can prove the following:

Lemma 3.1.5.

In a product space, the polytopes (i.e. fuzzy convex hulls of finite fuzzy sets) are of the product type and hence for each finite subset S of the product,

$$\mathcal{L}(S) = \prod_{\alpha \in I} \mathcal{L}(\pi_\alpha(S))$$

Lemma 3.1.6.

$f: (X, \mathcal{L}_1) \longrightarrow (Y, \mathcal{L}_2)$ is FCC if and only if

$\mathcal{L}_2(f(S)) \subseteq f(\mathcal{L}_1(S))$ for each finite fuzzy set S of X.

Proposition 3.1.7.

The projection map π_α of a product to its factors is both FCP and FCC.

Proof:

In a product space $X = \prod X_\alpha$, a fuzzy alignment can be generated by the convex fuzzy sets of the form

$$\left\{\pi_\alpha^{-1}(C_\alpha) \mid C_\alpha \text{ is a convex fuzzy set in } X_\alpha ,\ \alpha \in I\right\},$$

it follows that each π_α is FCP.

That each π_α is FCC is a consequenceof the two lemmas and the remark above.

Proposition 3.1.8.

A nonempty product space $\prod_{\alpha \in I} (X_\alpha, \mathcal{L}_\alpha, T_\alpha)$ is locally fuzzy convex if and only if each factor is locally fuzzy convex.

Proof:

Suppose each X_α is locally fuzzy convex. Let a_λ be a fuzzy point in $X = \prod X_\alpha$ and consider a basic fuzzy neighbourhood

$$\pi_{\alpha_1}^{-1}(U_1) \cap \pi_{\alpha_2}^{-1}(U_2) \cap \ldots \cap \pi_{\alpha_n}^{-1}(U_n)$$

of a_λ in X where π_α is the projection map from X to X_α. Now U_i is a fuzzy neighbourhood of $(a_{\alpha_i})_\lambda$ in X_{α_i} for $i = 1,2,3,\ldots,n$, and since each X_{α_i} is locally fuzzy convex, U_i contains a fuzzy convex neighbourhood C_i of $(a_{\alpha_i})_\lambda$. i.e. $(a_{\alpha_i})_\lambda \in C_i \subset U_i$. Then

$$\pi_{\alpha_1}^{-1}(C_1) \cap \pi_{\alpha_2}^{-1}(C_2) \cap \ldots \cap \pi_{\alpha_n}^{-1}(C_n)$$

is a convex fuzzy neighbourhood of a_λ contained in

$$\pi_{\alpha_1}^{-1}(U_1) \cap \pi_{\alpha_2}^{-1}(U_2) \cap \ldots \cap \pi_{\alpha_n}^{-1}(U_n).$$

Thus every fuzzy neighbourhood of a_λ contains a convex fuzzy neighbourhood and hence X is locally fuzzy convex.

Conversely let $(a_\alpha)_\lambda$ be a fuzzy point in X_α. Then we can choose a fuzzy point a_λ in X such that $\pi_\alpha(a_\lambda) = (a_\alpha)_\lambda$. Let U_α be a fuzzy neighbourhood of $(a_\alpha)_\lambda$ in X_α. Then $\pi_\alpha^{-1}(U_\alpha)$ is a fuzzy neighbourhood of a_λ in X. Since X is locally fuzzy convex, there exists a convex fuzzy neighbourhood 'C' of a_λ contained in $\pi_\alpha^{-1}(U_\alpha)$. Since π_α is FCC, $\pi_\alpha(C)$ is a convex fuzzy neighbourhood of $(a_\alpha)_\lambda$ in X_α contained in U_α. Hence X_α is locally fuzzy convex.

Proposition 3.1.9.

Quotient of a locally ftfcs is a locally ftfcs if the quotient map is an FCC, F-open map.

Proof:

Let $(X, \mathcal{L}, T)$ be a locally ftfcs. Let f be the quotient map from X to Y. Let G be an open fuzzy neighbourhood of a fuzzy point a_λ, $0 < \lambda \leq 1$ in Y. Then we can find a point b in X such that f(b) = a. Then clearly $f(b_\lambda) = a_\lambda$. Therefore $f^{-1}(G)$ is an open fuzzy neighbourhood of b_λ in X. Since X is locally fuzzy convex, there exists a convex fuzzy neighbourhood C of b_λ

such that $b_\lambda \in C \subset f^{-1}(G)$.

Then $f(b_\lambda) \in f(C) \subset G$

i.e. $a_\lambda \in f(C) \subset G$.

Since f is an FCC, F-open map, f(C) is a convex fuzzy neighbourhood of a_λ in Y and hence Y is a locally ftfcs.

Remark:

If X is a fuzzy topological vector space with fuzzy convexity as defined in 1.1, then the quotients are locally ftfcs, if the quotient map is F-open. This is because, if X is a fuzzy topological vector space, then a quotient map f is a linear map and under a linear map the image of a convex fuzzy set is a convex fuzzy set [11].

Chapter 4

SEPARATION AXIOMS IN FUZZY CONVEXITY SPACES

4.1. Fuzzy Hemispaces or Fuzzy Half Spaces

Definition 4.1.1.

Let $(X, \mathcal{L})$ be a fuzzy convexity space. A subset H of X is called a fuzzy hemispace (fuzzy half space, fuzzy biconvex set) if H is a convex fuzzy set with a convex fuzzy complement.

Note:

$\emptyset$ and X are fuzzy hemispaces.

Proposition 4.1.2.

Let $(X, \mathcal{L}_1)$ and $(Y, \mathcal{L}_2)$ be fuzzy convexity spaces. If $f:(X, \mathcal{L}_1) \longrightarrow (Y, \mathcal{L}_2)$ is FCP and H is a fuzzy hemispace in Y, then $f^{-1}(H)$ is a fuzzy hemispace in X.

Proof:

H is a fuzzy hemispace of Y. Then H and its complement H' are both convex fuzzy sets in Y. Therefore $f^{-1}(H)$ and $f^{-1}(H')$ are both convex fuzzy sets in X.

Now to show that $f^{-1}(H)$ is a fuzzy hemispace in X, we have to prove that $[f^{-1}(H)]'$ is a convex fuzzy set. For every $x \in X$,

$$\begin{aligned}
\mu_{f^{-1}(H')}(x) &= \mu_{H'}(f(x)) \\
&= 1 - \mu_H(f(x)) \\
&= 1 - \mu_{f^{-1}(H)}(x) \\
&= \mu_{[f^{-1}(H)]'}(x)
\end{aligned}$$

$$\therefore \quad f^{-1}(H') = [f^{-1}(H)]'$$

Now $f^{-1}(H')$ is a convex fuzzy set and hence $[f^{-1}(H)]'$ is a convex fuzzy set. Therefore $f^{-1}(H)$ is a fuzzy hemispace in X.

Proposition 4.1.3

Let $(X, \mathcal{L})$ be an fcs and M be a crisp subspace of X and if H is a fuzzy hemispace in X, then $H \cap M$ is a fuzzy hemispace in M.

Proof:

H is a fuzzy hemispace in X. Clearly $H \cap M$ is a convex fuzzy set in M and we have to show that the complement of $H \cap M$ (i.e. $(H \cap M)'$) is also a convex fuzzy set.

For $x \in M$, we have

$$\begin{aligned}
\mu_{(H \cap M)'}(x) &= 1 - \mu_{H \cap M}(x) \\
&= 1 - \min\left\{\mu_H(x), \mu_M(x)\right\} \\
&= 1 - \mu_H(x) \text{ since } x \in M \\
&= \mu_{H'}(x).
\end{aligned}$$

Now,

$$\begin{aligned}
\mu_{H' \cap M}(x) &= \min\left\{\mu_{H'}(x), \mu_M(x)\right\} \\
&= \mu_{H'}(x) \text{ since } x \in M.
\end{aligned}$$

$$\therefore \quad \mu_{(H \cap M)'}(x) = \mu_{H' \cap M}(x) \text{ for every } x \in M$$

$$\therefore \quad (H \cap M)' = H' \cap M$$

Since $H' \cap M$ is a convex fuzzy set in M, we have the complement of $H \cap M$ is also a convex fuzzy set. Hence $H \cap M$ is a fuzzy hemispace in M.

