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SOME LATTICE PROBLEMS IN FUZZY SET THEORY AND FUZZY TOPOLOGY

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By

S. BABUSUNDAR

DEPARTMENT OF MATHEMATICS AND STATISTICS
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
COCHIN - 682 022

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bona fide work done by Shri S.Babusundar, under my guidance and supervision in the Department of Mathematics and Statistics, Cochin University of Science and Technology, and has not been included in any other thesis submitted for the award of any degree.

Cochin -22
March 21, 1989

T.THRIVIKRAMAN

Professor and Head
Department of Mathematics
and Statistics
Cochin University of Scince
and Technology

DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text.

Cochin -22
March 21, 1989

(S. BABUSUNDAR)

Lecturer
Department of Mathematics
and Statistics
Cochin University of Scince
and Technology

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CONTENTS

INTRODUCTION

A theory of fuzzy sets, was introduced by L.A.ZADEH [30] as an alternative to classical theory of sets. He took the closed unit interval [0,1] as the membership set. J.A.GOGUEN [13] considered order structures beyond the unit interval for the membership set. He considered fuzzy subsets as generalized characteristic functions. Thus the ordinary set theory is a special case of fuzzy set theory where the membership set is {0,1}. GOGUEN suggested that a complete and distributive lattice would be a minimum structure for the membership set. Thereafter, many mathematicians, while developing fuzzy set theory have used different lattice-structures for the membership sets, like 1) Completely distributive lattice with 0 and 1 by T.E.GANTNER, R.C. STEINLAGE and R.H.WARREN [12], 2) Complete and completely distributive lattice with order reversing involution by BRUCE HUTTON and IVAN REILLY [8], 3) Complete and completely distributive non-atomic Boolean algebra by MIRA SARKAR [20], 4) Complete chain by ROBERT BADARD [2] and F.CONRAD [10], 5) Complete Brouwerian lattice with its dual also Brouwerian by ULRICH HOHLE [25], 6) Complete Boolean algebra by ULRICH HOHLE [26], 7) Complete and distributive lattice by S.E.RODABAUGH [21] and S.P.LOU [17].

Fuzzy topology was introduced by C.L.CHANG [9] and the theory of fuzzy topology was developed by many mathematicians, thereafter. BRUCE HUTTON [6,7], BRUCE HUTTON and IVAN REILLY [8] observed that the lattice of all fuzzy subsets on a set, has all the properties required of the membership set in the point-fuzzy set approach and hence the underlying set could be dispensed with. They defined a fuzzy topology as a subset of the membership lattice closed for finite meet and arbitrary join operations, and containing 0, 1. RICHARD LOWEN [18] vastly modified the definition of fuzzy topology, given by C.L.CHANG [9] and obtained a fuzzy version of Tychonoff therom, but he lost the concept that fuzzy topology generalizes topology.

We believe that every fuzzy generalization should be formulated in such a way that it contain the ordinary set theoretic notion as a special case. Therefore we take the definition of fuzzy topology in the line of C.L.CHANG [9] with an arbitrary complete and distributive lattice as the membership set. Almost all the results proved and presented in this thesis can, in a sense, be called generalizations of corresponding results in ordinary set theory and set topology. However the tools and the methods have to be in many of the cases, new.

In the first chapter of this thesis, we study the properties of induced functions between the lattices of fuzzy subsets, where the underlying sets and the corresponding membership sets are allowed to vary. Complete and completely distributive lattices are taken as membership sets. Let X and Y be two sets and L,M be the corresponding membership sets. Let L(X) and M(Y) denote the lattices of fuzzy subsets of X and Y respectively. Let f:X--->Y and g:L--->M be two given functions. Corresponding to the pair of functions (f,g), a function E:L(X)--->M(Y) and another F:M(Y)--->L(X) are defined. S.E.RODABAUGH [21] had used (f,g), g satisfying some more properties, to define morphisms in the category of L-fuzzy topologies: FUZZ. Necessary and sufficient conditions on the pair of functions (f,g) are investigated so that the induced functions E and F are one to one, onto, lattice homomorphism and t-homomorphism. A t-homomorphism is a {0,1}-homomorphism [GEORGE GRATZER, 14] which preserves arbitrary join operation. Also minimal conditios on the pair of functions, are derived so that E and F are inverses of each other. The collection of all fuzzy topologies [cf.,C.L.CHANG, 9] is found to be a lattice under the order relation of set inclusion. Conditions on f and g are further investigated so that f and g naturally induce a function E':(L,X)--->(M,Y) and another, F′:(M,Y)--->(L,X). Properties of E′ and F′ are also studied with reference to properties of f and g.

Some results in lattice theory, are developed which are required for further studies in the later chapters. A complete lattice L is considered, in general. The concept of join and meet irreducible element [GRATZER,14,p.60] is adapted to define t-irreducible elements,t-irreducible subsets and minimal t-irreducible subsets. Existence of t-irreducible elements in a complete, complented and distributive lattice is studied in detail. The minimal t-irreducible subsets of the Boolean algebra of subsets of a set, are characterised. It is shown that in a lattice if every nonzero element belongs to some minimal t-irreducible subset of the lattice, then the lattice must be a chain. In chapter IV, minimal t-irreducible subsets of the membership lattice are shown to be intimately connected with the dual atoms in the lattice of fuzzy topologis.

A.K.KATSARAS [16] introduced fuzzy filters, taking the membership set to be the closed unit interval [0,1]. P.SRIVASTAVA and R.L.GUPTA [23] observed that the behaviour of ultrafuzzy filters, is radically different from the ordinary set theory. We, in the third chapter, study the properties of fuzzy filters and ultrafuzzy filters with reference to the structure of the membership lattice. We begin the study of fuzzy filters by taking a complete and distributive lattice as the membership set and any ordinary set, as the underlying set. Some necessary and sufficent conditions, for a fuzzy filter to be an ultrafuzzy filter, are derived. When the membership lattice is further assumed to be complemented, many characterisations of ultrafuzzy filters are obtained, analogous to those available for ultrafilters. Principal fuzzy filters are introduced and found that unlike in the ordinary set theory, principal fuzzy filters on fuzzy singletons are not ultrafuzzy filters. Only when the membership value in the fuzzy singleton is an atom, the principal fuzzy filter is an ultrafuzzy filter. Thus it is observed that if the membership lattice is the unit interval [0,1], then no principal fuzzy filter is an ultrafuzzy filter.

In the final chapter we study some of the lattice properties of the lattice of fuzzy topologies on a fixed set X. Membership set is taken to be a complete and distributive lattice L. The collection (L,X) of all fuzzy topologies on a set, ordered by set inclusion is a complete lattice. It is in general, not distributive. This lattice is atomic. The dual atoms in (L,X) are designated as ultrafuzzy topologies. The existence of minimal t-irreducible subsets in L, is proved to be necessary and sufficient for (L,X) to have dual atoms. A significant observation is that ultrafuzzy toplogies do not exist if, the membership set is taken to be the closed unit interval [0,1]. A necessary and sufficient condition for (L,X) to be dually atomic is derived. In the light of the above characterisation, a few necessary conditions on L, is found out for (L,X) to be dually atomic. It is found that if $L \neq \{0,1\}$ and if X contains atleast two elements, then (L,X) is not dually atomic. However the lattice of topologies on a set is dually atomic [FROLICH, 11]. Finally, an attempt is made to solve the problem of complementation in the lattice of fuzzy topologies on a set. It is proved that in general, the lattice of fuzzy topologies is not complemented. Complements of some fuzzy topologies are found out. It is observed that (L,X) is not uniquely complemented. However, a complete analysis of the problem of complementation in the lattice of fuzzy topologies is yet to be found out.

CHAPTER I

PROPERTIES OF INDUCED FUNCTIONS

1.1 Preliminaries

Many order structures are being used as membership sets in fuzzy set theory and fuzzy topology. Through out this chapter we consider complete and completely distributive lattices only, as membership sets. Let X and Y be two sets and, L, M be the membership sets respectively. "0" and "1" commonly denote the least and largest element in any lattice. The symbols "$\geq$","$\wedge$" and "V" are also commonly used to denote the order relation, meet operation and join operation in any latice. "I" denotes an arbitrary index set and "i" is a general member of I. We take join of members of empty set =0. i.e., $V\emptyset = 0$

1.1.1 Lattice of fuzzy sets

Definitions

A function a: X--->L is called a fuzzy subset of X. For a point x in X, a(x) is called the membership value of x in the fuzzy subset a.

Let L(X) denote the collection of all fuzzy subsets of X. In L(X) we can define an order as follows: for a,b in L(X), a $\leq$ b iff a(x) $\leq$ b(x) for all x in X.

1.1.2 Remark

L(X) is a complete and distributive lattice. Clearly $\leq$ is a partial order and L(X) is a lattice under this order relation. We have that for a,b in L(X), $(a\wedge b)(x) = a(x) \wedge b(x)$ and $(a\vee b)(x) = a(x) \vee b(x)$, for each x in X. Let $\{ a(i) \mid i$ in $I \}$ be a subset of L(X). Then $((\vee a(i))(x) = \vee a(i)(x)$, similarly $(\wedge a(i))(x) = \wedge a(i)(x)$ for each x in X. Thus L(X) is closed for arbitrary join and meet operations. Moreover the constant fuzzy subsets taking the membership values 0 and 1 of L respectively are the least and the largest elements in L(X). Further L(X) is distributive, since if a,b,c are in L(X), then

$$(a\wedge(b\vee c))(x) = a(x)\wedge(b(x)\vee c(x))$$

$$= (a(x)\wedge b(x))\vee(a(x)\wedge b(x)) \quad \text{..L is distributive}$$

$$= (a\wedge b)(x)\vee(a\wedge c)(x)$$

$$= ((a\wedge b)\vee(a\wedge c))(x) \quad \text{..for each x in X}$$

Further, if L is assumed to be complemented then L(X) is also complemented.

1.1.3 Definition

Let A be a subset of X. Characteristic function on A is a fuzzy subset of X defined by

$$\mathrm{Char}(A)(x) = \begin{cases} 1 & \text{if } x \text{ belongs to } A \\ 0 & \text{otherwise.} \end{cases}$$

1.1.4 Note

Denoting the two element sublattice {0,1} of L by 2, 2(X) denotes the lattice of all characteristic functions on subsets of X. 2(X) is also the lattice of all fuzzy subsets of X with the membership set {0,1}. 2(X) is a complete and complemented distributive sublattice of L(X).

1.1.5 Induced functions

Definition

Let f:X--->Y and g:L--->M be two given functions. Define functions E:L(X)--->M(Y) and F:M(Y)-->L(X) as follows: for a in L(X) and y in Y,

$$E(a)(y) = g(\mathrm{V}f(y)) \text{ and } F(b)(x) = \mathrm{V}g^{-1}(bf(x))$$

for x in X and b in M(Y).

1.3.2 Definition

Let f:X--->Y and h:M--->L be two given functions. Define function H:M(Y)--->L(X) as follows: for d in M(Y) and x in X, H(d)(x) = hdf(x).

1.2 Properties of E

1.2.1 Theorem

(i) E is one to one iff f and g are one to one,

(ii) E is onto iff f and g are onto,

and hence

E is a bijection iff f and g are bijections.

Proof

i) **Sufficiency:** Let f and g be one to one and a,b belong to L(X). Let E(a) = E(b). We want to show that a = b, equivalently a(x) = b(x) for each x. Suppose not, then there exists an x in X such that $a(x) \neq b(x)$. Since f is one to one there exists a unique y in Y such that $f^{-1}(y) = \{x\}$. Then $af^{-1}(y) = a(x) \neq b(x) = bf^{-1}(y)$. Now since g is one to one

$$E(a)(y) = g(\vee af^{-1}(y)) = g(a(x)) \neq g(b(x)) = Eb(y).$$

i.e., there exists a y such that $E(a)(y) \neq E(b)(y)$, hence $E(a) \neq E(b)$, a contradiction. Thus a(x) = b(x) for all x in X. i.e., a = b and Therefore, E is one to one.

Necessary: Let E be one to one. We want to show that a) f is one to one and b) g is one to one

a) Suppose f is not one to one, then there exists w,x in X such that f(w) = f(x) = z in Y (say). Let a = Char({w}) and

$b = Char(\{x\})$. Then a, b belong to L(X) and $a \neq b$. However,

$$E(a)(y) = g(\vee a f^{-1}(y)) = \begin{cases} g(1) & \text{if } y = z \\ 0 & \text{if } y \neq z \end{cases}$$

and $$E(b)(y) = g(\vee b f^{-1}(y)) = \begin{cases} g(1) & \text{if } y = z \\ 0 & \text{if } y \neq z. \end{cases}$$

Thus, though $a \neq b$, $E(a) = E(b)$, hence E is not one to one, a contradiction. Therefore, f must be one to one.

b) Suppose g is not one to one, let $g(l) = g(m)$ for some l,m in L and $l \neq m$. Consider the constant fuzzy subsets $\underline{l},\underline{m}$ of X taking the membership values l and m respectively on X. Then though $l \neq m$, for y in Y

$$E(\underline{l})(y) = g(\vee \underline{l} f^{-1}(y)) = \begin{cases} 0 & \text{if } y \text{ is not in } f(X) \\ g(l) & \text{if } y \text{ is in } f(X) \end{cases}$$

and $$E(\underline{m})(y) = g(\vee \underline{m} f^{-1}(y)) = \begin{cases} 0 & \text{if } y \text{ is not in } f(X) \\ g(l) & \text{if } y \text{ is in } f(X) \end{cases}$$

i.e., $E(\underline{l}) = E(\underline{m})$ and hence E is not one to one. Therefore, g must be one to one.

ii) **Sufficiency:** Let f and g be onto, and d belongs to M(Y). Since g is onto, there exists l(y) in L such that $g(l(y)) = d(y)$ for each y in Y. Since f is onto, for each y in Y, $f^{-1}(y)$ is nonempty. Now define a in L(X) as follows: for x in X, a(x) = l(y) if f(x) = y. Clearly E(a) = d. Since d is an arbitrary element in M(Y), E is onto.