4.2 FS_o Spaces

Definition 4.2.1.

An fcs $(X, \mathcal{L})$ is said to be FS_o if fuzzy convex hulls of distinct fuzzy points are distinct (Two fuzzy points a_λ and b_μ are distinct if $a \neq b$).

Example:

Let $X = \{a, b, c\}$

Choose $\mathcal{L}_1 = \left\{ \emptyset, X, \{a\}, \{a,b\}, \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow 1 \\ c \longrightarrow \frac{1}{2} \end{array} \right\}$

Then $(X, \mathcal{L}_1)$ is an FS_o space.

Choose $\mathcal{L}_2 = \left\{ \emptyset, X, \{a\}, \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow \frac{1}{4} \end{array} \right\}$

Then $(X, \mathcal{L}_2)$ is not an FS_o space since the fuzzy convex hulls of b_λ, $0 < \lambda \leqslant \frac{1}{2}$ and c_μ, $0 < \mu \leqslant \frac{1}{4}$ is the same fuzzy set $\begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow \frac{1}{4} \end{array}$.

Proposition 4.2.2.

If $(X, \mathcal{L})$ is an FS_o space, then given two distinct fuzzy points a_λ and b_μ, there exist distinct convex fuzzy sets F_1 and F_2. i.e., $F_1 \neq F_2$ such that $a_\lambda \in F_1$ and $b_\mu \in F_2$.

Note:

$$a_\lambda \in F_1 \Longleftrightarrow \lambda \leqslant \mu_{F_1}(a)$$

Note:

The converse of Proposition 4.2.2 is not true.

Example:

Let $X = \{a, b\}$ and

$$\mathcal{L} = \left\{ \emptyset, X, b_{1/2}, \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow \frac{3}{4} \end{array} \right\}$$

Now for distinct fuzzy points a_λ, b_μ in X, there exist distinct convex fuzzy sets $\begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow \frac{3}{4} \end{array}$ and X containing both of them. But there do not exist distinct fuzzy convex hulls containing each of them.

Proposition 4.2.3.

If $(X, \mathcal{L})$ is an FS_o space, then any $M \subset X$ is also an FS_o space.

Proof:

$(X, \mathcal{L})$ is an FS_o space. Let a_λ, b_μ be two distinct fuzzy points in M. Suppose F_1 and F_2 are the distinct fuzzy convex hulls of a_λ and b_μ in X respectively. Now we can prove that $F_1 \cap M$ and $F_2 \cap M$ are the distinct fuzzy convex hulls of a_λ and b_μ in M.

First we will prove that $F_1 \cap M$ and $F_2 \cap M$ are distinct convex fuzzy sets containing a_λ and b_μ in M. If possible assume that $F_1 \cap M = F_2 \cap M$.

Then $a_\lambda, b_\mu \in F_1 \cap M = F_2 \cap M$

$\therefore$ $a_\lambda, b_\mu \in F_1$ and $a_\lambda, b_\mu \in F_2$

But then $F_1 \cap F_2$ is fuzzy convex and since $a_\lambda \in F_1 \cap F_2 \subset F_1$ and F_1 is the convex hull containing a_λ, we get $F_1 \cap F_2 = F_1$. Similarly $F_1 \cap F_2 = F_2$.

$\therefore$ $F_1 = F_2$ and is a contradiction to the assumption that F_1 and F_2 are distinct.

$\therefore$ $F_1 \cap M \neq F_2 \cap M$

Now we will prove that $F_1 \cap M$ and $F_2 \cap M$ are the fuzzy convex hulls of a_λ and b_μ in M. If possible assume that there exist convex fuzzy sets F_1' and F_2' in M such that $a_\lambda \in F_1' \subset F_1 \cap M$ and $b_\mu \in F_2' \subset F_2 \cap M$.

Since F_1' and F_2' are convex fuzzy sets in M we have $F_1' = U \cap M$ and $F_2' = V \cap M$, where U and V are convex fuzzy sets in X.

$$\therefore \quad a_\lambda \in U \cap M \subset F_1 \cap M \qquad \text{and} \quad b_\mu \in V \cap M \subset F_2 \cap M \tag{1}$$

Now $a_\lambda \in U$ and $a_\lambda \in F_1$ and since F_1 is the fuzzy convex hull of a_λ in X, we have $F_1 \subset U$. Similarly $F_2 \subset V$.

$$\therefore \quad F_1 \cap M \subset U \cap M \text{ and } F_2 \cap M \subset V \cap M \tag{2}$$

From (1) and (2), we have

$$F_1 \cap M = U \cap M \text{ and } F_2 \cap M = V \cap M$$

$\therefore$ $F_1' = F_1 \cap M$ and $F_2' = F_2 \cap M$

$\therefore$ $F_1 \cap M$ and $F_2 \cap M$ are the fuzzy convex hulls of a_λ and b_μ in M.

$\therefore$ M is an FS_o space.

Proposition 4.2.4.

A nonempty product is FS_o if each factor is FS_o.

Proof:

Let $X = \prod_{\alpha \in I} X_\alpha$ and let each factor X_α be FS_o. Let a_λ and b_μ be two distinct fuzzy points in X. Then for some α, $(a_\alpha)_\lambda$ and $(b_\alpha)_\mu$ are distinct fuzzy points in X_α. Now X_α is FS_o. Therefore there exist distinct fuzzy convex hulls F_α and G_α containing $(a_\alpha)_\lambda$ and $(b_\alpha)_\mu$ respectively. Since F_α and G_α are fuzzy convex, $\pi_\alpha^{-1}(F_\alpha)$ and $\pi_\alpha^{-1}(G_\alpha)$ are fuzzy convex, π_α is the projection map from X to X_α.

$$\text{Now, } \mu_{\pi_\alpha^{-1}(F_\alpha)}(a_\lambda) = \mu_{F_\alpha}(\pi_\alpha(a_\lambda))$$

$$= \mu_{F_\alpha}((a_\alpha)_\lambda) \geqslant \lambda$$

$$\therefore \quad a_\lambda \in \pi_\alpha^{-1}(F_\alpha).$$

Similarly

$$b_\mu \in \pi_\alpha^{-1}(G_\alpha).$$

Now we will prove that $\pi_\alpha^{-1}(F_\alpha)$ and $\pi_\alpha^{-1}(G_\alpha)$ are distinct.

If possible assume that

$$\pi_\alpha^{-1}(F_\alpha) = \pi_\alpha^{-1}(G_\alpha)$$

Then

$$\mu_{\pi_\alpha^{-1}(F_\alpha)}(x) = \mu_{\pi_\alpha^{-1}(G_\alpha)}(x) \quad \forall\, x \in X$$

i.e. $\mu_{F_\alpha}(\pi_\alpha(x)) = \mu_{G_\alpha}(\pi_\alpha(x))$

$\therefore \quad F_\alpha = G_\alpha$

which is a contradiction since $F_\alpha \neq G_\alpha$.

$\therefore \quad \pi_\alpha^{-1}(F_\alpha) \neq \pi_\alpha^{-1}(G_\alpha)$

Now to prove that $\pi_\alpha^{-1}(F_\alpha)$ and $\pi_\alpha^{-1}(G_\alpha)$ are the fuzzy convex hulls of a_λ and b_μ respectively. If possible assume that there exists a convex set H in X such that $a_\lambda \in H \subset \pi_\alpha^{-1}(F_\alpha)$. Then we have

$$\pi_\alpha(a_\lambda) \in \pi_\alpha(H) \subset F_\alpha$$

i.e. $(a_\alpha)_\lambda \in \pi_\alpha(H) \subset F_\alpha$.

Since π_α is FCC, $\pi_\alpha(H)$ is a convex fuzzy set and hence it is a contradiction to the assumption that F_α is the fuzzy convex hull of $(a_\alpha)_\lambda$ in X_α. Hence $\pi_\alpha^{-1}(F_\alpha)$ is the fuzzy convex hull of a_λ in X. Similarly $\pi_\alpha^{-1}(G_\alpha)$ is the fuzzy convex hull of b_μ in X and we have

$$\pi_\alpha^{-1}(F_\alpha) \neq \pi_\alpha^{-1}(G_\alpha). \text{ Hence X is } FS_o.$$

Note:

The quotient of an FS_o space need not be an FS_o space.

Example:

Let $X = \{a,b,c\}$

$$\mathcal{L}_1 = \left\{ \emptyset, X, \{a\}, \{b\}, \begin{matrix} a \longrightarrow 1 \\ b \longrightarrow 1 \\ c \longrightarrow \frac{1}{2} \end{matrix} \right\}$$

Now $(X, \mathcal{L}_1)$ is an FS_o space.

Consider $Y = \{1,2\}$

Define $f : X \longrightarrow Y$ as

$$f(a) = 1 = f(b) \text{ and } f(c) = 2$$

Choose $\mathcal{L}_2 = \left\{ \emptyset, Y, \begin{matrix} 1 \longrightarrow 1 \\ 2 \longrightarrow \frac{1}{2} \end{matrix} \right\}$

Then f is a quotient map from $(X, \mathcal{L}_1)$ to $(Y, \mathcal{L}_2)$.

$\therefore$ $(Y, \mathcal{L}_2)$ is the quotient space of $(X, \mathcal{L}_1)$. But $(Y, \mathcal{L}_2)$ is not FS_0, since the fuzzy convex hulls of 1_λ, $0 < \lambda \leqslant 1$ and 2μ, $0 < \mu \leqslant \frac{1}{2}$ are the same convex fuzzy set $\begin{array}{l} 1 \longrightarrow 1 \\ 2 \longrightarrow \frac{1}{2} \end{array}$

4.3 FS_1 Spaces

Definition 4.3.1.

An fcs $(X, \mathcal{L})$ is said to be FS_1 if given two distinct fuzzy points a_λ, b_μ there exist distinct fuzzy convex hulls of each of them not containing the other. i.e. given a_λ, b_μ, $a \neq b$, there exist fuzzy convex hulls F_1 and F_2 of a_λ and b_μ respectively with $F_1 \neq F_2$ such that $a_\lambda \notin F_2$ and $b_\mu \notin F_1$.

Example:

Let X be any nonempty set and

$$\mathcal{L} = \{\emptyset, X\} \cup \{\{x\} \mid x \in X\}$$

Then $(X, \mathcal{L})$ is an FS_1 space.

Example:

Let $X = \{a, b, c\}$

$$\mathcal{L} = \left\{ \emptyset, X, \{a\}, \{b\}, \{c\}, a_{1/2}, \begin{matrix} a \longrightarrow \frac{1}{2} \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{matrix} \right\}$$

$(X, \mathcal{L})$ is an FS_1 space.

Proposition 4.3.2.

$(X, \mathcal{L})$ is an FS_1 space if and only if each fuzzy singleton in X is a convex fuzzy set. (Recall: a fuzzy point a_λ is a fuzzy singleton if $\lambda = 1$).

Proof:

If each fuzzy singleton in X is a convex fuzzy set then clearly $(X, \mathcal{L})$ is FS_1. To prove the other part, assume that there is a fuzzy singleton $\{a\}$ which is not a convex fuzzy set. Then the support of the fuzzy convex hull of $\{a\}$ consists of at least one more point 'b' with membership value μ, $0 < \mu \leqslant 1$. Then the two fuzzy points a_1 and b_μ, $\mu \in (0,1]$ cannot be separated by distinct fuzzy convex hulls, which is a contradiction to the assumption that $(X, \mathcal{L})$ is an FS_1 space. Hence every fuzzy singleton must be a convex fuzzy set.

Proposition 4.3.3.

An FS_1 space is always FS_o.

Proof:

Trivial.

Note:

An FS_o space need not be FS_1.

Example:

Let $X = \{a, b, c\}$

$$\mathcal{L} = \left\{ \emptyset, X, \{a\}, \{a,b\}, \begin{matrix} a \longrightarrow 1 \\ b \longrightarrow 1 \\ c \longrightarrow \frac{1}{2} \end{matrix} \right\}$$

$(X, \mathcal{L})$ is FS_o but not FS_1.

Proposition 4.3.4.

Let $(X, \mathcal{L})$ be an FS_1 space and $\mathcal{L}_2$ a fuzzy alignment on X such that $\mathcal{L}_1 \subset \mathcal{L}_2$. Then $(X, \mathcal{L}_2)$ is also an FS_1 space.

Proof:

It can be easily proved by Proposition 4.3.2.

Proposition 4.3.5.

Every subspace of an FS_1 space is an FS_1 space.

Proof:

$(X, \mathcal{L})$ is an FS_1 space. Let M be a crisp subspace of X. Let a_λ, b_μ be two distinct fuzzy points in M. Then there exist distinct fuzzy convex hulls F_1 and F_2 in X of a_λ and b_μ respectively such that $a \notin F_2$, $b_\mu \notin F_1$. Then we can prove as in the proof of Proposition 4.2.3 that $F_1 \cap M$ and $F_2 \cap M$ are distinct fuzzy convex hulls in M of a_λ and b_μ respectively and $a_\lambda \notin F_2 \cap M$, $b_\mu \notin F_1 \cap M$. Hence M is an FS_1 space.

Proposition 4.3.6.

A nonempty product is FS_1 if each factor is FS_1.

Proof:

Let $X = \prod_{\alpha \in I} X_\alpha$ and let each factor X_α be FS_1. Let a_λ, b_μ, λ, $\mu \in (0,1]$ be two distinct fuzzy points in X. Then for some α, $(a_\alpha)_\lambda$ and $(b_\alpha)_\mu$ are distinct fuzzy points in X_α. Then there exist distinct fuzzy convex hulls F_α and G_α in X_α for $(a_\alpha)_\lambda$ and $(b_\alpha)_\mu$

respectively and such that $(a_\alpha)_\lambda \notin G_\alpha$ and $(b_\alpha)_\mu \notin F_\alpha$. Then $\pi_\alpha^{-1}(F_\alpha)$ and $\pi_\alpha^{-1}(G_\alpha)$ are distinct fuzzy convex hulls in X for a_λ and b_μ respectively such that $b_\mu \notin \pi_\alpha^{-1}(F_\alpha)$ and $a_\lambda \notin \pi_\alpha^{-1}(G_\alpha)$. Hence X is FS_1.

Note:

Quotient of an FS_1 space need not be FS_1.

Example:

Let X = N

$$\mathcal{L}_1 = \{\emptyset, X\} \cup \{\{x\} \mid x \in X\}$$

Then $(X, \mathcal{L}_1)$ is an FS_1 space. Let $Y = \{0,1\}$ and define $f: X \longrightarrow Y$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is an odd number} \\ 1 & \text{if } x \text{ is an even number} \end{cases}$$

Let $\mathcal{L}_2$ be the indiscrete fuzzy alignment on Y. Then $(Y, \mathcal{L}_2)$ is a quotient of $(X, \mathcal{L}_1)$ and is not an FS_1 space.

4.4 FS_2 Spaces

Let $(X, \mathcal{L})$ be a fuzzy convexity space. We say that two fuzzy sets are separated by a fuzzy hemispace if one is contained in the fuzzy hemispace and the other in its complement which is also a convex fuzzy set.

Definition 4.4.1.

An fcs $(X,\mathcal{L})$ is said to be FS_2 if distinct fuzzy points can be separated by a fuzzy hemispace. i.e., if a_λ, b_μ are distinct fuzzy points, we can find a fuzzy hemispace H such that $a_\lambda \in H$ and $b_\mu \in 1-H$.

Example:

Let $X = \{a,b,c\}$

$$\mathcal{L} = \left\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \begin{matrix} a \longrightarrow \frac{1}{2} \\ b \longrightarrow 1 \end{matrix}, \; a_{1/2}, \; \begin{matrix} a \longrightarrow \frac{1}{2} \\ c \longrightarrow 1 \end{matrix} \right\}$$

$(X,\mathcal{L})$ is an FS_2 space.

Proposition 4.4.2.

An FS_2 space is always FS_1.

Proof:

Trivial.

Note:

The converse of Proposition 4.4.2 need not be true.