Necessary: Let E be onto. We want to show that a) f is onto and b) g is onto.

a) Suppose f is not onto, then there exists an z in Y such that $f^{-1}(z) = \emptyset$. Now let d = Char({z}). Then d is in M(Y). For any a in L(X), $E(a)(z) = g(\vee a f^{-1}(z)) = 0$. Therefore, d is not an image under E, as d(z) = 1. Thus E is not onto, a contradiction. Hence f must be onto.

b) Suppose g is not onto, then there exists an m in M such that m is not in g(L). Then the constant fuzzy set $\underline{m}$ of Y cannot be an image under E. Therefore, E is not onto, a contradiction. Hence g must be onto.
The proof of ii) is complete.

1.2.2 Theorem

If g is a non-constant function, then E is a latice homomorphism if and only if g is a homomorphism and f is one to one.

Proof

Necessary: Suppose g is a non-constant function, E is a lattice homomorphism, and f is not one to one, then there exist w and x in X such that $w \neq x$ and $f(w) = f(x) = z$ (say) Let $a = Char(\{w\})$and $b = Char(\{x\})$. Then $a \wedge b = 0$ in $L(X)$, but $E(a \wedge b)(z) = g(V(a \wedge b)f^{-1}(z)) = g(0)$, whereas $E(a)(z) = E(b)(z) = g(1)$. Since for a non-constant lattice homomorphism g, $g(0) \neq g(1)$, we have that $E(a \wedge b) \neq E(a) \wedge E(b)$ Therefore, E isa not a lattice homomorphism. Thus f must be onto.

Suppose g is not a homomorpism, then there exist l,m in L such that either

$$g(l \vee m) \neq g(l) \vee g(m) \text{ or } g(l \wedge m) \neq g(l) \wedge g(m).$$

Correspondingly for the constant fuzzy subsets $\underline{l}$ and $\underline{m}$

either $$E(\underline{l} \vee \underline{m}) \neq E(\underline{l}) \vee E(\underline{m})$$

or $$E(\underline{l} \wedge \underline{m}) \neq E(\underline{l}) \wedge E(\underline{m}).$$

Therefore, for E to be a lattice homomorphism, g must be a lattice homomorphism.

Sufficiency: Let f be one to one and g be a lattice homomorphism. Then for a,b in $L(X)$ and y in Y

$$\begin{aligned} E(a \vee b)(y) &= g(V(a \vee b)f^{-1}(y)) \\ &= g((af^{-1}(y)) \vee (b(f^{-1}(y)))) \quad \text{..f is one to one} \\ &= g(af^{-1}(y)) \vee g(gf^{-1}(y)) \quad \text{..g is homomorphism} \\ &= E(a)(y) \vee E(b)(y) \end{aligned}$$

Similarly, $E(a \wedge b)(y) = E(a)(y) \wedge E(b)(y)$. Hence E is a lattice homomorphism.

1.2.3 Note

If g is a constant function then E is a lattice homomorphism irrespective of f being one to one or not, since E, in this case will also be a constant function.

1.2.4 Definition

Let J and K be two complete lattices and h:J--->K be a function such that

i) h is a homomorphism,

ii) h(0) = 0 and h(1) = 1,

and iii) $h(\vee l(i)) = \vee h(l(i))$ where {l(i)| i in I} is an arbitrary subset of J.

Then h is called a t-homomorphism.

["t" in the above definition is indicative of the fact that a t-homomorphism takes a fuzzy topology to a fuzzy topology. In particular when j = K = 2(X), a t-homomorphism takes a topology to another topology on X].

1.2.5 Theorem

E is a homomorphism with E(0) = 0 and E(1) = 1 iff g is a homomorphism , g(0) = 0 and g(1) = 1, and f is a bijection.

Proof

Necessary: From the theorem (1.2.2) f must be one to one and g must be a homomorphism when E is a homomorphism. Clearly if $g(0) \neq 0$, then $E(0) \neq 0$ and if $g(1) \neq 1$ then $E(1) \neq 1$. Thus $g(0) = 0$ and $g(1) = 1$, are necessary. Now if f is not onto then there exists an z in Y such that $f(z) = 0$. But then $E(1)(z) = 0$ and hence $E(1) \neq 1$, a contradiction. Hence f must be onto, as well.

Sufficiency: From the theorem (1.2.2), if f is a bijection and g is a homomorphism with $g(0) = 0$ and $g(1) = 1$, then E is a homomorphism. Further, for all y in Y, since f is onto

$$E(1)(y) = g(V1f^{-1}(y)) = g(1) = 1$$
$$\text{and } E(0)(y) = g(V0f^{-1}(y)) = g(0) = 0.$$

The proof is complete.

1.2.6 Theorem

Let f be a bijection. Then E is a t-homomorphism iff g is a t-homomorphism.

Proof

In the light of the theorem (1.2.5), to complete the proof, it is enough to show that E preserves arbitrary join operation iff g does so.

Necessary: Suppose g doesnot preserve arbitrary join operation, then there exists $\{l(i) \mid i \text{ in } I\}$, an arbitrary subset of L such that $g(\vee l(i)) \neq \vee g(l(i))$. Consider the constant fuzzy subsets $\underline{l}(i)$, for each i. We then have that $E(\vee \underline{l}(i)) \neq \vee E(\underline{l}(i))$. i.e., E doesnot preserve arbitrary join operation.

Sufficiency: Let g preserves arbitrary join operation and $\{a(i) \mid i \text{ in } I\}$ be an arbitrary subset of L(X). Then for each y in Y,

$$E(\vee a(i))(y) = g(\vee(\vee a(i) f^{-1}(y))$$

$$= g(\vee(a(i) f^{-1}(y))) \quad \text{..f is a bijection}$$

$$= \vee g(a(i) f^{-1}(y)) \quad \text{..g preserves arbitrary join operation}$$

$$= \vee E(a(i))(y)$$

i.e., $E(\vee a(i)) = \vee E(a(i))$

1.2.7 Observation

Taking L = M and g to be the identity function, we have that g is a t-homomorphism, which is also a bijection. In the light of the above theorem, we have that, corresponding to every function f:X--->Y, there exists a function E:L(X)--->L(Y) such that

i) E is one to one iff f is one to one

ii) E is onto iff f is onto

and iii) E is an onto t-isomorphism iff f is a bijection.

Taking X = Y and f to be the identity function, there exists a naturally induced function $E:L(X) \longrightarrow M(X)$, corresponding to every function $g:L \longrightarrow M$ such that

i) E is one to one iff g is one to one

ii) E is onto iff g is onto

and iii) E is a t-homomorphism iff g is a t-homomorphism.

1.2.8 Definition [cf. C.L.CHANG, 9]

A subset T of L(X) is said to be a fuzzy topology on X if

i) $0,1 \in T$,

ii) $a,b \in T$ implies $a \wedge b \in T$

and iii) $a(i) \in T$ for i in I implies $\vee a(i) \in T$.

1.2.9 Remark

Let (L,X) denote the collection of all fuzzy topologies on X. Ordered by set inclusion (L,X) is a lattice. For S,T in (L,X), $S \wedge T = S \cap T$ and $S \vee T$ is the smallest fuzzy topology containing S and T. This is meaningful, since arbitrary intersection of fuzzy topologies is a fuzzy topology and L(X) is the largest element in (L,X). The smallest fuzzy topology on X is {0,1}. More lattice properties of (L,X) are studied in Chapter IV.

1.2.10 Note

A t-homomorphism $E:L(X) \longrightarrow M(Y)$ induces a function

$E':(L,X) \longrightarrow (M,Y)$ where $E'(T) = \{E(t) \mid t \in T\}$ for each T in (L,X). Clearly $E'(T)$ is a fuzzy topology on Y, for each T in (L,X).

The following observations on E', are immediate:

i) $E'(0) = 0$

ii) $E'(1) = 1$ iff E is onto, and

iii) $E'(VT(i)) = V(E'(T(i))$ for T(i) in (L,X) and I in I.

1.2.11 Theorem

Let $E:L(X) \longrightarrow M(Y)$ be a t-homomorphism and $E':(L,X) \longrightarrow (M,Y)$ be the induced function [1.2.10]. Then

i) E' is one to one iff E is one to one

ii) E' is onto iff E is onto

iii) E' is a homomorphism iff $E^{-1}(d)$ is either a singleton or empty for all d in M(Y) and $d \neq 0,1$

iv) E' is an onto t-isomorphism iff E is a bijection.

Proof

i) **Necessary:** Let E' be one to one and $a,b \in L(X)$ be such that $E(a) = E(b)$. consider $S = \{0,a,1\}$ and $T = \{0,b,1\}$. S and T belong to (L,X) and $E'(S) = E'(T)$. Since E' is one to one, S = T and hence a = b. i.e., E is one to one.

Sufficiency: Let E be one to one and S,T ∈ (L,X) be such that S ≠ T. Then there exist an a ∈ L(X), which belongs to only one of them. Assume that a (S only. Then E(a) ∈ E′(S) but E(a) ∉ E′(T). Therefore, E′ is one to one.

ii) **Necessary:** Let E′ be onto and d ∈ M(Y).Let T = {0,d,1}. Then T ∈ (M,Y) and there exists S in (L,X) such that E′(S) = T. Thus there exists an s in S such that E(s) = d. Hence E is onto.

Sufficiency: Let E be onto and T ∈ (M,Y). Let S be the set of all s in S such that E(s) ∈ T. Clearly 0, 1 belong to S. If a, b ∈ S, then E(a), E(b) ∈ T and hence E(a)∧E(b) ∈ T. Since E(a)∧E(b) = E(a∧b), a∧b belongs to S. Finally if {a(i)| i in I} is a subset of S, then E(a(i)) ∈ T, for each i, hence V(E(a(i)) ∈ T. Since V(E(a(i)) = E(Va(i)), Va(i) ∈ S. Thus S is a fuzzy toplogy and trivially E′(S) = T. Therefore, E′ is onto.

iii) **Necessary:** Let E′ be a lattice homomorphism. Suppose, there exists d in M(Y) such that d ≠ 0,1 and E (d) ≠ 0, is not a singleton, then there exist a,b in L(X) such that E(a) = E(b) = d. Let R = {0,a,1} and S = {0,b,1}. Then R,S ∈ (L,X), R ≠ S and R ∧ S = {0,1}. However, E′(R) = E′(S) = {0,d,1}. Thus E′(R^S) ≠ E′(R) ∧ E′(S), a contradiction to the assumption that E′ is a homomorphism.

Therefore, $E^{-1}(d)$ must be either a singleton or empty whenever E′ is a homomorphism.

Sufficiency: Let E be such that $E^{-1}(d)$ is either a singleton or empty, for all d in M(Y)\{0,1}. Recalling the result (1.2.10)(iii), it remains only to prove that E′ preserves meet operation. Let R,S belong to (L,X). Clearly E′(R∧S) is a subset of E′(R)∧E′(S). Let d (E′(R)∧E′(S). If d is either 0 or 1, then d belongs to E′(R∧S) also, and if d ≠ 0,1 then there exists s in S and t in T such that E(s) = E(t) = d. Thus {s,t} is a subset of $E^{-1}(d)$. Since $E^{-1}(d)$ is nonempty, it must be a singleton. Therefore, s = t. i.e., t ∈ R∧S. Thus E(t) =d, and d ∈ E′(R∧S). Hence E′(R∧S) = E′(R)∧E′(S) and E′ preserves meet operation and the proof is complete.

iv) follows from i), ii), and iii) above and the result (1.2.10).

1.2.12 Theorem

If f:X--->Y is a bijection, then the following are equivalent, for the induced functions.

i) g:L--->M is an onto t-isomorphism,

ii) E:L(X)--->M(Y) is an onto t-isomorphism,

and iii) E′:(L,X)--->(M,Y) is an onto t-isomorphism.

1.3 Properties of F

1.3.1 Theorem

i) If g is a bijection, then F is one to one iff f is onto

ii) F is one to one does not imply either g is one to one or g is onto. But, if F is one to one, then g is one to one implies g is onto.

and iii) F is one to one implies f is onto.

Proof

i) **Necessary:** Suppose F is not one to one and f is not onto then there exists an z in Y such that $z \notin f(X)$. Let $a = Char(\{z\})$. Now $a \in M(Y)$ and $a \neq 0$ but $F(a) = F(0)$, which is a contradiction to the assumption that F is one to one. Hence f must be onto.

Sufficiency: Let $a, b \in M(Y)$ and $F(a) = F(b)$. Then for each x in X, $F(a)(x) = F(b)(x)$. i.e., $Vg^{-1}af(x) = Vg^{-1}bf(x)$. This implies that $af(x) = bf(x)$, since g is a bijection. Thus $a = b$, since f is onto. Therefore, F is one to one.

ii) Consider the following example.