Example:

Let $X = \{a, b, c\}$

$$\mathcal{L} = \left\{ \emptyset, X, \{a\}, \{b\}, \{c\}, a_{1/2}, \begin{matrix} a \longrightarrow \frac{1}{2} \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{matrix} \right\}$$

$(X, \mathcal{L})$ is FS_1 but not FS_2 since the only hemispaces are $a_{1/2}$ and its complement; these will not separate a_λ and c_μ.

Proposition 4.4.3.

Let $(X, \mathcal{L}_1)$ be an FS_2 space and $\mathcal{L}_2$ a fuzzy alignment on X such that $\mathcal{L}_1 \subset \mathcal{L}_2$ then $(X, \mathcal{L}_2)$ is also an FS_2 space.

Proof:

Trivial.

Proposition 4.4.4.

Every subspace of an FS_2 space is an FS_2 space.

Proof:

Let $(X, \mathcal{L})$ be an FS_2 space. Let M be a subspace of X. Let a_λ, b_μ, $\lambda, \mu \in (0,1]$ be two distinct fuzzy points in M. Since $(X, \mathcal{L})$ is FS_2 there exists a fuzzy hemispace H such that $a_\lambda \in H$ and $b_\mu \in 1-H$. Then $H \cap M$ is a fuzzy hemispace in M (by Proposition 4.1.3) and $H \cap M$ separates a_λ and b_μ in M.

Proposition 4.4.5.

A nonempty product is FS_2 if each factor is FS_2.

Proof:

Let $X = \prod_{\alpha \in I} X_\alpha$ and let each factor X_α be FS_2. Let a_λ, b_μ, $\lambda, \mu \in (0,1]$ be two distinct fuzzy points in X. Then for some α, $(a_\alpha)_\lambda$ and $(b_\alpha)_\mu$ are distinct fuzzy points in X_α and hence there exists a fuzzy hemispace H in X_α such that H separates $(a_\alpha)_\lambda$ and $(b_\alpha)_\mu$. Then $\pi_\alpha^{-1}(H)$ separates a_λ and b_μ in X and $\pi_\alpha^{-1}(H)$ is clearly a hemispace. Therefore X is FS_2.

Note:

Quotient of an FS_2 space need not be an FS_2 space.

Example:

Let $X = \{a, b, c\}$

$$\mathcal{L}_1 = \left\{ \emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \begin{matrix} a \longrightarrow \frac{1}{2} \\ b \longrightarrow 1 \end{matrix}, a_{1/2}, \begin{matrix} a \longrightarrow \frac{1}{2} \\ c \longrightarrow 1 \end{matrix} \right\}$$

$(X, \mathcal{L}_1)$ is an FS_2 space.

Consider $Y = \{1,2\}$ and define $f: X \longrightarrow Y$ as $f(a) = 1 = f(b)$ and $f(c) = 2$.

Choose $\mathcal{L}_2 = \{\emptyset, Y, \{1\}\}$

Then f is a quotient map from X to Y and $(Y, \mathcal{L}_2)$ is not FS_2.

4.5. FS_3 Spaces

Definition 4.5.1.

An fcs $(X,\mathcal{L})$ is FS_3 if any convex fuzzy set and a fuzzy point (not in it) such that the supports are disjoint, can be separated by a fuzzy hemispace. i.e. given a convex fuzzy set A in X and a fuzzy point a_λ such that the supports of a_λ and A are disjoint, then there exists a fuzzy hemispace H such that $A \subset H$ and $a_\lambda \in 1-H$.

Example:

Let X = N, the set of natural numbers.

Let $a \in N$ be a chosen point.

Choose $\mathcal{L} = \left\{ \emptyset, X, \{a\}, \begin{array}{l} x \longrightarrow 1 \ \forall \ x \neq a \\ x \longrightarrow 0 \text{ if } x = a \end{array} \right\}$

Then $(X, \mathcal{L})$ is an FS_3 space.

Example:

Let $X = \{a, b, c\}$

$$\mathcal{L} = \left\{ \emptyset, X, \{a\}, \begin{array}{l} b \longrightarrow \frac{1}{2} \\ c \longrightarrow \frac{1}{2} \end{array} \quad \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow \frac{1}{2} \end{array} \quad \{b, c\} \right\}$$

$(X, \mathcal{L})$ is an FS_3 space.

The above example shows that an FS_3 space need not be FS_2, FS_1 and FS_o.

Note:

An FS_2 space need not be FS_3.

Example:

Let $X = \{a, b, c\}$

$$\mathcal{L} = \left\{ \emptyset, X, \{a\}, b_{1/2}, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow \frac{1}{2} \end{array} \right\}$$

$(X, \mathcal{L})$ is FS_2 but not FS_3.

Note:

If $(X, \mathcal{L}_1)$ is an FS_3 space and $\mathcal{L}_2$ a fuzzy alignment such that $\mathcal{L}_1 \subset \mathcal{L}_2$, then $(X, \mathcal{L}_2)$ need not be an FS_3 space.

Example

Let $X = \{a, b, c\}$

$$\mathcal{L}_1 = \left\{ \emptyset, X, \{a\}, \begin{array}{l} b \longrightarrow \frac{1}{2} \\ c \longrightarrow \frac{1}{2} \end{array}, \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow \frac{1}{2} \end{array} \quad \{b, c\} \right\}$$

$(X, \mathcal{L}_1)$ is FS_3.

Choose $$\mathcal{L}_2 = \left\{ \emptyset, X, \{a\}, \begin{array}{l} b \longrightarrow \frac{1}{2} \\ c \longrightarrow \frac{1}{2} \end{array}, \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow \frac{1}{2} \end{array} \{b, c\}, \; c_{1/2} \right\}$$

Now $\mathcal{L}_1 \subset \mathcal{L}_2$ and $(X, \mathcal{L}_2)$ is not an FS_3 space.

Proposition 4.5.2.

Any fuzzy convex subspace of an FS_3 space is an FS_3 space.

Proof:

Let M be a fuzzy convex subspace of an FS_3 space $(X, \mathcal{L})$ and let a_λ, $\lambda \in (0, 1]$ be a fuzzy point in M and A, a convex fuzzy set in M such that the supports of a_λ and A are disjoint. Then $A = M \cap L$ where L is a convex fuzzy seti

Since supports of a_λ and A are disjoint, we have supports of a_λ and L are disjoint. Since X is FS_3, there exists a fuzzy hemispace H such that $L \subset H$ and $a_\lambda \in 1-H$. Then $H \cap M$ is a fuzzy hemispace in M such that $H \cap M$ separates a_λ and A. Therefore, M is an FS_3 space.

Proposition 4.5.3.

A nonempty product is FS_3 if each factor is FS_3.

Proof:

Let $X = \prod_{\alpha \in I} X_\alpha$ and let each factor X_α be FS_3. Let a_λ, $\lambda \in (0,1]$ be a fuzzy point in X and A, a convex fuzzy set in X such that the supports of a_λ and A are disjoint. Since the fuzzy alignment generated by sets of the form $\{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha,$ a convex fuzzy set in $X_\alpha\}$, where π_α is the projection map from X to X_α; we can take A as $A = \bigcap_{\alpha \in I} \pi_\alpha^{-1}(U_\alpha)$. Now for some α, $(a_\alpha)_\lambda$ is a fuzzy point in X_α and the supports of $(a_\alpha)_\lambda$ and U_α are disjoint. Since X_α is FS_3, $(a_\alpha)_\lambda$ and U_α can be separated by a fuzzy hemispace H_α. Then $H = \bigcap_{\alpha \in I} \pi_\alpha^{-1}(H_\alpha)$ separates $A = \bigcap_{\alpha \in I} \pi_\alpha^{-1}(U_\alpha)$ and a_λ in X. Hence X is FS_3.

Proposition 4.5.3.

Quotient of an FS_3 space need not be an FS_3 space.