1.3.2 Example: Let L = M = X = Y = [0,1] and f:X--->Y be the identity function. Let $\{p(n) \mid n = 1,2,...\}$ be a set of distinct prime numbers. Let A(n) denote the set of all p(n)-adic rationals in (0,1) and let $\{q(n) \mid n = 1,2,,,\}$ be the set of rationals in (0,1). Now define for each x in [0,1], subsets :

$$B(x) = \begin{cases} \{x\} & \text{if } x \text{ is a irrational} \\ \{0\} & \text{if } x = 0 \\ A(n) \cap [0,x] & \text{if } x = q(n) \\ (U\{A(n) \not\ni B(q(n))\}) \cup \{1\} & \text{if } x = 1 \end{cases}$$

Now { B{x}! x ∈ [0,1] } is a partition of [0,1] such that VB(x) = x , for every x in X. Define g:L--->M such that for each l in L, g(l) = x if l ∈ B(x). Then g is onto, but not one to one. However for a,b in M(Y), F(a) = F(b) implies a = b since, F(a) = F(b) implies F(a)(x) = F(b)(x), for each x in X.

$$\text{i.e., } Vg^{-1}af(x) = Vg^{-1}bf(x)$$

$$\text{i.e., } Vg^{-1}a(x) = Vg^{-1}b(x) \qquad \therefore f \text{ is the identity.}$$

But this implies that a(x) = b(x) for each x. Thus F is one to one, though g is not one to one. Moreover, here F is a bijection (identity function).

1.3.3 Example: Let L = M = {0,1} and g:L--->M be the constant function 1. Let X and Y be two sets such that there exists an onto function f:X--->Y. Now if for some a,b in M(Y), F(a) = F(b) then F(a)(x) = F(b)(x) for all x in X.

$$\text{i.e., } Vg^{-1}af(x) = Vg^{-1}bf(x).$$

Thus we have $\vee g^{-1}af(x) = \vee g^{-1}bf(x) = 0$ or 1 for each x.

$\vee g^{-1}af(x) = \vee g^{-1}bf(x) = 0$ implies $af(x) = bf(x) = 0$ and $\vee g^{-1}af(x) = \vee g^{-1}bf(x) = 1$ implies $af(x) = bf(x) = 1$ for $x \in X$. Since f is onto , a = b. i.e., F is one to one, though g is not onto.

Suppose F and g are one to one but g is not onto, then there exists m in M such that $m \notin g(L)$. Let $g(0) = n$. Then $n \neq m$. Consider the constant fuzzy subsets $\underline{n}$ and $\underline{m}$ of Y, $\underline{n} \neq \underline{m}$ but $F(\underline{n}) = F(\underline{m})$, which is a contradiction to the assumption that F is one to one. Hence g must be onto. Proof of ii) is complete.

iii) If f is not onto, then there exists an z in Y such that $z \notin f(X)$. Let a = Char({z}). Then $a \in M(Y)$ and $a \neq 0$, but $F(a) = F(0)$. Thus F is not one to one, a contradiction. Hence the result.

1.3.4 Definition

Let $h:L \longrightarrow M$ be a function and L/h denote the set $\{A(m) = h^{-1}(m) \mid m \in M\}$. A(m)'s are called the fibers of h at m. Define $\vee$ and $\wedge$ operation in L/h as follows: for $m,n \in M$

$$A(m) \vee A(n) = A(m \vee n) \text{ and } A(m) \wedge A(n) = A(m \wedge n).$$

Thus L/h is a lattice which is complete and completely distributive with least element A(0) and the largest element

$A(1)$. A function $J: L/h \to L$ defined by $J(A(m)) = \vee A(m)$ is called the join function on the fibers of h. The fibers of h are distinct iff h is either onto or $M \neq h(L)$ is a singleton.

1.3.5 Theorem

F is one to one iff i) f is onto,

ii) the fibers of g are distinct,

and iii) join function on the fibers of g is one to one.

Proof

Necessary: f must be onto follows from the theorem (1.3.1). Suppose the fibers are not distinct then there exist m,n in M such that $m \neq n$ and $g^{-1}(m) = g^{-1}(n)$. Consider the constant fuzzy subsets $\underline{m}$ and $\underline{n}$ in $M(Y)$, we have $\underline{m} \neq \underline{n}$ but $F(\underline{m}) = F(\underline{n})$. Thus F is not one to one, a contradiction. Therefore, the fibers of g must be distint.

Suppose the join function on the fibers of g is not one to one, then there exist m,n in M such that $\vee g^{-1}(m) = \vee g^{-1}(n)$ but $m \neq n$. Thus the constant fuzzy subsets $\underline{m}$ and $\underline{n}$ are different but $F(\underline{m}) = F(\underline{n})$, a contradiction to the assumption that F is one to one. Hence the join function on the fibers of g must be one to one.

Sufficiency: Let $F(c) = F(d)$ for some c,d in $M(Y)$. Then $F(c)(x) = F(d)(x)$ for all x in X. i.e., $Vg^{-1}c(x) = Vg^{-1}d(x)$ for each x in X. This implies $c(x) = d(x)$ for each x, since join function on the fibers of g is one to one. Thus $c = d$. Since c and d are arbitrary, F is one to one.

1.3.6 Theorem

i) If g is one to one then F is onto iff f is one to one.

ii) F is onto, does not imply that g is one to one.

iii) f is one to one does not imply that F is onto.

iv) F is onto iff f is one to one and the join function on the fibers of g is one to one.

Proof

i) **Necessary:** Suppose F is onto and f is not one to one. Then there exist w,x in X such that $f(w) = f(x)$. Then for all d in $M(Y)$, $F(d)(w) = F(d)(x)$. Thus no c in $L(X)$ with $c(w) \neq c(x)$ can be an image under F. Therefore, F is not onto, a contradiction. Thus f must be one to one, irrespective of g being one to one or not.

Sufficiency: Let f and g be one to one and $a \in L(X)$. Define $d:Y \longrightarrow M$ such that, for y in Y

$$d(y) = \begin{cases} ga(x) & \text{if } f(x) = y \\ 0 & \text{otherwise.} \end{cases}$$

"d" is well defined, since f is one to one. For each x in X $F(d)(x) = Vg^{-1}df(x) = Vg^{-1}ga(x) = a(x)$, since g is one to one. Thus F is onto.

ii) In example (1.3.2), F is onto while g is not one to one. Hence ii) holds.

iii) Let X be a set with atleast two points and Y = X; f:X--->Y be the identity function; M = {0,1}; L = {0,l,1}, where 0<l<1 ; g:L--->M be the constant function taking one on L and $\underline{l}$ be the constant fuzzy subset on X taking the value l. Now for all d in M(Y) and x in X,

$$F(d)(x) = \begin{cases} 1 & \text{if } d(x) = 1 \\ 0 & \text{if } d(x) = 0 \end{cases}$$

Then $F(d) \neq \underline{l}$ for all d in M(Y). Therefore, F is not onto. Hence iii) holds.

iv) Necessity: f must be one to one, follows from the proof of i). Suppose the join function on the fibers of g is not onto, then there exists l in L such that $Vg^{-1}(m) \neq l$ for all m in M. But then the constant fuzzy subset $\underline{l}$ in L(X) cannot be an image under F. Therefore, F is not onto, a contradiction. Hence the necessity.

Sufficiency: Let a $\in$ L(X). Since join function on the fibers of g is onto, for each x in X, and a(x) $\in$ L, there exists

$m(x) \in M$ such that $Vg^{-1}m(x) = a(x)$. Define $d \in M(Y)$ by

$$d(y) = \begin{cases} m(x) & \text{if } f(x) = y \\ 0 & \text{otherwise,} \end{cases}$$

for each y in Y. "d" is well defined since f is one to one. And for each x in X,

$$F(d)(x) = Vg^{-1}df(x) = Vg^{-1}m(x) = a(x).$$

i.e., $F(d) = a$. Thus F is onto. The proof is complete.

1.3.7 Theorem

i) g is an onto isomorphism, implies F is a homomorphism

ii) F is a homomorphism , does not imply that

a) g is one to one,

b) g is a homomorphism,

and c) g is onto.

iii) F is a homomorphism iff join function on the fibers of g, is a homomorphism.

Proof

i) Let g be an onto isomorphism. Then, for any c,d in M(Y),

$$\begin{aligned} F(cVd)(x) &= Vg^{-1}(cVd)f(x) \\ &= g^{-1}(cf(x)V(df(x))) && \text{.. g is one to one} \\ &= g^{-1}(cf(x))\ V\ g^{-1}(df(x)) && \text{.. g is onto isomorphism} \\ &= F(c)(x)\ V\ F(d)(x), \text{ for each x in X.} \end{aligned}$$

Thus $F(c \vee d) = F(c) \vee F(d)$. Similarly $F(c \wedge d) = F(c) \wedge F(d)$. Hence F is a homomorphism.

ii) In example (1.3.2), F is an identity homomorphism while g is neither one to one nor a homomorphism. So it remains to prove only that "g is onto" is not necessary for F to be a homomorphism. Consider $L = [0,1]$ and $M = \{0\} \cup [0.5, 1]$ and $g: L \to M$, defined as

$$g(l) = 0.5(l) + 0.5, \quad \text{for } l \text{ in } L.$$

Then g is an isomorphism and not onto M. Consider c,d in M(Y) and x in X.

$$
\begin{aligned}
F(c \vee d)(x) &= \vee g^{-1}(c \vee d)f(x) \\
&= \vee g^{-1}(cf(x) \vee df(x)) \\
&= \begin{cases} g^{-1}cf(x) \vee g^{-1}df(x) & \text{if } cf(x), df(x) \neq 0 \\ g^{-1}cf(x) & \text{if } df(x) = 0 \\ g^{-1}df(x) & \text{if } cf(x) = 0 \end{cases} \\
&= F(c)(x) \vee F(d)(x), \quad \text{since } cf(x) = 0
\end{aligned}
$$

implies $F(c) = 0$ and $df(x) = 0$ implies $F(d)(x) = 0$.

i.e., $F(c \vee d) = F(c) \vee F(d)$.

$$
\begin{aligned}
\text{And } F(c \wedge d)(x) &= \vee g^{-1}(c \wedge d)f(x) \\
&= \vee g^{-1}(cf(x) \wedge df(x)) \\
&= \begin{cases} (g^{-1}cf(x)) \wedge (g^{-1}df(x)) & \text{if } af(x) = 0 \\ 0 & \text{othewrwise} \end{cases} \\
&= F(c)(x) \wedge F(d)(x) \text{ for each } x \text{ in } X.
\end{aligned}
$$

Thus, $F(c \wedge d) = F(c) \wedge F(d)$ and hence F is a homomorphism.

iii) Necessity: Suppose join function on the fibers of g is not a homomorphism, then there exists m,n in M such that

either $\quad \vee g^{-1}(m \vee n) \neq (\vee g^{-1}(m)) \vee (\vee g^{-1}(n))$

or $\quad \vee g^{-1}(m \wedge n) \neq (\vee g^{-1}(m)) \wedge (\vee g^{-1}(n))$.

Correspondingly for the constant fuzzy subsets $\underline{m}$ and $\underline{n}$ in M(Y),

either $\quad F(\underline{m} \vee \underline{n}) \neq F(\underline{m}) \vee F(\underline{n})$

or $\quad F(\underline{m} \wedge \underline{n}) \neq F(\underline{m}) \wedge F(\underline{n})$.

i.e., F is not a homomorphism. Hence the necessity.

Sufficiency: Let $c,d \in M(Y)$. Then for each x in X,

$F(c \vee d)(x) = \vee g^{-1}(c \vee d)f(x)$

$= \vee g^{-1}(cf(x) \vee df(x))$

$= (\vee g^{-1}(cf(x)) \vee (\vee g^{-1}(df(x)))$ ..the join function is a homomorphism

$= F(c)(x) \vee F(d)(x)$.

Similarly, $F(c \wedge d)(x) = F(c)(x) \wedge F(d)(x)$, for each x in X.

Thus F is a homomorphism.

1.3.8 Theorem

If $g^{-1}(0) \subset \{0\}$ and the join function on the fibers of g is a t-homomorphism, then F is a t-homomorphism.

[Prof is straight forward and hence is omitted]

1.3.9 Theorem

i) $F(0) = 0$ iff $g^{-1}(0) \subset \{0\}$

ii) g is a t-homomorphism and $g^{-1}(0) \subset \{0\}$, do not imply that F is a homomorphism

Proof

i) Can be easily proved.

ii) Consider the following example. Let $L = \{0,l,1\}$, where $0 < l < 1$ and $M = \{0,m,n,1\}$, where m and n are atoms in M. Define $g:L \longrightarrow M$ as follows: $g(0) = 0$, $g(1) = 1$, and $g(l) = m$. Then g is a t-homomorphism and $g^{-1}(0) \subset \{0\}$. Let X be an arbitrary set, $Y = X$ and $f:X \longrightarrow Y$ be the identity function. Consider the constant fuzzy subsets $\underline{m}$ and $\underline{n}$ of Y. $\underline{m} \vee \underline{n} = 1$. Therefore, $F(\underline{m} \vee \underline{n})(x) = 1$ while $F(\underline{m})(x) = l$ and $F(\underline{n})(x) = 0$. Thus $F(\underline{m} \vee \underline{n}) \neq F(\underline{m}) \vee F(\underline{n})$ and hence F is not a homomorphism. The proof of ii) is complete.

1.3.10 Remark

If g is an one to one and onto lattice homomorphism then f is an onto t-homomorphism. But the converse is not true, follows from example (1.3.2), where F is an onto t-isomorphism, while g is neither one to one nor a lattice homomorphism.

1.3.11 Theorem

$F \circ E$ is the identity function iff f and g are one to one.

Proof

Necessity: Suppose f is not one to one, then there exist w,x in X such that $w \neq x$ and $f(w) = f(x)$. Let a be the fuzzy subset of X defined by, y in for X,

$$a(y) = \begin{cases} 0 & \text{if } y = w \\ 1 & \text{if } y \neq w. \end{cases}$$

Now,

$$\begin{aligned} (F \circ E)(a)(w) &= Vg^{-1}(g(Vaf^{-1}f(w))) \\ &= Vg^{-1}(g(Va(\{ w,x \}))) \\ &= Vg^{-1}(g(1)) \\ &= 1 \neq a(w). \end{aligned}$$

Therefore, $(F \circ E)(a) \neq a$. i.e., $F \circ E$ is not the identity function on L(X). Hence f must be one to one.

Suppose g is not one to one, then there exist an l in L such that $Vg^{-1}(g(l)) \neq l$. Let $\underline{l}$ be the constant fuzzy subset of X, with the membership value l. Then $(F \circ E)(\underline{l}) \neq \underline{l}$, and hence $F \circ E$ is not the identity function. Thus g must be one to one.

Sufficiency: Let $a \in L(X)$. Then for each x in X,

$$
\begin{aligned}
(F\circ E)(a)(x) &= \vee g^{-1}(E(a)(x)) \\
&= \vee g^{-1}(g(\vee a f^{-1}(f(x)))) \\
&= \vee g^{-1} g(a(x)) && \text{..f is one to one} \\
&= a(x) && \text{..g is one to one}
\end{aligned}
$$

Thus $F\circ E$ is the identity function on $L(X)$.

1.3.12 Definition

A subset S of a complete lattice is said to be upper complete if $\vee S$ belongs to S.

1.3.13 Theorem

$E\circ F$ is the identity function on $M(Y)$ if and only if

i) f is onto,

ii) g is onto, and

iii) the fibers of g are upper complete.

Proof

Necessity: i) Suppose f is not onto, then there exists an z in Y such that $f^{-1}(z) = \emptyset$. Consider $d = \mathrm{Char}(\{z\})$ in $M(Y)$. $d \neq 0$ but $F(d)(x) = \vee g^{-1} d f(x) = 0$ for all x in X. Therefore,

$$(E\circ F)(d)(y) = E(0)(y) = 0,$$

for all y in Y. Thus $(E\circ F)(d) \neq d$. Hence $E\circ F$ is not the identity function, and hence f must be onto.

ii) Suppose g is not onto, then there exists m in M such that $m \notin g(L)$. Then, for the constant fuzzy subset $\underline{m}$ in $M(Y)$, and for each x in X,

$$F(\underline{m})(x) = \vee g^{-1}\underline{m}f(x) = \vee g^{-1}(m) = 0.$$

Therefore, $F(\underline{m}) = 0$. Thus, for each y in Y,

$$\begin{aligned}(E\circ F)(\underline{m})(y) &= E(0)(y)\\ &= g(\vee 0 f^{-1}(y))\\ &= g(0).\end{aligned}$$

Since $g(0) \neq m$, $(E\circ F)(\underline{m}) \neq \underline{m}$. Thus $E\circ F$ is not the identity function. Hence g must be onto.

iii) Suppose there exist an m in M such that the fiber $g^{-1}(m)$ is not upper complete, then $\vee g^{-1}(m) = 1$(say) is not in $g^{-1}(m)$. i.e., $g(1) \neq m$. Now for the constant fuzzy subset $\underline{m}$ of $M(Y)$ and for each x in X,

$$F(\underline{m})(x) = \vee g^{-1}\underline{m}f(x) = \vee g^{-1}(m) = 1.$$

Hence for each y in Y,

$$(E\circ F)(\underline{m})(y) = g(\vee(f(\underline{m})(f^{-1}(y)))) = g(1) \neq m.$$

Thus $(E\circ F)(\underline{m}) \neq \underline{m}$. i.e., $E\circ F$ is not the identity function. Hence iii) is necessary.

Sufficiency: Let $d \in M(Y)$. Then for each y in Y,

$$
\begin{aligned}
(E\circ F)(d)(y) &= g(VF(d)f^{-1}(y)) \\
&= g(V(Vg^{-1}df(f^{-1}(y)))) \\
&= g(V(Vg^{-1}d(y))) && \text{.. f is onto} \\
&= g(l) && \text{where } l = Vg^{-1}d(y) \\
&= d(y). && \text{.. fibers of g is upper complete}
\end{aligned}
$$

Therefore, $E\circ F(d) = d$ for each d in $M(Y)$. i.e., $E\circ F$ is the identity function on $M(Y)$. The proof is complete.

1.3.14 Remark

E and F are inverses if and only of f and g are bijections, since fibers of g are upper complete when g is a bijection.

1.3.15 Note

A t-homomorphism $F: M(Y) \longrightarrow L(X)$ takes a fuzzy topology of Y to a fuzzy topology of X. Hence F induces a function $F': (M,Y) \longrightarrow (L,X)$ defined as $F'(T) = \{ F(t) \mid t \in T \}$, for T in (M,Y). Then the following observations are immediate.

i) $F'(0) = 0$.

ii) $F'(1) = 1$ iff F is onto.

iii) $F'(VT(i)) = VF'(T(i))$, where $\{ T(i) \mid i \text{ in } I \} \subset (M,Y)$.

iv) F′ is onto iff F is onto.

v) F′ is one to one iff F is one to one.

vi) F′ is a homomorphism iff $F^{-1}(d)$ is either a singleton or empty, for each d in M(Y)\{0,1}.

vii) F′ is an onto t-homomorphism iff F is a bijection.

Now we can state the following theorem.

1.3.16 Theorem

If f:X--->Y is a bijection, then the following are equivalent, for the induced functions:

i) g:L--->M is an onto t-isomorphism.

ii) F:M(Y)--->L(X) is an onto t-isomorphism.

iii) F′:(M,Y)--->(L,X) is an onto t-isomorphism.

1.4 Properties of H [1.3.2].

1.4.1 Theorem

i) H is one to one iff f is onto and h is one to one, and ii) H is onto iff f is one to one and h is onto.

Proof

i) **Necessary:** Suppose f is not onto, then there exists an z in Y such that z is not in f(X). Let d = Char({z}) in M(Y). Then d ≠ 0 and for any x in X,

$$H(d)(x) = hdf(x) = h(0),$$

$$\text{and } H(0)(x) = h0f(x) = h(0).$$

Thus $H(d) = H(0)$. Hence H is not one to one, a contradiction. Therefore, f must be onto.

Suppose h is not one to one, then there exist m,n in M such that $m \neq n$ and $h(m) = h(n)$. Let $\underline{m}$ and $\underline{n}$ be the constant fuzzy subsets in Y, taking the membership values m and n, respectively. Though $\underline{m}$ and $\underline{n}$ are different, for each x in X,

$$H(\underline{m})(x) = h\underline{m}f(x) = h(m),$$

$$\text{and } H(\underline{n})(x) = h\underline{n}f(x) = h(n).$$

Since $h(m) = h(n)$, $H(\underline{m}) = H(\underline{n})$, a contradiction. Therefore, h must be one to one.

Sufficiency: Let $c,d \in M(Y)$ be such that $H(c) = H(d)$. i.e., for each x in X,

$$H(c)(x) = H(d)(x).$$

$$\text{i.e., } hcf(x) = hdf(x).$$

This implies, $cf(x) = df(x)$ ..h is one to one

Hence $c = d$, as f is onto. Thus H is one to one.

ii) **Necessary:** Suppose f is not one to one, then there exist w,x in X such that $w \neq x$ and $f(w) = f(x)$. Let a in $L(X)$ be the Characteristic function on $\{w\}$. Then $a(w) \neq a(x)$. However, for any d in $M(Y)$,

$$H(d)(w) = hdf(w) = hdf(x) = H(d)(x).$$

Thus, a cannot be an image under H. Hence H is not onto, a contradiction. Therefore, f must be one to one.

Suppose h is not onto, then there exist an l in L\h(M) and the constant fuzzy subset $\underline{l}$ of X, cannot be an image under H. Thus H is not onto, a contradiction. Hence h must be onto.

Sufficiency: Let a $\in$ L(X). Then $h^{-1}(a(x)) \neq \emptyset$, for all x in X, since h is onto. Let l(x) be a representative element, from $h^{-1}(a(x))$ for each x in X. Now define $d: Y \longrightarrow M(Y)$ as follows: for y in Y,

$$d(y) = \begin{cases} l(x) & \text{if } f(x) = y \\ 0 & \text{otherwise.} \end{cases}$$

Clearly d $\in$ M(Y) and for every x in X,

$$H(d)(x) = hdf(x) = h(l(x)) = a(x).$$

i.e., H(d) = a. Hence, H is onto. The proof is complete.

1.4.2 Theorem

H is lattice homomorphism if and only if h is so.

Proof

Necessary: Suppose h is not a homomorphism, then there exist m,n in M such that either

i) $h(m \vee n) \neq h(m) \vee h(n)$, or

ii) $h(m \wedge n) \neq h(m) \wedge h(n)$.

Let $\underline{m}$ and $\underline{n}$ be the constant fuzzy subsets of Y, taking the membership values m and n, respectively. We have that for each x in X,

$$H(\underline{m})(x) = h\underline{m}f(x) = h(m),$$

$$H(\underline{n})(x) = h\underline{n}f(x) = h(n),$$

$$H(\underline{m} \vee \underline{n})(x) = h(\underline{m} \vee \underline{n})f(x) = h(m \vee n),$$

and
$$H(\underline{m} \wedge \underline{n})(x) = h(\underline{m} \wedge \underline{n})f(x) = h(m \wedge n).$$

Thus in case i)

$$H(\underline{m} \vee \underline{n}) \neq H(\underline{m}) \vee H(\underline{n}),$$

and in case ii)

$$H(\underline{m} \wedge \underline{n}) \neq H(\underline{m}) \wedge H(\underline{n}).$$

Therefore, in either case, H is not a homomorphism, which is a contradiction. Hence h must be a homomorphism.

Sufficiency: Let h be a homomorphism and $c,d \in M(Y)$. Then for each x in X,

$$\begin{aligned} H(c \vee d)(x) &= h(c \vee d)f(x) \\ &= h(cf(x) \vee df(x)) \\ &= h(cf(x)) \vee h(df(x) \quad \text{.. h is a homomophism} \\ &= H(c)(x) \vee H(d)(x). \end{aligned}$$

Similarly, we can show that,for each x in X,

$$H(c \wedge d)(x) = H(c)(x) \wedge H(d)(x).$$

Thus H is a homomorphism.

1.4.3 Theorem

H is a t-homomorphism iff h is so.

Proof

Necessary: Since, $H(0)(x) = h(0)$ and $H(1)(x) = h(1)$, for all x in X,

$$H(0) = 0 \text{ and } H(1) = 1$$

imply that

$$h(0) = 0 \text{ and } h(1) = 1.$$

From the theorem (1.4.2) h must be a homomorphism. It remains to prove that h preserves arbitrary join operation.

Suppose h does not preserve arbitrary join operation, then there exists a subset $\{ m(i) \mid i \text{ in } I \}$ of M such that $h(\vee m(i)) \neq \vee h(m(i))$. But then for the constant fuzzy subsets $\underline{m}(i)$, for i in I, $H(\vee \underline{m}(i))(x) \neq \vee H(\underline{m}(i))(x)$, for each x in X. Thus H does not preserve arbitrary join operation, a contradiction. Hence h must preserve arbitrary join operation.

Sufficiency: Let h be a t-homomorphism. Then for each x in X,

i) $$H(0)(x) = h0f(x) = h(0) = 0,$$
$$H(1)(x) = h1f(x) = h(1) = 1.$$

ii) for d(i) in M(Y),

$$H(Vd(i))(x) = h(Vd(i))f(x)$$
$$= h(Vd(i)f(x))$$
$$= V(hd(i)f(x)) \quad .. \text{ h is a t-homomorphism}$$
$$= VH(d(i))(x).$$

From i), ii) and the theorem (1.4.2), H is a t-homomorphism. The proof is complete.

We state the following theorem without proof.

1.4.4 Theorem

i) H is a t-isomorphism iff h is a t-isomorphism and f is onto.

ii) H is an onto t-isomorphism iff h is an onto t-isomorphism and f is a bijection.

1.4.5 Observation

Taking L = M and h:M--->L to be the identity function, we have that, corresponding to every function f:X--->Y, there exist a function H:M(Y)--->L(X) such that

i) H is one to one iff f is onto,

ii) H is onto iff f is one to one,

and iii) H is a t-homomorphism for any f.

Taking X = Y and f:X--->Y to be the identity function, we have that , corresponding to every function h:M--->L, there exists a function $H:M(Y) \longrightarrow L(X)$ such that

i) H is one to one iff h is one to one,

ii) H is onto iff h is onto,

iii) H is a lattice homomorphism iff h is a lattice homomorphism, and

iv) H is a t-homomorphism iff h is a t-homomorphism.

1.4.6 Note

The image of a fuzzy topology on Y, under a t-homomorphism $H:M(Y) \longrightarrow L(X)$, is a fuzzy topology on X. Hence H in this case, induces a function $H':(M,Y) \longrightarrow (L,X)$ such that $H'(U) = \{ H(u) \mid u \text{ in } U \}$, for U in (M,Y).

The following observations on H'are immediate.

i) $H'(0) = 0$ always,

ii) $H'(1) = 1$ iff H is onto,

and iii) H' preserves arbitrary join operation.