Example:

Let X = N, the set of Natural numbers.

$$\mathcal{L}_1 = \{\emptyset, X, \{1\}, A, B, C, D\}$$

where the fuzzy sets A,B,C,D are given by

$$A : \begin{array}{l} 1 \longrightarrow 0 \\ x \longrightarrow 1 \ \forall \ x \geqslant 2, \end{array} \qquad B : \begin{array}{l} x \longrightarrow \frac{1}{2} \ \forall \ x \text{ even} \\ x \longrightarrow 0 \ \forall \ x \text{ odd} \end{array}$$

$$C : \begin{array}{l} 1 \longrightarrow 0 \\ x \longrightarrow \frac{1}{2} \ \forall \ x \text{ even} \\ x \longrightarrow 1 \ \forall \ x \geqslant 3, \ x \text{ odd} \end{array} \qquad D : \begin{array}{l} 1 \longrightarrow 1 \\ x \longrightarrow \frac{1}{2} \ \forall \ x \text{ even} \\ x \longrightarrow 0 \ \forall \ x \geqslant 3, \ x \text{ odd} \end{array}$$

Consider $Y = \{a, b\}$ and

$$\mathcal{L}_2 = \{\emptyset, Y, b_{1/2}\}$$

Define $f : (X, \mathcal{L}_1)$ to $(Y, \mathcal{L}_2)$ as

$$f(x) = \begin{cases} a \text{ if } x \text{ is an odd number} \\ b \text{ if } x \text{ is an even number} \end{cases}$$

Then f is a quotient map from $(X, \mathcal{L}_1)$ to $(Y, \mathcal{L}_2)$.

Now $(Y, \mathcal{L}_2)$ is not an FS_3 space since the fuzzy point a_λ and the convex fuzzy set $b_{1/2}$ cannot be separated by a fuzzy hemispace.

4.6 FS$_4$ Spaces

Definition 4.6.1.

An fcs $(X, \mathcal{L})$ is FS_4 if two disjoint convex fuzzy sets can be separated by a fuzzy hemispace.

i.e. given two disjoint convex fuzzy sets A and B there exist a fuzzy hemispace H such that $A \subset H$ and $B \subset 1-H$.

Note:

Two fuzzy sets are said to be disjoint if their supports are disjoint.

Example 1.

Let X = N, the set of Natural numbers, and let $a \neq b$ be points of N chosen arbitrarily.

$$\text{Choose, } \mathcal{L} = \left\{ \emptyset, X, \{a\}, \{b\}, \begin{array}{l} x \longrightarrow 1 \ \forall\ x \neq a \\ x \longrightarrow 0 \text{ if } x = a \end{array} \right\}$$

$(X, \mathcal{L})$ is an FS_4 space.

Example 2.

Let $X = \{a, b, c\}$

$\mathcal{L}_1 = \{\emptyset, X, \{a\}, a_{1/2}, b_{1/2}, \{b, c\}\}$

$(X, \mathcal{L}_1)$ is an FS_4 space.

Note:

The above example 2 shows that an FS_4 space need not be FS_3.

Note:

If $(X, \mathcal{L}_1)$ is an FS_4 space and $\mathcal{L}_2$ a fuzzy alignment on X such that $\mathcal{L}_1 \subset \mathcal{L}_2$, then $(X, \mathcal{L}_2)$ need not be an FS_4 space.

Example:

Let $X = \{a, b, c\}$

$\mathcal{L}_1 = \{\emptyset, X\ \{a\}, a_{1/2}, b_{1/2}, \{b, c\}\}$

$(X, \mathcal{L}_1)$ is an FS_4 space.

Choose $\mathcal{L}_2 = \left\{\emptyset, X, \{a\}, a_{1/2}, b_{1/2}, \{b, c\}, \{c\}, \begin{array}{l} a \longrightarrow \frac{1}{2} \\ c \longrightarrow 1 \end{array}\right\}$

Now $\mathcal{L}_1 \subset \mathcal{L}_2$ and $(X, \mathcal{L}_2)$ is not FS_4.

Proposition 4.6.2.

Any fuzzy convex subspace of an FS_4 space is an FS_4 space.

Proof:

Let $(X, \mathcal{L})$ be an FS_4 space. Let $(M, \mathcal{L}_M)$ be a fuzzy convex subspace of $(X, \mathcal{L})$. Let F_1 and F_2 be disjoint convex fuzzy sets in M. Since M is a convex fuzzy set, F_1 and F_2 are disjoint convex fuzzy sets in X and since X is FS_4, there exists a fuzzy hemispace H such that $F_1 \subset H$ and $F_2 \subset 1-H$. Then $H \cap M$ is a fuzzy hemispace in M such that $F_1 \subset H \cap M$ and $F_2 \subset 1-(H \cap M)$. Hence $(M, \mathcal{L}_M)$ is FS_4.

Proposition 4.6.3.

A nonempty product is FS_4 if each factor is FS_4.

Proof:

Let $X = \prod_{\alpha \in I} X_\alpha$ and let each factor X_α be FS_4. Let A and B be two convex fuzzy sets in X. Then A and B can be written as

$$A = \bigcap_{\alpha \in I} \pi_\alpha^{-1}(U_\alpha) \text{ and } B = \bigcap_{\alpha \in I} \pi_\alpha^{-1}(V_\alpha)$$

where π_α is the projection map from X to X_α and U_α's and V_α's are convex fuzzy sets in X_α. Since A and B are disjoint convex fuzzy sets, U_α and V_α will be disjoint for at least one α. Since each X_α is FS_4, U_α and V_α can be separated by a fuzzy hemispace H_α, $\alpha \in I$. Then the hemispace $\pi_\alpha^{-1}(H_\alpha)$ separates A and B in $\prod X_\alpha$. Hence $\prod X_\alpha$ is FS_4.

Note 4.6.4.

Quotient of an FS_4 space need not be FS_4.

Chapter 5

SEPARATION AXIOMS IN FUZZY TOPOLOGY FUZZY CONVEXITY SPACES*

5.1. FNS_o, FNS_1, FNS_2 Spaces

In a topological convexity structure (crisp) a number of separation properties were considered by Van de Vel [33]. The separation involve closed convex sets. We introduce the "fuzzy neighbourhood separation properties" FNS_o, FNS_1, FNS_2, FNS_3 and FNS_4 in a fuzzy topology fuzzy convexity space of which the last three are in some sense analogous to NS_2, NS_3 and NS_4 introduce by Van de Vel.

Definition 5.1.1.

(Let $(X, \mathcal{L}, T)$ be an ftfcs. Then $(X, \mathcal{L}, T)$ is said to be

(i) FNS_o if for any two distinct fuzzy points there exists a closed convex fuzzy neighbourhood containing one and not containing the other.

(ii) FNS_1 if for any two distinct fuzzy points there exists a closed convex fuzzy neighbourhood of each of them not containing the other.

* Some of the resul ts of this chapter will appear in the Jl Fuzzy Mathematics(1994).

(iii) FNS_2 if for any two distinct fuzzy points there exist disjoint closed convex fuzzy neighbourhoods of each of them.

From the above definition 5.1.1 one can notice that $(X, \mathcal{L}, T)$ is $FNS_2 \Longrightarrow FNS_1 \Longrightarrow FNS_0$. Also we can show that $FNS_0 \not\Longrightarrow FNS_1 \not\Longrightarrow FNS_2$.

Examples

1) Let X=N, the set of natural numbers.

$$\mathcal{L} = \{\emptyset, X\} \cup \{\{x\} \mid x \in X\}$$

T, the discrete fuzzy topology on X.

Then $(X, \mathcal{L}, T)$ is FNS_2.

2) Let $X = \{a, b, c\}$

$$\mathcal{L} = \left\{ \emptyset, X, \{a\}, \begin{matrix} a \longrightarrow \frac{1}{2} \\ b \longrightarrow 1 \end{matrix}, a_{1/2} \right\}$$

$$T = \left\{ \emptyset, X, \{a\}, a_{1/2}, \{c\}, \{a, c\}, \{b, c\}, \begin{matrix} a \longrightarrow \frac{1}{2} \\ b \longrightarrow 1 \end{matrix} \quad \begin{matrix} a \longrightarrow \frac{1}{2} \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{matrix} \quad \begin{matrix} a \longrightarrow \frac{1}{2} \\ c \longrightarrow 1 \end{matrix} \right\}$$

$(X, \mathcal{L}, T)$ is FNS_0 but not FNS_1.