The proof of the follwing theorem is on the same line of proof of the theorem (1.2.11), hence we state it without proof.

1.4.7 Theorem

If $f:X \longrightarrow Y$ is a bijection, then the following are equivalent for the induced functions.

i) $h:M \longrightarrow L$ is an onto t-isomorphism

ii) $H:M(Y) \longrightarrow L(X)$ is an onto t-isomorphism

iii) $H':(M,Y) \longrightarrow (L,X)$ is an onto t-isomorphism.

CHAPTER II

SOME LATTICE PROPERTIES

In this chapter a complete lattice L is taken and a special class of subsets of it, called t-irreducible subsets are introduced and studied. These subsets play a vital role in the study of the lattice of fuzzy topologies on a fixed set. In Chapter IV, it is shown that the existence of minimal t-irreducible subsets in the membership lattice is a necessary and sufficient condition for the lattice of fuzzy topologies on a set to hsve dual atoms. t-irreducible subsets in the Boolean lattice of all subsets of a set are characterised.

Through out this chapter, L generally denote a complete lattice and I is used as an arbitrary index set, with i as its general member. 0 and 1 denote the least and the largest element in L. For l in L, l′ denotes the complement of l.

2.1 t-irreducible subsets

2.1.1 Definitions

A nonempty subset R, not containing 0, of L is said to be t-irreducible if no element of R can be written as the finite meet or arbitrary join of members of L\R.

If further, no proper subset of R is t-irreducible then R is said to be minimal t-irreducible.

A nonzero element l in L is said to be a t-irreducible element if { l } is a t-irreducible subset of L.

An element a in L is called an atom if 0 is the only element in L smaller than l, and is called a dual atom if 1 is the only element in L greater than 0.

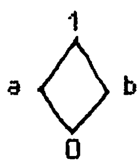
A lattice is said to be atomic if every element in it is the join of some atoms in it, and it is called dually atomic if every element in L, can be writen as the meet of dual atoms above it.

2.1.2 Note

t-irreducible elements are meet and join irreducible elements in the language of GRATZER [14].

2.1.3 Examples

In L(1) = { 0,1 }, 1 is a t-irreducible element.

In L(2) = 

, { a,b }, { a }, { b }, { b,1 } and { a,1} are the t-irreducible subsets, and { a }, { b } are the minimal t-irreducible subsets; a and b are the t-irreducible elements.

2.1.4 Theorem

If 1 belongs to a minimal t-irreducible subset R, then R = {1}. i.e., 1 is a t-irreducible element.

Proof

Let R be a minimum t-irreducible subset containing 1. Suppose R ≠ { 1 }. Let S= R\{ 1 }. Then we shall show that S is t-irreducible, which is a contradiction to the assumption that R is minimal, and thus completing the proof. Let {a(i)| i in I} and {b(1),b(2),...,b(n)} be two arbitrary subset of L\S and let a = Va(i) and b = b(1)∧b(2)∧....∧b(n). Now if for some i, a(i) = 1 then a = 1 and a ∉ S, and if a(i) ≠ 1 for all i, then a(i) ∈ L\R for all i in I, and so a ∉ R, hence a ∉ S. Considering {b(1),.. b(n)}, if b(j) =1, for all j, then b = 1 and hence b ∉ S, and if there exist b(j), not equal to 1, then{b(j)| b(j) ≠ 1 & j = 1,.,n} is a subset of L\R and b = ∧{ b(j)|b(j) ≠ 1 & j = 1,..,n }. Hence b ∉ R and therefore, b ∉ S. Altogether we have proved that S is t-irreducible.

2.1.5 Theorem

In L an element which is both an atom and a dual atom is a t-irreducible element.

Proof

Let l be an atom and a dual atom in L. Let if possible l= Vl(i), for l(i) in L, i in I, and l(i) $\neq$ l. But then l(i) $<$ l for all i, and hence l(i) = 0, as l is an atom. Therefore, l $\neq$ Vl(i), a contradiction. Now suppose l(1)$\wedge$ l(2)$\wedge$... $\wedge$l(n) and l(j) $\neq$ l for j = 1,2,....,n. Then l(j) $>$ l for all i, hence l(j) = 1 for all j, as l is a dual atom. Therefore, l $\neq$ l(1) $\wedge$ l(2) $\wedge$...$\wedge$l(n), a contradiction. Thus l is a t-irreducible element.

2.1.6 Lemma

If 1 of L is t-irreducible then 1 has a unique immediate predecessor.

Proof

Let 1 be t-irreducible. Then 1 must have an immediate predecessor, for otherwise, 1 is the join of all elements, smaller than it, which is impossible as 1 is t-irreducible. We further claim that the immediate predecesssor is unique. for otherwise, 1 will be the join of these Immediate predecessors of 1, which is impossible.

2.1.7 Theorem

Let every nonzero element in L is contained in some minimal t-irreducible subsets of L. Then L is a chain of t-irreducible elements and 0.

Proof

Since every nonzero element of 1 belongs to some minimal t-irreducible subsets, 1 must be a t-irreducible element, by the theorem (2,1.4). But then by the lemma(2.1.6) 1 has a unique immediate predecessor: 1/2 (say). If 1/2 is not the 0 of L , then L\{1} is a complete sublattice of L, with minimal t-irreducible subsets, containing every nonzero element, hence by the same argument, 1/2 is t-irreducible and has an immediate predecessor 1/3 (say), and so on. If L is finite, clearly L is lattice isomorphic to the finite chain:

$$C(n) = \{ 1, 1/2, 1/3, ..., 1/n, 0 \},$$

under usual order, for some natural number, n. Suppose L is infinite then we can pick, t-irreducible elements 1/n, for n = 1,2,,, such that 1/(n+1) is the immediate predecessor of 1/n. Thus the infinite chain:

$$C = \{ 1, 1/2, 1/3,,,,, \},$$

under usual order, is a sublattice of L. If L\C is finite, then L is isomorphic to either C U { 0 } or C U C(n), for some natural number,n where each member of C is bigger than every member of C(n), and hence, L is a chain of t-irreducible elements and 0. But if L\C is infinite, L\C contains atleast one more copy of C. In this case, let

{ L(i)|i in I }be the set of all copies of C that could be found in L, by the same process, one after another. Thus if L(i) is found after L(j), for i,j in I, then every member of L(j) is greater than every member of L(i). Clearly D = U { L(i)| i in I }, is a chain. Then L\D is finite, for otherwise, there still exists another copy of C in L, which is not included in { L(i)|i in I }, a contradiction. Thus L\D is finite, therefore, L is either D U { 0 } or D U C(n), for some n, where every member of D is greater than every member of C(n). Thus L is a chain of t-irreducible elements and 0.

2.2 t-completion

2.2.1 Definitions

A subset E of L, is said to be t-complete if E is closed for finite meet and arbitrary join operations.

A t-complete subset of L containing a given subset is called its t-completion.

2.2.2 Theorem

Every subset of L has a t-completion.

Proof

Let E be a subset of L. Let ə be the set of all t-complete subsets of L containing E. Clearly ə is nonempty, as L is in ∂. Let $D = \wedge \{F \mid F \text{ is in } \partial\}$. Then E is contained in D and if d(1),d(2),,,d(n) are in D, then $d(j) \in F$, for all F in ∂ and for all j = 1,2,,,n. Since each F in ∂ is complete, $\wedge d(j) \in F$, and hence $\wedge d(j) \in D$. Similarly d(i) belongs to D for i in I, implies $\vee d(i) \in D$. Thus D is t-complete and D is the t-completion of E.

2.2.3 Note

Complement of every t-complete subset of a lattice is t-irreducible. Moreover if R is a minimal t-irreducible subset and $r \in R$, then the t-completion of $(L \setminus R) \cup \{r\}$ is L.

2.3 t-irreducible subsets in Boolean lattice

2.3.1 Theorem

Every t-irreducible element, not equal to 1 in a Boolean lattice is a dual atom and an atom.

Proof

Let L be a Boolean lattice and l in L be a t-irreducible element. Let l′ be the complement of l in L. Suppose $m \in L$ and $m > l$, then m′ exists and $m' \leq l'$, but $m' \neq l'$, since $m \neq l$ [complement in a Boolean lattice is unique]. Thus $m' < l'$ and hence $m' \wedge l = 0$. But then $m' \vee l \neq 1$, for otherwise, $m' = l'$. Therefore,

$$(m' \vee l) \wedge m = (m' \wedge m) \vee (l \wedge m) = l.$$

Since l is t-irreducible, $m' \vee l = l$, but then $m' \leq l < m$, which implies, $m = 1$ and $m' = 0$. Thus l is a dual atom.

Now consider an n in L such that $n < l$. Then the n' is such that $n' \geq l'$, but $n' \neq l'$, since complement is unique. Hence $n' > l'$. We have $n' \vee l = 1$ and hence $n' \wedge l \neq 0$, for otherwise, $n' = l'$. Then

$$(n' \wedge l) \vee n = (n' \vee n) \wedge (l \vee n) = 1 \wedge l = l.$$

Thus $n < l \leq n'$, which implies that $n = 0$ and $n' = 1$. Therefore, l is an atom in L. The proof of the theorem is complete.

2.3.2 Theorem

Let l be a t-irreducible element in a Boolean lattice L. If $l \neq 1$ then l' is also t-irreducible.

Proof

From the theorem (2.3.1), if l is a t-irreducible element, then l is an atom and a dual atom. In view of the theorem (2.1.5), it is enough to prove that l' is an atom and a dual atom. Suppose $m \in$ L and $m < l'$ then

$$l \wedge m \leq l \wedge l' = 0. \qquad \ldots(1).$$

Thus $(l \vee m) < 1$, since complements are unique. Then $l \vee m = l$, as l is dual atom. Thus $m \leq l$. Therefore, from (1), $m = l \wedge m \leq l \wedge l' = 0$. i.e., $m = 0$. Therefore, l' is an atom.

Let n in L be such that $n > l'$. We have $n \vee l = 1$ ..(2) Since complement is unique, $l \wedge n \neq 0$. Thus $0 < l \wedge n \leq l$. But then $l \wedge n = l$, as l is an atom. Thus $n \geq l$. and hence $n = 1$, follows from (2). Thus l' is a dual atom. This completes the proof.

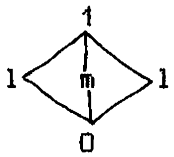
2.3.3 Theorem

In a Boolean lattice L, t-irreducible elements exist iff L = L(1) or L(2) of examples (2.1.3).

Proof

If L = L(1) or L(2) of example (2.1.3) then L has t-irreducible elements. Conversely, let L be a Boolean lattice with t-irreducible elements and $L \neq L(1)$. If $L \neq L(1) \neq \phi$ and L is a Boolean lattice then 1 is no more t-irreducible. Let l in L be a t-irreducible element. Then by the theorem(2.3.2), l' is also t-irreducible. But then by the theorem (2.3.1) l and l' are atoms and dual atoms in L. Then L(2) is lattice isomorphic to a sublattice of L. Suppose there exists m in L such that m is not in $\{0, l, l', 1\}$

then m must be incomparable with l and l′, as they are atoms and dual atoms. By the same reason, $l \wedge m = l' \wedge m = 0$ and

$l \vee m = l' \vee m = 1$. But then 

is isomorphic to a sublattice of L, which is a contradiction to the assumtion that L is a Boolean lattice (hence distributive). Hence there does not exists an m not in $\{0, l, l', 1\}$. Thus L(2) is isomorphic onto L. The proof is complete.

2.3.4 Theorem

Let X be any set with atleast two elements.Let $x, y \in X$, be such that $x \neq y$. Then,

$$R(x,y) = \{ A \quad X \mid x \in X \text{ and } y \notin A\}$$

is a minimal t-irreducible subset of the Boolean lattice: P(X) of all subsets of X.

Proof

Let $S = P(X) \setminus R(x,y)$, and $\{A(i) \mid i \text{ in } I\} \subset S$ and $\{B(j) \mid j = 1,2,,,n\}$, be arbitrary subsets of S. Define $A = \cup \{A(i) \mid i \text{ in } I\}$ and $B = \wedge \{B(j) \mid j = 1,2,,,n\}$. Since $A(i) \in S$, $A(i) \notin R(x,y)$ and hence for each i, either $x \notin A(i)$ or $\{x,y\} \subset A(i)$. If $x \notin A(i)$, for all i in I then $x \notin A$ and hence $A \notin R(x,y)$. And if $\{x,y\} \subset A(i)$ for some i, then $\{x,y\} \subset A$, and hence $A \notin R(x,y)$. Thus arbitrary join of members of S doesnot belong to R(x,y).

B(j)'s belong to S implies $B(j) \notin R(x,y)$, for $j = 1,..,n$. Then for each j, either $x \notin B(j)$ or $\{x,y\} \subset B(j)$. If for some j, $x \notin B(j)$ then $x \notin B$, and hence $B \notin R(x,y)$. And if $\{x,y\} \subset B(j)$ for all j, then $\{x,y\} \subset B$, and hence $B \notin R(x,y)$. Thus finite meet of members of S does not belong to R(x,y). Therefore, R(x,y) is a t-irreducible subset of P(X).

It remains to show that R(x,y) is minimal. If R(x,y) is a singleton set [only when $X = \{x,y\}$], then R(x,y) is clearly a minimal t-irreducible subset. Suppose R(x,y) is not a singleton set, let Q be a proper subset of R(x,y), such that Q is t-irreducible. Let $T = P(X)\setminus Q$. There exists an A in T such that A is in R(x,y) also. Then $y \notin A$. Let $E = A' \cup \{ x \}$. Then $x,y \in E$, and hence E is in S and so in T,also. $E \wedge A = \{ x \}$. Since Q is t-irreducible, $\{ x \}$ is in T. Thus T contains all the singletons, as S contains all singletons except $\{x\}$. Thus every subset of X can be written as join of members of T, and hence Q cannot be t-irreducible, a contradiction.

Therefore, R(x,y) is minimal t-irreducible subset of P(X).

2.3.5 Theorem

If X contains atleast two points then a subset R of P(X) is minimal t-irreducible if and only if R′ is an ultratopology on X.

Proof

Necessary: Let R be a minimal t-irreducible subset of P(X). Since,X contains atleast two points, P(X) is not lattice isomorphic to L(1) of example (2.1.3). Hence X, the largest element in P(X) is not t-irreducible. By the theorem (2.1.4), X is not in R. Let S = P(X)\R. Clearly, ∅,X ∈ S, and by the note (2.2.3), S being the complement of a t-irreducible subset, is t-complete, i.e., S is closed for finite meet (intersection) and arbitrary join (union) operations. Therefore, S is a topology on X. Further, let T be a topology on X, finer than S. T, being topology, is closed for finite intersection (meet) and arbitrary union (join operations), and hence T′ is t-irreducible. But then T′ is a subset of R and R is minimal t-irreducible, hence T′= R or T′= ∅, equivalently, T = S or T = P(X). Since T is an arbitrary topology containing S, S is an ultratopology on X.

Sufficiency: Let U be an ultratopology and R = P(X)\U. Then clearly, R is t-irreducible. Suppose R is not minimal t-irreducible then there exists a proper subset Q of R such that Q is t-irreducible. But then Q′ is a proper topology strictly finer than U, a contradiction to the assumption that U is an ultratopology. Therefore, R is minimal t-irreducible.

2.3.6 Note

The complement of R(x,y) in the theorem (2.3.4) is the ultratopology: T = P(X\{x}) U F(y), where F(y) is the principal ultrafilter at y. Since , every ultratopology is not of the form given above [FROLICH, 11], every minimal t-irreducible subset of P(X) is not of the form R(x,y), for some x,y in X. However, if X is finite, then every ultrafilter on X is a principal ultrafilter, and hence every minimal t-irreducible subset is of the form R(x,y), for some x,y in X.

CHAPTER III

FUZZY FILTERS AND ULTRAFUZZY FILTERS

Let L be a complete and distributive lattice, and X be any set. In this chapter fuzzy filters on X are defined on the lines of definition given by A.K.KATSARAS [16] and P.SRIVASTAVA and R.L.GUPTA [23], by taking L to be the membership set, instead of the closed unit interval [0,1]. Ultrafuzzy filters are defined and characterized in terms of properties of the membership lattice. Study is extended to the case when the membership lattice is further, complemented as well.

We denote the complement of an element a by a', and I is commonly used to denote an arbitrary index set with i denoting a general element in it.

3.1 Definitions:

A nonempty subset F of L(X) is said to be a fuzzy filter if

i) $0 \notin F$

ii) $a, b \in F$ implies $a \wedge b \in F$

and iii) $a \in F$, $b \in L(X)$ and $b \geq a$, imply $b \in F$.

A nonempty subset B of L(X) is said to be a fuzzy filterbase if i) $0 \notin B$, and ii) $a,b \in B$ implies that there exists c in B such that $c \leq a \wedge b$.

A subset B of L(X) is said to be a base for a fuzzy filter F, if $F = \{a \in L(X) \mid a \geq b$, for some b in B$\}$.

Let $x \in X$ and $l \in L$, be such that $l \neq 0$. Let s(x,l), denote the fuzzy subset defined by, for y in X,

$$s(x,l) = \begin{cases} 0 & \text{if } y \neq x \\ 1 & \text{if } y = x. \end{cases}$$

s(x,l) is called the fuzzy singleton at x.

A fuzzy filter is said to be an ultrafuzzy filter if it is not properly contained in any other fuzzy filter.

Let a be a nonzero fuzzy subset of X. Then the subset P(a) of L(X) defined by $P(a) = \{b \in L(X) \mid b \geq a\}$, is a fuzzy filter on X, called the principal fuzzy filter at a.

3.2 Existence of ultrafuzzy filters

3.2.1 Theorem

Every fuzzy filterbase B, determines a fuzzy filter F, uniquely such that B is a base for the fuzzy filter F.

Proof

Let B be a fuzzy filterbase and

$F = \{a \in L(X) | a \geq b$, for some b in B$\}$.

Clearly, F is nonempty and $0 \notin F$, as B is a subset of F and $0 \notin B$. Further, if $a, b \in F$, then there exists c,d in B such that $c \leq d$ and $d \leq b$. But then there exists an e in B such that $e \leq c \wedge d$. Thus $e \leq c \wedge d \leq a \wedge b$, and therefore, $a \wedge b$ belongs to F.

Trivially, for every a in F, if b in L(X) is such that $b \geq a$, then $b \in F$. Thus F is a fuzzy filter and B is a base for F. Uniqueness is immediate from the definition of F.

3.2.2 Theorem

Every fuzzy filter is contained in an ultrafuzzy filter.

Proof

Let F be a fuzzy filter and ∂ be the set of all fuzzy filters containing F. ∂ is nonempty, as F belongs to it. ∂ is partially ordered under set inclusion. Let $\{U(i) | i$ in $I\}$ be a chain in ∂. Let $U = \bigcup U(i)$. We claim that U is an upper bound for the chain.

i) $0 \notin U$, since $0 \notin U(i)$, for each i.

ii) If $a,b \in U$, then $a \in U(i)$ and $b \in U(j)$, for some i,j in I. Then either $U(i) \subset U(j)$ or $U(j) \subset U(i)$, since U(i), U(j) belong to a chain. Let us assume that $U(i) \subset U(j)$. Thus $a,b \in U(j)$, and hence $a \wedge b \in U(j) \subset U$.

iii) Let $a \in U$ and b in L(X) be such that $b \geq a$. Then a is in U(i) for some i in I. But then b belongs to that U(i) and hence $b \in U$.

From i),ii) and iii), U is a fuzzy filter. Clearly U is bigger than U(i) for each i in I and U contains F. Hence U is an upper bound for the chain. Thus we have proved that every chain in ∂ has an upper bound, hence by Zorn's lemma, ∂ contains maximal elements. i.e., there exist ultrafuzzy filters containing F.

3.2.3 Theorem

A fuzzy filter U is an ultrafuzzy filter iff a in L(X) and $a \wedge u \neq 0$, for all u in U, imply that a is in U.

Proof

Necessay: Let U be an ultrafuzzy filter and a in L(X) be such that $a \wedge u \neq 0$ for all u in U. Let $B = \{ a \wedge u \mid u \in U \}$. Now B is nonempty, $0 \notin B$ and B is closed for meet operation, since, for u,v in U, $u \wedge v$ is in U and $(a \wedge u) \wedge (a \wedge v) = a \wedge (u \wedge v)$.

Thus B is a fuzzy filterbase and contains a, since 1 is in U. Hence by the theorem (3.2.1), there exists a fuzzy filter F containing B. But then U is a subset of F, since for every u in U, u is $\geq$ a $\wedge$ u. U being an ultrafuzzy filter, this implies that U = F. Thus a is in U.

Sufficiency: Let U be a fuzzy filter containing all a in L(X) such that a $\wedge$ u $\neq$ 0, for all u in U. Suppose F is a fuzzy filter containing U. Then for every f $\in$ F, f$\wedge$u $\neq$ 0, for all u in U, since u $\in$ F, also. But then by the hypothesis, f is in U, for every f in F. Thus F is a subset of U and Therefore, F = U. Since, F is arbitrary, U is an ultrafuzzy filter. This completes the proof.

3.2.4 Theorem

If F is a fuzzy filter such that for all a in L(X), either a or a′ (if exists), belongs to F, then F is an ultrafuzzy filter on X.

Proof

Suppose G is a fuzzy filter containing F. Let a $\in$ G. Then either i) a′ exists, or ii)a′ does not exists.

Case i) By hypothesis, either a is in F or a′ is in F. a′ cannot be in F, for otherwise, a′ will also belong to G, which is impossible, as a $\in$ G. Thus a is in F.

Case ii). By hypothesis, a belongs to F. Thus in either case, a $\in$ G implies a $\in$ F. Therefore, F contains G, and hence F = G. Since G is an arbitrary fuzzy filter containing F, F is an ultrafuzzy filter. The proof is complete.

3.2.5 Remark

If there exists a,b in L(X), such that a $\wedge$ b = 0 and a′and b′ do not exist, then no fuzzy filter on X can satisfy the hypothesis of the theorem (3.2.4). This would imply that the hypothesis is not a necessary condition for a fuzzy filter to be an ultrafuzzy filter.

3.2.6 Theorem

If L is not complemented and X contains atleast two elements, then no ultrafuzzy filter on X, satisfies the hypothesis of theorem (3.2.4).

Proof

Let 1 in L be such that 1′ doesnot exists. Let x,y (X be such that x $\neq$ y. Consider the fuzzy singletons s(x,1) and s(y,1), of X. clearly s(x,1) and s(y,1) do not have complements in L(X), and s(x,1) $\wedge$ s(y,1) = 0. Hence by the remark (3.2.5), we have the theorem.

3.2.7 Remark

Suppose L is not complemented and F is a fuzzy filter on X such that for all complemented elements a in L(X), either a or a′ is in F. This does not imply that F is an ultrafuzzy filter on X.

Let L = [0,1] and X be any set. Let, for a fixed z in X, F = {a $\in$ L(x)| a $\geq$ s(z,1)} for a fixed z in X. Then F is a fuzzy filter on X. Let a be a complemented fuzzy subset of X. Then a $\wedge$ a′ = 0 and a $\vee$ a′ = 1. i.e., for every x in X, a(x) $\wedge$ a′(x) = 0 and a(x)$\vee$a′(x) = 1. Thus for each x in X either a(x) = 1 and a′(x) = 0, or a(x) = 0 and a′(x) = 1. Thus, either a $\geq$ s(z,1) or a′ $\geq$ s(z,1). Therefore, either a or a′ is in F. But F is not an ultrafuzzy filter, since {a $\in$ L(x)| a $\geq$ s(z,0.5)} is a fuzzy filter bigger than F.

3.2.8 Theorem

Let F be an ultrafuzzy filter on X, and a in L(X) be a complemented fuzzy subset. Then either a $\in$ F or a′ $\in$ F.

Proof

Let a′ $\notin$ F. Then we claim that a $\wedge$ b $\neq$ 0 for all b in F. For if there exists b in F such that a $\wedge$ b = 0, then

$$b = b \wedge (a \vee a') = (b \wedge a) \vee (b \wedge a') = b \wedge a'.$$

Thus a′ $\geq$ b, and hence a′ $\in$ F, a contradiction. Hence the claim.

Let $B = \{a \wedge b \mid b \in F\}$. Then B is a fuzzy filterbase, containing a. Let U be the fuzzy filter generated by B (given by the theorem 3.2.1). Then clearly F is contained in U, and since F is an ultrafuzzy filter, F = U. Therefore, $a \in F$. Hence the theorem.

3.2.9 Theorem

Let U be an ultrafuzzy filter on X. Then for a,b in L(X), if $a \vee b \in U$, implies either $a \in U$ or $b \in U$.

Proof

Let a,b in L(X) be such that $a \vee b$ is in U. Let $a \notin U$. Then we claim that $b \wedge u \neq 0$, for all u in U. For otherwise, there exists some u in U such that $b \wedge u = 0$. But then $(a \vee b) \wedge u \in U$, would imply that

$$(a \vee b) \wedge u = (a \wedge u) \vee (b \wedge u) = a \wedge u \in U.$$

Thus a is in U, as $a \geq a \wedge u$, a contradiction. Therefore, $b \wedge u \neq 0$ for all u in U. Consider $B = \{ b \wedge u \mid u \in U \}$. B is a fuzzy filterbase containing b. Clearly the fuzzy filter W, generated by B contains U. Thus W = U, as U is an ultrafuzzy filter. Therefore, $b \in U$. Hence the theorem.

3.3 Principal fuzzy filter.

3.3.1 Therem

Let $a \in L(X)$. A principal fuzzy filter at a on X, is an ultrafuzzy filter iff a is a fuzzy singleton s(x,l), for some x in X, such that l is an atom in L.

Proof

Necessary: Let P be the principal fuzzy filter at a. Suppose P is an ultrafuzzy filter, then there exists a unique x in X such that $a(x) > 0$. for otherwise, the fuzzy singleton $s(x,a(x)) < a$, and hence the principal fuzzy filter at s(x,a(x)), will be bigger than P, a contradiction. Further a(x) must be an atom, for otherwise, there exists an m in L such that $0 < m < a(x)$. Then again the principal fuzzy filter at s(x,m) would be bigger than P, a contradiction. Thus a = s(x, a(x)), where a(x) is an atom.

Sufficiency: Let l be an atom in L and x be a fixed element in X. Let P be the principal fuzzy filter at s(x,l). Suppose, U is a fuzzy filter containg P. As $s(x,l) \in U$, for every u in U, $s(x,l) \wedge u = s(x, l \wedge u(x)) \in U$. Thus $l \wedge u(x) \neq 0$, therefore, $l \wedge u(x) = l$, since l is an atom. Thus $u(x) \geq l$. i.e., $u \geq s(x,l)$. Therefore, u is in P. Hence P = U. Since U is arbitrary, P is an ultrafuzzy filter. The proof of the theorem is complete.