3) Let $X = \{a, b, c\}$

$$\mathcal{L} = \left\{ \emptyset, X, \{a\}, \{b,c\}, \begin{matrix} a \longrightarrow \frac{1}{2} \\ b \longrightarrow 1 \end{matrix}, \begin{matrix} a \longrightarrow \frac{1}{2} \\ c \longrightarrow 1 \end{matrix}, a_{1/2}, \{b\}, \{c\} \right\}$$

$$T = \left\{ \emptyset, X, \{b,c\}, \{a\}, \begin{matrix} a \longrightarrow \frac{1}{2} \\ c \longrightarrow 1 \end{matrix} \quad \begin{matrix} a \longrightarrow \frac{1}{2} \\ b \longrightarrow 1 \end{matrix} \right.$$

$$\left. \{a,b\}, a_{1/2}, \{b\}, \{c\}, \begin{matrix} a \longrightarrow \frac{1}{2} \\ b \longrightarrow 1 \\ c \longrightarrow 1 \end{matrix}, \{a,c\} \right\}$$

$(X, \mathcal{L}, T)$ is FNS_1.

Proposition 5.1.2.

Any subspace of an FNS_i space is FNS_i for every $i = 0, 1, 2$.

Proof:

Let us prove when $i = 2$. Let $(X, \mathcal{L}, T)$ be an FNS_2 space and let $(M, \mathcal{L}_M, T_M)$ be a subspace of $(X, \mathcal{L}, T)$. Let a_λ, b_μ be two distinct fuzzy points in M. Then a_λ and b_μ can be considered as distinct fuzzy points in X and X is FNS_2. Therefore there exist disjoint closed convex fuzzy neighbourhoods U and V for a_λ and b_μ respectively. Then $U \cap M$ and $V \cap M$ are disjoint closed convex fuzzy neighbourhoods of a_λ and b_μ in M respectively. Therefore $(M, \mathcal{L}_M, T_M)$ is FNS_2. Similarly we can prove the result for i = 0 and 1.

Proposition 5.1.3.

A nonempty product is FNS_i, if each factor is FNS_i for every i = 0,1,2.

Proof:

Consider the case when i = 2.

Let $(X_\alpha, \mathcal{L}_\alpha, T_\alpha)_{\alpha \in I}$ be a family of FNS_2 spaces. Let a_λ , b_μ be two distinct fuzzy points in X, where $(X,\mathcal{L},T) = \prod_{\alpha \in I}(X_\alpha, \mathcal{L}_\alpha, T_\alpha)$ Then for some α, $(a_\alpha)_\lambda$ and $(b_\alpha)_\mu$ are distinct fuzzy points in X_α and each X_α is FNS_2, then there exist disjoint closed convex fuzzy neighbourhoods U_α and V_α in X_α for $(a_\alpha)_\lambda$ and $(b_\alpha)_\mu$ respectively. Then $U = \pi_\alpha^{-1}(U_\alpha)$ and $V = \pi_\alpha^{-1}(V_\alpha)$ are disjoint closed convex fuzzy neighbourhoods in X of a_λ and b_μ respectively.

$\therefore$ $(X, \mathcal{L}, T)$ is FNS_2.

Similarly one can prove the result when i = 0 and 1.

Note 5.1.4.

An FCP, F-continuous image of an FNS_i space need not be an FNS_i space, for every i = 0,1 and 2

Example:

Let X be any space with more than one point, having discrete fuzzy convexity $\mathcal{L}_1$ and discrete fuzzy topology T_1. Consider any map $f : (X, \mathcal{L}_1, T_1) \longrightarrow (Y, \mathcal{L}_2, T_2)$, where T_2 is any fuzzy topology on Y other than the indiscrete fuzzy topology and $\mathcal{L}_2$ is the indiscrete fuzzy convexity on Y. Then f is an FCP, F-continuous map from X to Y. Clearly X is FNS_i for every i = 0,1,2. But Y is not FNS_i for the corresponding i = 0,1,2 respectively, because, any proper closed subset A of Y is not a convex fuzzy set. Therefore there is no proper closed convex fuzzy set in Y. Hence Y is not FNS_i for every i = 0,1 and 2.

Note:

The quotient of an FNS_i space need not be FNS_i for every i = 0,1 and 2.

5.2 Pseudo FNS_3 and FNS_3 Spaces

Definition 5.2.1.

An ftfcs $(X,\mathcal{L},T)$ is pseudo FNS_3 if for each closed convex fuzzy set A in X and a fuzzy point a_λ (not in it) such that supports of a_λ and A are disjoint, then there exists a closed convex fuzzy neighbourhood V of A such that $a_\lambda \notin V$.

Definition 5.2.2.

An ftfcs $(X,\mathcal{L},T)$ is FNS_3 if for each closed convex fuzzy set A in X and a fuzzy point a_λ (not in it) such that the supports of a_λ and A are disjoint, then there exist disjoint closed convex fuzzy neighbourhoods U of a_λ and V of A.

Note:

From the above definition one can notice that $FNS_3 \Longrightarrow$ Pseudo FNS_3 and pseudo $FNS_3 \not\Longrightarrow FNS_3$.

Example (i):

Let X be any set with discrete fuzzy topology T and the fuzzy convexity given by

$$\mathcal{L} = \{\emptyset, X\} \cup \{\{x\} \mid x \in X\}$$

Then $(X, \mathcal{L}, T)$ is FNS_3.

Example (ii):

Let X = N, the set of Natural numbers with

$$T = \left\{\emptyset, X, \{a\}, \begin{matrix} x \longrightarrow 1 & \forall\, x \neq a \\ x \longrightarrow 0 & \text{if } x = a \end{matrix}\right\}$$

$$\mathcal{L} = \left\{\emptyset, X, \{a\}\right\} \text{ where } a \in N$$

$(X, \mathcal{L}, T)$ is pseudo FNS_3 but not FNS_3.

Example (iii):

Let $X = [0,1]$

$$T = \{\emptyset, X, A, B, C, D, E\}$$

$$\mathcal{L} = \{\emptyset, X, A^c, B^c\}$$

where A, B, C, D, E, A^c and B^c are the fuzzy sets

$$A : \begin{cases} x \longrightarrow 1, & 0 \leqslant x \leqslant \frac{1}{2} \\ x \longrightarrow \frac{1}{2}, & \frac{1}{2} < x \leqslant 1 \end{cases} \qquad B: \begin{cases} x \longrightarrow 1, & 0 \leqslant x \leqslant \frac{1}{2} \\ x \longrightarrow 0, & \frac{1}{2} < x \leqslant 1 \end{cases}$$

$$C : \begin{cases} x \longrightarrow 0, & 0 \leqslant x \leqslant \frac{1}{2} \\ x \longrightarrow \frac{3}{4}, & \frac{1}{2} < x \leqslant 1 \end{cases} \qquad D: \begin{cases} x \longrightarrow 1, & 0 \leqslant x \leqslant \frac{1}{2} \\ x \longrightarrow \frac{3}{4}, & \frac{1}{2} < x \leqslant 1 \end{cases}$$

$$E : \begin{cases} x \longrightarrow 0, & 0 \leqslant x \leqslant \frac{1}{2} \\ x \longrightarrow \frac{1}{2}, & \frac{1}{2} < x \leqslant 1 \end{cases}$$

$$A^c: \begin{cases} x \longrightarrow 0, & 0 \leqslant x \leqslant \frac{1}{2} \\ x \longrightarrow \frac{1}{2}, & \frac{1}{2} < x \leqslant 1 \end{cases} \qquad B^c: \begin{cases} x \longrightarrow 0, & 0 \leqslant x \leqslant \frac{1}{2} \\ x \longrightarrow 1, & \frac{1}{2} < x \leqslant 1 \end{cases}$$

$(X, \mathcal{L}, T)$ is pseudo FNS_3 but not FNS_3.

Note:

An FNS_3 space and so a pseudo FNS_3 space need not be FNS_2.

Example 1.

Let $X = \{a, b, c\}$

$\mathcal{L} = T = \{\emptyset, X, \{a\}, \{b, c\}\}$

$(X, \mathcal{L}, T)$ is FNS_3 but not FNS_2.

Example 2:

Here we give another example of a pseudo FNS_3 space which is not FNS_2.