3.3.2 Remark

If a principal fuzzy filter at a fuzzy singleton is not an ultrafuzzy filter then there exists a finer principal fuzzy filter. If in L, every nonzero element, either is an atom or has an atom below it, then every principal fuzzy filter is contained in a principal ultrafuzzy filter. Unlike in the ordinary set theory, principal fuzzy filters on fuzzy singletons are not maximal. One may note that principal fuzzy filter at every fuzzy singleton is an ultrafuzzy filter iff L = { 0,1 }.

3.4 Fuzzy filters with complemented membership lattice.

3.4.1 Theorem

If L is complemented, then for every fuzzy filter F on X, the following are equivalent.

i) F is an ultrafuzzy filter.

ii) For a in L(X), either a is in F or a' is in F.

iii) For a,b in L(X), if $a \vee b \in F$ implies $a \in F$ or $b \in F$.

Proof

i) implies ii): Suppose F is an ultrafuzzy filter, $a \in L(X)$ and $a' \notin F$. Then $a \wedge b \neq 0$, for all $b \notin F$, for otherwise, let $a \wedge b = 0$ for some b in F. Then $b = b \wedge (a \vee a') = (b \wedge a) \vee (b \wedge a')$; $(b \wedge a') = b \wedge a'$. Which implies $a' \geq b$. Hence a' is in F, a contradiction. Thus $a \wedge b \neq 0$ for all b in F. Hence by the theorem (3.2.3), $a \in F$. i.e.,. for all a in L(X) either $a \in F$ or $a' \in F$.

ii) implies iii): Let $a, b \in L(X)$, be such that $a \vee b$ is in F. Suppose $a \notin F$. Then by ii) a' belongs to F. Thus a' and $a \vee b$ are in F. Hence, $a' \wedge (a \vee b) = a' \wedge b \in F$. Thus $b \in F$, since $b \geq b \wedge a'$. Therefore, $a \vee b \in F$ implies either a or b, belongs to F.

iii) implies i): Let F be a fuzzy filter on X satisfying iii) Suppose G is a ultrafuzzy filter containing F. Then, for a in G, since $a \vee a' = 1$, and $1 \in F$, by iii), either $a \in F$ or $a' \in F$. But a' cannot belong to F, since $a \in G$, and F is a subset G. Thus $a \in F$. Therefore, G is subset of F. Hence F is an ultrafuzzy filter.

CHAPTER IV

LATTICE OF FUZZY TOPOLOGIES

4.1 Introduction

Let L be a complete and distributive lattice, and X be any set. We know that the collection of all fuzzy subsets of X, denoted by L(X), is a complete and distributive lattice. The smallest and the largest element in any lattice is commonly denoted by 0 and 1, respectively. We take the definition of fuzzy topology as given by C.L.CHANG [9], but an arbitrary, complete and distributive lattice L, replaces the membership set [0,1]. FROLICH [11] proved that ulratopologies exists and they are of the form $P(X\setminus\{x\}) \cup F$, where x is a fixed point in X and F is an ultrafilter on X, not containing {x}. Theorem (4.3.4), given below, leads to a generalization of FROLICH's result. FROLICH [11] also proved that the lattice of topologies on a set is dually atomic. We, however prove that the lattice of fuzzy topologies on a set, is not dually atomic, if L is different from {0,1}.

JURIS HARTMANIS [15] proved that the lattice of topologies on a finite set is complemented, and raised the question about complementation in the lattice of topologies on an arbitrary set. This problem was solved affirmatively by A.K.STEINER [24] and VAN ROOIJ [28], independently. We prove that the lattice of fuzzy topologies on a set, in general is not complemented.

In this chapter we discuss about atoms, dual atoms and complements in the lattice of all fuzzy topologies: (L,X). The smallest element in (L,X) is { 0,1 }, and the largest element is L(X).

We denote an arbitrary index set by I and i as its general member. For an element a in any lattice, a′ denote the complement of a.

4.1.1 Definitions

A subset T of L(X) is called a fuzzy topology [cf. C.L.CHANG, 9] if

i) $0,1 \in T$,

ii) $a,b \in T$ implies $a \wedge b \in T$,

and iii) $a(i) \in T$, for i in I, implies $\vee a(i) \in T$.

A subset B of L(X) is called fuzzy toplogical base, if B is closed for finite meet operation and $\vee\{b \mid b \in B\} = 1$.

A subset B of L(X) is called a base for a fuzzy toplogy T, if every member of T, is the join of some members of B.

4.2 Properties of (L,X)

4.2.1 Theorem

The lattice of fuzzy topologies (L,X) on a set X, is atomic.

Proof

The atoms in (L,X) are of the form T(a) = {0,a,1}, for a in L(X)\{0,1}. Let S be an arbitrary fuzzy topology on X, such that S $\neq \emptyset$. Then S = V{T(a)| a $\in$ S\{0,1}}. Thus (L,X) is atomic.

4.2.2 Remark

Since L is complete, {0,1}, is a sublattice of L. Thus the set of all characteristic functions on the subsets of X, is a sublattice of L(X). Therefore, if the subsets of X are identified by their characteristic functions, then every subset of X is a fuzzy subset of X, and every toplogy on X is a fuzzy topology on X. Thus the lattice of topologies on X is lattice isomorphic to a sublattice of the lattice of fuzzy topologies on X. R.VAIDYANATHASWAMY [27], and A.K.STEINER [24], showed that the latice of toplogies in general, is not distributive.

Hence (L,X) is not distributive.

4.2.3 Theorem

The lattice of fuzzy topologies (L,X), on a set X is complete.

Proof

We shall show that arbitrary meet of fuzzy topologies on X is a fuzzy topology. Then by the theorem (3) [BIRKHOFF, 3, p.112], (L,X) is complete, as 1 $\in$ (L,X). Let {T(i)| i in I}, be an arbitrary subset of (L,X). Let T = $\bigwedge$ {T(i)| i in I}. Since 0,1 $\in$ T(i), for each i, 0,1 $\in$ T also. If a,b are in B then a,b are in T(i), for every i in I. Then for each i, a $\wedge$ b $\in$ T(i). Thus a $\wedge$ b $\in$ T. Further, if { a(j) | j in J } is an arbitrary subset of T, then {a(j)| j in J} is a subset of T(i) for all i. Thus V{a(j)| j in J} belongs to T(i), for each i. Therefore, V{a(j)| j in J} belongs to T. Thus T is a fuzzy topology on X. Clearly, T = $\wedge$ {T(i)| i in I}. Thus (L,X) is closed for arbitrary meet operation. Hence the theorem.

4.3 Dual atoms in (L,X)

4.3.1 Definition

Dual atoms in the lattice of fuzzy topologies are designated as ultrafuzzy topologies. Equivalently, U $\in$ (L,X), is an ultrafuzzy topology iff for any T in (L,X), U$\subset$T implies T = U.

4.3.2 Theorem

Let U be an ultrafuzzy topology on X. Then
i) atleast one fuzzy singleton does not belong to U,
ii) if two fuzzy singletons: s(x,l), s(y,m) donot belong to U, then x = y,
and iii) the set of all l in L, such that $s(x,l) \notin U$, for some x in X, is a minimal t-irreducible subset [2.1.1] of L.

Proof

i) Suppose all fuzzy singletons belong to U. Since for a in L(X), a = V{s(x,a(x))| x in X},and so $a \in U$. Thus U = L(X), a contradiction. Therefore, atleast one fuzzy singleton doesnot belong to U.

ii) Let if possible s(x,l) and s(y,m) donot belong to U, such that $x \neq y$. Let S = { 0,s(y,m),1 }. Then S is in (L,X). Let T = U V S. Then a base for T is given by $\{ u \wedge s \mid u \in U, s \in S \} = U \cup \{ s(y,m \wedge u(y)) \mid u \in U \}$. Clearly s(x,l) doesnot belong to T, since $x \neq y$, but $U \leq T$, a contradiction to the assumption that U is an ultrafuzzy topology. Hence x = y, if s(x,l) and s(y,m) donot belong to U.

iii) Let $R = \{l \in L \mid s(x,l) \notin U\}$. From i) we have that R is nonempty, and uniqueness of x, follows from ii). Now suppose

$r(i) \notin R$, then $s(x,r(i)) \in U$, for i in I. Thus if $\vee r(i) = r$, then $\vee s(x,r(i)) = s(x,r) \in U$, and hence $r \notin R$. Therefore, arbitrary join of members of cmplement of R, does not belong to R. Further if $r(j) \notin R$, for $j = 1,2,,,,,..,n$, then $s(x, r(j))$ belongs to U, for all j. Then if $r = r(1)\wedge...\wedge r(n)$ then $\wedge s(x,r(j)) = s(x,r) \in U$, and hence $r \notin R$. Thus, finite meets of members of the complement of R, do not belong to R. Therefore, R is t-irreducible. It remains to show that R is minimal.

Let Q be a subset of R such that $Q \neq R$, and Q is t-irreducible, then if $r \in R\backslash Q$ and if T is the smallest fuzzy topology containing U and $s(x,r)$, then $s(x,q) \notin T$ for all q in Q, but $U \subset T$. Thus U is not an ultrafuzzy toplogy, a contradiction. Therefore, R is minimal t-irreducible.

4.3.3 Remark

Corresponding to every ultrafuzzy topologu U on X, there exists a unique x in X and a minimal t-irreducible subset R of L such that $\{s(x,r) \mid r \text{ in } R\} \cap U = \emptyset$. Moreover in ordinary set topology, if U is an ultratopology on X then there exists a unique singleton $\{ x \} \notin U$., and the corresponding minimal t-irreducible subset is $\{ 1 \}$, as L in this case, is $\{0,1\}$.

4.3.4 Theorem

Let X be a set with atleast two points. Then, ultrafuzzy topologies on X, exist iff minimal t-irreducible subsets exist in the membership lattice.

Proof

Necessary: Suppose U is an ultrafuzzy topology, then by the theorem (4.3.2) there exists a unique x in X, such that $\{l \in L \mid s(x,l) \notin U\}$, is minimal t-irreducible subset in the membership lattice L. Hence the Necessity.

Sfficiency: Let R be a minimal t-irreducible subset of L. We have two cases: either 1 is in R or 1 is not in R.

Case i): 1 is in R. Then by the theorem (2.1.4) R = {1}. Let x be a fixed point in X and F be an ultrafuzzy filter [3.1.5] not containg s(x,1), equivalently, let F be an ultrafuzzy filter containing the lattice complement of s(x,1) in L(X). Define $U(x,1) = \{ a \text{ in } L(X) \mid a(x) \neq 1 \}$,and U = U(x,1) U F. We shall show that U is an ultrafuzzy topology.

i) clearly $0,1 \in U$

ii) Let $a,b \in U$. If $a, b \in U(x,1)$ then $a(x) \wedge b(x) \neq 1$ and hence $a \wedge b \in U(x,1)$; if $a,b \in F$, then $a \wedge b \in F$, and if $a \in U(x,1)$ and $b \in F$, then $a(x) \wedge b(x) \neq 1$, as $a(x) \neq 1$, hence

$a \wedge b$ belongs to U(x,1). Thus, if a,b $\in$ U, then $a \wedge b \in$ U.

iii) Let {a(i)| i in I} be an arbitrary subset of U. Suppose there exists some k $\in$ I such that a(k) is in F, then $\vee a(i)$ is in F, since $\vee a(i) \geq a(k)$. And if, a(i) is in U(x,1) for all i, then $a(i)(x) \neq 1$, for all i. Then $\vee a(i)(x) \neq 1$ as 1 is t-irreducible. Therefore, $\vee a(i) \in$ U(x,1). Thus in all cases, $\vee a(i)$ belongs to U.

From i),ii), and iii) U is a fuzzy topology.

To prove that U is an ultrafuzzy topology, let T be a fuzzy topology containing U. If T $\neq$ U, then there exists a in T such that a is not in U. Thus a $\notin$ U(x,1) and a $\notin$ F. Therefore, a(x) = 1 and there exists b in F, by the theorem (3.2.3), such that $a \wedge b = 0$. Now if $d = b \vee s(x,1)$, then $d \geq b$, and hence d is in F. Thus a and d, belong to T, and so $a \wedge d$ belongs to T. But

$$a \wedge d = a \wedge (b \vee s(x,1)) = (a \wedge b) \vee (a \wedge s(x,1)) = s(x,1).$$

Thus, s(x,1) belongs to T. Since U(x,1) contains all fuzzy singletons, except s(x,1), T contains all fuzzy singleton of X. But then from the proof of the theorem (4.3.2)(i), T = L(X). Thus U is an ultrafuzzy topology on X.

Case ii) 1 $\notin$ R. Let U(x,R) = {a $\in$ L(X)| a(x) $\notin$ R}, where x is a fixed point of X. We shall first show that U(x,R) is a fuzzy topology on X.

i) Clearly $0,1 \in U(x,R)$.

ii) If a,b are in $U(x,R)$, i.e., $a(x)$, $b(x) \notin R$, then $a(x) \wedge b(x) \notin R$, since R is t-irreducible. Thus $a \wedge b \in U(x,R)$.

iii) If $a(i) \in U(x,r)$, for i in I, then $a(i)(x) \notin R$, for all i. Therefore, $\vee a(i)(x) \notin R$, since R is t-irreducible. Thus $\vee a(i)$ belongs to $U(x,R)$.

From i),ii) and iii), $U(x,R)$ is a fuzzy topology.

Let T in (L,X) be such that T contains U and $T \neq U(x,R)$. Then there exists a in L(X) such that $a \notin U(x,R)$, and $a \in T$. But then $a(x) \in R$. Let $a(x) = r$. $S(x,1) \in U(x,R)$, $s(x,1)$ and a, are in T, and hence

$$s(x,1) \wedge a = s(x,a(x)) = s(x,r) \in T.$$

Let $Q = (L \backslash R) \cup \{ r \}$. Since R is minimal t-irreducible, from the note (2.2.3) the lattice completion of Q is L. Thus the lattice completion of $\{s(x,q) \mid q \text{ in } Q\}$ is $\{s(x,l) \mid l \text{ in } L\}$. thus T contains all fuzzy singletons, since $\{s(y,i) \mid y \text{ in } X, \ y \neq x, \ l \in L \backslash \{0\}\} \subset U(x,R)$, and $\{s(x,q) \mid q \text{ in } Q\} \subset T$. Therefore, $T = L(X)$, follows from the proof of the theorem (4.3.2)(i). Thus $U(x,R)$ is an ultrafuzzy topology, since T is arbitrary.

4.3.5 Remark

Dual atoms in (L,X) are of the form $U(x,1) \cup F$ or $U(x,R)$, for x in X, F is an ultrafuzzy filter not containing s(x,1), and R is a minimal t-rreducible subset of L. When L = {0,1}, the only minimal t-irreducible subset is { 1 }. Thus in this case, all dual atoms (ultratopologies) are of the form U(x,1) U F. But when L = { 0,1 }, U(x,1) corresponds to P(X\{x}) and F corresponds to an ultrafilter on X, not containing { x }, which is the point-ultrfilter characterisation of ultratopologies, due to FROLICH [11].

4.3.6 Theorem

(L,X) is dually atomic (2.1.1) iff Given any T in (L,X), such that $T \neq L(X)$, and $a \notin T$, there exists an ultrafuzzy topology U, such that $T \subset U$, and $a \notin U$.

Proof

Necessary: Let (L,X) be dually atomic. Let T in (L,X) and a in L(X) be such that $a \notin T$. Then there exists ultrafuzzy topologies U(i) in (L,X), for i in I, such that $T = \wedge U(i) = \bigcap U(i)$. Clearly, T is a subset of U(i) for all i in I, and since $a \notin T$, there exists i in I such that $a \notin U(i)$, hence the necessity.

Sufficiency: Let T in (L,X) be such that $T \neq L(X)$. For each a in L(X) such that a is not in T, let U(a) be an ultrafuzzy topology finer than T and not containing a. Then clearly

T = $\bigwedge$ { U(a) | a $\notin$ T }. Since T is arbitrary, (L,X) is dually atomic.

4.3.7 Lemma

If (L,X) is dually atomic then for each l in L, l $\neq$ 0, there exists a minimal t-irreducible subset, containing l.

Proof

Given any l in L, l $\neq$ 0, there always exists fuzzy topology not containing the fuzzy singleton s(x,l), for some fixed x in X [unless X is a singleton and L= {0,1}]. Then by the theorem (4.3.6), there exists an ultrafuzzy topology U on X such that s(x,l) $\notin$ U. Now by the theorem (4.3.2)(iii),

{m (L| s(y,m) $\notin$ U, for y in X}

is minimal t-irreducible. Clearly l belongs to this minimal t-irreducible subset, and the proof is complete.

4.3.8 Lemma

If (L,X) is dually atomic, then for every l,m in L\{0}, there exists minimal t-irreducible subset R(l,m), such that l is in R(l,m), and m $\notin$ R(l,m).

Proof

In the light of the theorem (4.3.6), and the lemma (4.3.7), to complete the proof, it is enough to prove that there exists a fuzzy topology containing s(x,m), and not containing s(x,1), for some fixed x in X.

Let T = {a ∈ L(X)| a(x) ∈ {0,m,1}, for x in X}. Then M = {0,m,1} is a complete, distributive sublattice of L. Further T = M(X). and M(X) is a complete sublattice of L(X). Thus T is a fuzzy topology on X. Clearly, s(x,m) is in T and s(x,1) ∉ T, for x in X. The proof is complete.

4.3.9 Lemma

If (L,X) is dually atomic then L is a chain of t-irreducible elements and 0.

Proof

If (L,X) is dually atomic, then by the lemma(4.3.7), every nonzero element of L is contained in some minimal t-irreducible subset of L. Then by the theorem (2.1.7) L is a chain of t-irreducible elements and 0.

4.3.10 Theorem

The lattice of fuzzy topologies (L,X) is dually atomic iff either L = {0,1} or X is a singleton and L is a chain of t-irreducible elements and zero.

Proof

Necessary: Suppose (L,X) is dually atomic. Then by the lemma

(4.3.9), L must be a chain of t-irreducible elements and 0. Suppose L $\neq$ {0,1} and X is not a singleton, then there exists l in L such that $0 < l < 1$. Let z be a fixed element in X. Let a in L(X) be such that, for x in X,

$$a(x) = \begin{cases} 1 & \text{if } x \neq z \\ l & \text{if } x = z. \end{cases}$$

Let T be the fuzzy topology of all constant fuzzy subsets of X. Since X is not a singleton, T $\neq$ L(X). Since L(X) is dually atomic, by the theorem (4.3.6), there exists an ultrafuzzy topology U bigger than T and not containing a, as a is not in T. Now, since support of a is X, and L is a chain, $a \wedge b \neq 0$, for all b in L(X)\{0}. Hence by the theorem (3.2.3) a is in every ultrafuzzy filter on X. Hence U cannot be of the form: U(x,1) U F as defined in the proof of the therem (4.3.4). Thus U = U(x,m), for some x in X, and m in L. But then the constant fuzzy subset $\underline{m}$, with membership value m on X, does not belong to U(x,m), and hence T is not a subset of U, a contradiction. Now by remark (4.3.5), T does not have an ultrafuzzy topology above it, and not containing a. Then by the theorem (4,3,6), (L,X) is not dually atomic. Thus, if (L,X) is dually atomic, then either X is a singleton or L = {0,1}.

Sufficiency: Suppose L = { 0,1 }. Then L(X) is lattice isomorphic onto to the Boolean lattice P(X), and (L,X) is lattice isomorphic to the lattice of all toplogies on X. The lattice of topologies is dually atomic [FROLICH,11]. Hence (L,X) is dually atomic.

Suppose X is a singleton, and L is a chain of t-irreducible elements and 0. Let $X = \{x\}$. It may be noted that s(x,1) in this case is the "1" of L(X). Hence every T in (L,X), contains s(x,1). Moreover, {l} is minimal t-irreducible for every l in L\{0}. Thus the ultrafuzzy topologies (dual atoms) in (L,X), are of the form

$$U(x,l) = \{a \in L(X) \mid a(x) \neq l\} = \{s(x,m) \mid m \neq l\} \cup \{0\},$$

for each nonzero l in L. Thus if $T \in (L,X)$, and $T \neq L(X)$ then $T = \bigwedge \{ U(x,l) \mid s(x,l) \notin T \}$. Therefore, (L,X) is dually atomic.

4.3.11 Remark

From the theorem (4.3.10), we see that (L,X) is dually atomic only in the trivial cases. i.e., $L = \{0,1\}$ and X is a singleton. Thus in general (L,X) is not dually atomic. This is a clear departure from the corresponding results for the lattice of topologies.

4.4 Complements in (L,X)

4.4.1 Lemma

Let $S,T \in (L,X)$. Then $B = \{s \wedge t \mid s \in S, t \in T\}$ is a base for the fuzzy topology $S \vee T$.

Proof

We shall first show that B is a fuzzy topological base. Clearly S and T are subsets of B, hence U{b! b in B} = 1 Further, if a, b $\in$ B, then there exist r,s in S and t, u in T, such that a = r$\wedge$t and b = s$\wedge$u, then a$\wedge$b = (r$\wedge$s) $\wedge$ (t$\wedge$u). Since r$\wedge$s $\in$ S and t$\wedge$u $\in$ T, a$\wedge$b is in B. Thus B is a fuzzy topological base.

Since S V T is the smallest fuzzy topology containing S and T, it is closed for finite meet of members from S and T. Hence B is a subset of S V T. Thus B generates S V T.

4.4.2 Theorem

If L is a chain and if (L,X) is complemented, then every l in L\{0,1} is a t-irrreducible elements.

Proof

Suppose there exists an l in L\{0,1} such that , l is not t-irreducible. Since L is a chain, l cannot be written as the finite meet of members other than l. Thus there exist l(i) in L, l(i) $\neq$ l, for i in I such that Vl(i) = l. ..(1)

Let S = { 0,$\underline{l}$,1 }, where $\underline{l}$ is the constant fuzzy subset of X, with the membership value l. S is a fuzzy topology on X. Since (L,X) is complemented, there exists a T in (L,X) such that $T \wedge S = \{0,1\}$, and $T \vee S = L(X)$. Thus all fuzzy singletons of X belong to $S \vee T$. Therefore, by the lemma (4.4.1) every fuzzy singleton can be written as arbitrary join of members of $B = \{s \wedge t \mid s \in S \text{ and } t \in T\}$.

Let s(x,m) be a fuzzy singleton of X, where $m < l$. We claim that s(x,m) is in T.

We have, either s(x,m) is in B or $s(x,m) = \vee a(i)$, for some a(i)'s in B.

Case i): $s(x,m) \in B$, implies there exist s in S and t in T such that $s(x,m) = s \wedge t$. Since s is 0 or $\underline{l}$ or 1, t is either s(x,n), where $l \wedge n = m$, or s(x,m). Since L is a chain, $l \wedge n = m$, implies $l = m$ or $n = m$. since $m < l$, $n = m$. i.e., $t = s(x,m) \in T$.

case ii) $s(x,m) = \vee a(i)$, for a(i) in B, implies $a(i) = s(x,n(i))$, for some n(i) in L. But then $\vee n(i) = m$. From case i), $s(x,n(i)) \in B$ implies $s(x,n(i)) \in T$. Then $\vee s(x,n(i)) = s(x,m)$ is in T, since T is a fuzzy topology.

Thus in either case, s(x,m) is in T. Hence the claim.

Now , from (1), we have that $l(i) < l$ for each i. Thus from the above argument s(x,l(i)) is in T, for each i in I and for each x in X. Since T is a fuzzy topology, $\underline{1} = \vee \{ s(x,1(i)) \mid x \text{ in } X \text{ and } i \text{ in } I \}$, belongs to T. Thus $T \wedge S$ is no more {0,1}, a contradiction. Hence 1 must be t-irreducible. The proof of the theorem is complete.

4.4.3 Remark

From the theorem, (4.4.2), in the fuzzy set topology with closed unit interval [0,1] as the membership lattice, the lattice of fuzzy topologies on a set is not complemented.

4.4.4 Theorem

If L is a lattice of t-irreducible elements and 0, then in (L,X), every atom has a dual atom as its complement and vice versa. [X contains atleast two elements].

Proof

Let T(a) = {0,a,1} be an atom in (L,X). If there exists x in X such that $0 < a(x) < 1$, then for such an x, consider the ultrafuzzy topology

$$U(x,a(x)) = \{ b \in L(X) \mid b(x) \neq a(x) \}.$$

Clearly, $a \notin U(x,a(x))$, and hence $T(a) \wedge U(x,a(x)) = \{0,1\}$. Further U(x,a(x)) contains all fuzzy singletons, except s(x,a(x)). Since s(x,1) is in U(x,a(x)), s(x,a(x)) is in

$T(a) \vee U(x,a(x))$, as $s(x,1) \wedge a = s(x,a(x))$. Therefore, $T(a) \vee U(x,a(x)) = L(X)$. i.e., $U(x,a(x))$ is a complement of $T(a)$.

Suppose for all x in $a(x) = 1$ or 0, then pick an x such that $a(x) = 1$ and y such that $a(y) = 0$. Such x,y exist as a is neither 0 nor 1. Let $U(x,1) = \{b \in L(X) \mid b(x) \neq 1\}$, and F be an ultrafuzzy filter containing $s(y,1)$. Let $U = U(x,1) \cup F$. From the proof of the theorem (4.3.4), we have that U is an ultrafuzzy topology. Since, a is not in U, $U \wedge T(a) = \{0,1\}$. Further, since all fuzzy singletons except $s(x,1)$ are in $U(x,1)$ and F contains $s(y,1) \vee s(x,1)$, $s(x,1) = a \wedge (s(x,1) \vee s(y,1))$, belongs to $T(a) \vee U$, $T(a) \vee U = L(X)$. Thus U is a complement of $T(a)$.

Now, let U be an ultrafuzzy topology. Since, minimal t-irreducible elements in L are singletons, by the theorem (4.3.2), there exists a unique x in X and l in L, such that $s(x,l)$ is the only fuzzy singleton, not in U. But then $T = \{0,s(x,1),1\}$ is an atom in (L,X) and a complement of U. Hence the teorem.

4.4.5 Note

PAUL S. SCHNARE [22] proved that complement in the lattice of topologies is not unique. From the proof the theorem (4.4.4), it is evident that a fuzzy topology can have more than one complement in the lattice of fuzzy topologies on a set.

4.4.6 Concluding remark

The problem of complementation in the lattice of fuzzy topologies on a set, though solved in the negative, a complete study of the problem with particular emphasis on the structure of the membership set, still remains.

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