Let X = N, the set of Natural numbers and let

$$T = \{\emptyset, X, A, B, C, D, E\}$$
$$\mathcal{L} = \{\emptyset\} \cup \{Y \subset I^X \mid A \subseteq Y\}$$

where A,B,C,D,E are the fuzzy sets,

$$A: \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow \frac{1}{2} \\ x \longrightarrow 0 \quad \forall\, x \geq 3 \end{array} \qquad B: \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow \frac{1}{2} \\ 3 \longrightarrow 0 \\ x \longrightarrow 1 \ \forall\, x \geq 4 \end{array} \qquad C: \begin{array}{l} 1 \longrightarrow \frac{3}{4} \\ 2 \longrightarrow \frac{3}{4} \\ 3 \longrightarrow 1 \\ x \longrightarrow 0 \ \forall\, x \geq 4 \end{array}$$

$$D: \begin{array}{l} 1 \longrightarrow \frac{3}{4} \\ 2 \longrightarrow \frac{3}{4} \\ x \longrightarrow 1 \ \forall\, x \geq 3 \end{array} \qquad E: \begin{array}{l} 1 \longrightarrow 0 \\ 2 \longrightarrow 0 \\ 3 \longrightarrow 0 \\ x \longrightarrow 1 \quad \forall\, x \geq 4 \end{array} .$$

Now the only proper closed convex fuzzy sets in X are A^c, B^c and E^c, where A^c, B^c, E^c are the complements of A,B,E respectively. Now there is no fuzzy point a_λ in X such that supports of a_λ and A^c are disjoint. Any fuzzy point disjoint with B^c is of the form a_λ where $a > 3$ and $0 < \lambda \leq 1$. Now $B^c \subset C \subset E^c$ where $C \in T$.

Therefore $(X,\mathcal{L},T)$ is pseudo FNS_3 and is not FNS_2, because a_λ and b_λ , $0 < \lambda \leq 1$ are distinct fuzzy points in X, where $a \neq b$ and $a,b \geq 4$. But any closed convex fuzzy set containing a_λ contains b_λ also.

Proposition 5.2.3.

Fuzzy closed fuzzy convex subspaces of an FNS_3 space is FNS_3.

Proof:

Let $(X,\mathcal{L},T)$ be an FNS_3 space. Let M be a fuzzy closed fuzzy convex subspace of X. Let a_λ be a fuzzy point in M and A, a closed convex fuzzy set in M such that the supports of a_λ and A are disjoint. Then $A = M \cap L$, where L is a closed convex fuzzy set in X. Since supports of a_λ and A are disjoint, we have supports of a_λ and L are disjoint in X and X is FNS_3, hence there exist disjoint closed convex fuzzy neighbourhoods U,V of a_λ and L respectively. Then $M \cap U$ and $M \cap V$ are disjoint closed convex fuzzy neighbourhoods of a_λ and A respectively in M.

Proposition 5.2.4.

Fuzzy closed fuzzy convex subspace of a pseudo FNS_3 space is pseudo FNS_3.

Proof:

Similar to Proposition 5.2.3.

Note:

An FCP, F-continuous image of an FNS_3, pseudo FNS_3 space need not be FNS_3, pseudo FNS_3 respectively.

Example:

Let X be any set with more than one point having discrete fuzzy topology T_1 and discrete fuzzy convexity $\mathcal{L}_1$. Consider

$f : (X, \mathcal{L}_1, T_1) \longrightarrow (Y, \mathcal{L}_2, T_2)$, where T_2 is the indiscrete fuzzy topology and $\mathcal{L}_2$, the discrete fuzzy convexity on Y. Now X is FNS_3 and hence pseudo FNS_3. Clearly f is an FCP, F-continuous map from X to Y. But Y is not pseudo FNS_3 and not FNS_3.

Proposition 5.2.5.

A nonempty product of FNS_3 spaces is an FNS_3 space.

Proof:

Let $(X_\alpha, \mathcal{L}_\alpha, T_\alpha)$ be a family of FNS_3 spaces. Let $(X, \mathcal{L}, T) = \prod_{\alpha \in I} (X_\alpha, \mathcal{L}_\alpha, T_\alpha)$. Let a_λ be a fuzzy point in X and A, a closed convex fuzzy set in X such that the supports of a_λ and A are disjoint. Let π_α be the projection from X to X_α. Then we can take A as $A = \bigcap_{\alpha \in I} \pi_\alpha^{-1}(U_\alpha)$, where U_α is a closed convex fuzzy set in X_α. Now for some α, the supports of $(a_\alpha)_\lambda$ and U_α are disjoint. Since X_α is FNS_3, there exist closed convex fuzzy neighbourhoods V_α of $(a_\alpha)_\lambda$ and W_α of U_α such that $(a_\alpha)_\lambda \notin W_\alpha$ and V_α and U_α are disjoint. Then $U = \pi_\alpha^{-1}(V_\alpha)$ and $W = \pi_\alpha^{-1}(W_\alpha)$ are closed convex fuzzy neighbourhoods of a_λ and A in X respectively such that $a_\lambda \notin W$ and U and A are disjoint.

Proposition 5.2.6.

A nonempty product of pseudo FNS_3 spaces is a pseudo FNS_3 space.

Proof:

Similar to Proposition 5.2.5.

Proposition 5.2.7.

The quotient of an FNS_3, pseudo FNS_3 space is FNS_3, pseudo FNS_3 respectively if the quotient map is an FCC, F-closed and F-open map.

Proof:

Let $(X, \mathcal{L}, T)$ be an FNS_3 space and let f be the quotient map from X to Y, which is FCC, F-closed and

F-open. Let a_λ be a fuzzy point in Y and let G be a closed convex fuzzy set in Y, such that the supports of a_λ and G are disjoint. Then we can choose a point 'b' in X such that $f(b) = a$ and then $f(b_\lambda) = a_\lambda$. Also $f^{-1}(G)$ is a closed convex fuzzy set in X and

$$\mu_{f^{-1}(G)}(b) = \mu_G(f(b)) = \mu_G(a) = 0,$$

since the supports of a_λ and G are disjoint.

$$\therefore \quad b_\lambda \notin f^{-1}(G).$$

Now X is FNS_3, therefore there exist closed convex fuzzy neighbourhoods U and V of b_λ and $f^{-1}(G)$ respectively such that $b_\lambda \notin V$ and $f^{-1}(G) \cap U = \emptyset$. Then $f(U)$ and $f(V)$ are closed convex fuzzy neighbourhoods of a_λ and G respectively in Y since f is an FCC, F-closed and F-open map. Also

$$\mu_{f(V)}(a) = \sup_{b \in f^{-1}(G)} \mu_V(b) = 0$$

$$\therefore \quad a_\lambda \notin f(V), \ 0 < \lambda \leqslant 1.$$

Also $G \cap f(U) = \emptyset$. Therefore Y is FNS_3 and hence pseudo FNS_3.

5.3 Semi FNS_4 and FNS_4 Spaces

Definition 5.3.1.

An ftfcs $(X,\mathcal{L},T)$ is semi FNS_4 if for each pair of disjoint closed convex fuzzy sets in X there is a closed convex fuzzy neighbourhood U of one of the closed convex fuzzy sets such that U and the other are disjoint.

Definition 5.3.2.

An ftfcs $(X,\mathcal{L},T)$ is FNS_4 if for each pair of disjoint closed convex fuzzy sets A and B in X there are closed convex fuzzy neighbourhoods U of A and V of B respectively such that U and B are disjoint and A and V are disjoint.

Note:

From the above definitions one can notice that

$(X,\mathcal{L},T)$ is $FNS_4 \implies$ Semi FNS_4 and

semi $FNS_4 \not\Longrightarrow FNS_4$ and semi $FNS_4 \not\Longrightarrow FNS_3$.

Example:

(i) Let X be any set with discrete fuzzy topology T and the fuzzy convexity given by

$$\mathcal{L} = \{\emptyset, X\} \cup \{\{x\} \mid x \in X\}$$

$(X,\mathcal{L},T)$ is FNS_4.

(ii) Let $X = N$, the set of Natural numbers

$$T = \{\emptyset, X, A, B, C, D, E, F, G\}$$
$$\mathcal{L} = X \cup \{Y \subset I^X \mid Y \subseteq K\}$$

where the fuzzy sets A,B,C,D,E,F,G and K are

$$A : \begin{array}{l} 1 \longrightarrow \frac{3}{4} \\ x \longrightarrow 1 \;\forall\; x \geq 2 \end{array} \qquad B : \begin{array}{l} 1 \longrightarrow 1 \\ 2 \longrightarrow \frac{3}{4} \\ x \longrightarrow 1 \;\forall\; x \geq 3 \end{array}$$

$$C : \begin{array}{l} 1 \longrightarrow \frac{3}{4} \\ 2 \longrightarrow \frac{3}{4} \\ x \longrightarrow 1 \;\forall\; x \geq 3 \end{array} \qquad D : \; x \longrightarrow 1 \;\forall\; x$$

$$E: \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ x \longrightarrow 1 \;\forall\; x \geq 2 \end{array} \qquad F: \begin{array}{l} 1 \longrightarrow \frac{1}{3} \\ x \longrightarrow 0 \;\forall\; x \geq 2 \end{array}$$

$$G: \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow \frac{3}{4} \\ x \longrightarrow 0 \;\forall\; x \geq 3 \end{array} \qquad K: \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow \frac{1}{2} \\ x \longrightarrow 0 \;\forall\; x \geq 3 \end{array}$$

Now consider

$$A^c : \begin{array}{l} 1 \longrightarrow \frac{1}{4} \\ x \longrightarrow 0 \;\forall\; x \geq 2 \end{array}$$

$$B^c : \begin{array}{l} 1 \longrightarrow 0 \\ 2 \longrightarrow \frac{1}{4} \\ x \longrightarrow 0 \;\forall\; x \geq 3 \end{array} \quad \text{and } E^c: \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ x \longrightarrow 0 \;\forall\; x \geq 2 \end{array}$$

A^c and B^c are disjoint closed convex fuzzy sets in X. Also A^c and E^c are disjoint closed convex fuzzy sets and also there are no other disjoint closed convex fuzzy sets in X. Now

$$A^c \subset E^c \text{ and } A^c \subset F \subset E^c, \text{ where } F \in T.$$

$\therefore$ X is semi FNS_4. But there is no closed convex fuzzy neighbourhood of B^c in X and hence X is not FNS_4.

(iii) In the above example (ii), $C^c : \begin{array}{l} 1 \longrightarrow \frac{1}{4} \\ 2 \longrightarrow \frac{1}{4} \\ x \longrightarrow 0 \;\forall\; x \geqslant 3 \end{array}$

is closed and is convex and any fuzzy point not in C^c is of the form a_λ, where $a \geqslant 3$ and $0 < \lambda \leqslant 1$. Then there is no closed convex fuzzy neighbourhood containing C^c and disjoint with a_λ.

$\therefore$ $(X, \mathcal{L}, T)$ is not FNS_3.

$\therefore$ Semi $FNS_4 \not\Longrightarrow FNS_3$.

Proposition 5.3.3.

Fuzzy closed fuzzy convex subspace of an FNS_4, semi FNS_4 space is FNS_4, semi FNS_4 respectively.

Proof:

Similar to Proposition 5.2.3.

Note:

A nonempty product of FNS_4 and semi FNS_4 spaces need not be FNS_4 and semi FNS_4 respectively.

Example:

We quote below the example from [33] in the crisp case and this will then be an example to prove the above Note in the fuzzy case also.

Consider the Real line R with crisp usual topology and usual convexity. It is FNS_4 and semi FNS_4. Now R^2 is not FNS_4, semi FNS_4, because consider the disjoint closed convex fuzzy sets

$$A = \{(a,b) \mid a \geqslant 0,\ b \geqslant 0,\ \ a.b \geqslant 1\}$$

and $B = \{(a,b) \mid a = 0\}$ in R^2

Now B has no closed convex fuzzy neighbourhood disjoint with A.

Note 5.3.4.

An FCP, F-continuous image of a semi FNS_4 space need not be semi FNS_4.

Example:

Let $X = \{a, b, c\}$ and $Y\ \{1, 2\}$

Let $T_1 = \{\emptyset, X, A_1, A_2, A_3, A_4, A_5, A_6\}$ and

$\mathcal{L}_1 = \{\emptyset, X, A_1, A_3, A_2{}^c\}$ where

$$A_1: \begin{array}{l} a \longrightarrow 1 \\ b \longrightarrow 1 \\ c \longrightarrow 0 \end{array} \qquad A_2: \begin{array}{l} a \longrightarrow \frac{3}{4} \\ b \longrightarrow \frac{3}{4} \\ c \longrightarrow 1 \end{array}$$

$$A_3: \begin{array}{l} a \longrightarrow 0 \\ b \longrightarrow 0 \\ c \longrightarrow 1 \end{array} \qquad A_4: \begin{array}{l} a \longrightarrow \frac{1}{2} \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow 0 \end{array}$$

$$A_5: \begin{array}{l} a \longrightarrow \frac{3}{4} \\ b \longrightarrow \frac{3}{4} \\ c \longrightarrow 0 \end{array} \qquad A_6: \begin{array}{l} a \longrightarrow \frac{1}{2} \\ b \longrightarrow \frac{1}{2} \\ c \longrightarrow 1 \end{array}$$

$$A_2^c: \begin{array}{l} a \longrightarrow \frac{1}{4} \\ b \longrightarrow \frac{1}{4} \\ c \longrightarrow 0 \end{array}$$

Consider $f: (X, \mathcal{L}_1, T_1) \longrightarrow (Y, \mathcal{L}_2, T_2)$ such that

$f(a) = 1,\ \ f(b) = 1$ and $f(c) = 2$.

Let $T_2 = \{\emptyset, Y, B_1, B_2, B_3, B_4\}$ where

$$B_1 : \begin{array}{l} 1 \longrightarrow 1 \\ 2 \longrightarrow 0 \end{array} \qquad B_2 : \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow 0 \end{array}$$

$$B_3 : \begin{array}{l} 1 \longrightarrow 0 \\ 2 \longrightarrow 1 \end{array} \qquad B_4 : \begin{array}{l} 1 \longrightarrow \frac{1}{2} \\ 2 \longrightarrow 1 \end{array}$$

Now $f^{-1}(B_i)$, i=1,2,3,4 is fuzzy open in X for each i. Therefore f is a fuzzy continuous map.

Let $\mathcal{L}_2 = \{\emptyset, Y, C_1, C_2, C_3\}$, where

$$C_1 : \begin{array}{l} 1 \longrightarrow 0 \\ 2 \longrightarrow 1 \end{array}, \quad C_2 : \begin{array}{l} 1 \longrightarrow \frac{1}{4} \\ 2 \longrightarrow 0 \end{array}, \quad C_3 : \begin{array}{l} 1 \longrightarrow 1 \\ 2 \longrightarrow 0 \end{array}$$

Now $f^{-1}(C_i)$, i=1,2,3 is a convex fuzzy set in X for each i. Therefore f is an FCP map.

Now A_3, A_1 and $A_2{}^c$ are the only proper closed convex fuzzy sets in which A_3 and $A_2{}^c$ are disjoint and A_3 and A_1 are disjoint. Now consider A_3 and $A_2{}^c$. $A_2{}^c \subset A_1$ and $A_2{}^c \subset A_4 \subset A_1$, where $A_4 \in T$.

$\therefore$ A_1 is a closed convex fuzzy neighbourhood of $A_2{}^c$ and A_3 and A_1 are disjoint. Then in the case of A_3 and A_1, A_1 itself is a closed convex fuzzy neighbourhood.

$\therefore$ $(X, \mathcal{L}, T)$ is semi FNS_4.

In $(Y, \mathcal{L}_2, T_2)$ consider the two disjoint closed convex fuzzy sets $B_1^c : \begin{matrix} 1 \longrightarrow 0 \\ 2 \longrightarrow 1 \end{matrix}$ and $B_3^c : \begin{matrix} 1 \longrightarrow 1 \\ 2 \longrightarrow 0 \end{matrix}$. Now there is no closed convex fuzzy neighbourhood separating them in Y. Hence $(Y, \mathcal{L}_2, T_2)$ is not semi FNS_4.

Proposition 5.3.4.

The quotient of an FNS_4, semi FNS_4 space is FNS_4, semi FNS_4 respectively if the quotient map is an F-closed, FCC and F-open map.

Proof:

Similar to Proposition 5.2.7.

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