SOME PROBLEMS IN TOPOLOGY AND ALGEBRA

A STUDY ON FUZZY SEMI INNER PRODUCT SPACES

THESIS SUBMITTED TO THE
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN MATHEMATICS
UNDER THE FACULTY OF SCIENCE

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NOVEMBER 1995

CERTIFICATE

Certified that the thesis entitled **A STUDY ON FUZZY SEMI INNER PRODUCT SPACES** is a bona fide record of work done by Sri.T.V.Ramakrishnan under my supervision and guidance in the Division of Mathematics, School of Mathematical Sciences, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

Cochin - 22
November 2, 1995

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DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

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ACKNOWLEDGEMENT

First and foremost, I would like to express my heart felt gratitude to Prof. T.Thrivikraman, my supervisor, whose inspiring guidance, constant care and abundant advice were instrumental in completing this thesis. I am particularly obliged to Dr. P.T. Ramanchandran of the University of Calicut for his substantial support and encouragement.

At this moment it is my pleasure to acknowledge my indeptness to all my friends and teachers of the School of Mathematical Sciences, for their help during the various occasions I also wish to express my appreciation to the office staff of the School for their co-operation and support.

A special note of gratitude goes to Data Systems for their immaculate and expedient typing of this thesis.

The financial support received from the Council of Scientific and Industrial Research, India, is greatefully acknowledged.

The blessing and encouragement from my parents, sisters and brother were with me through out.

T.V. RAMAKRISHNAN

CONTENTS

Chapter 0

INTRODUCTION

Mathematical models are often used to describe physical realities. However, the physical realities are imprecise while the mathematical concepts are required to be precise and perfect. Even mathematicians like H. Poincare worried about this. He observed that mathematical models are over idealizations, for instance, he said that only in Mathematics, equality is a transitive relation. A first attempt to save this situation was perhaps given by K. Menger in 1951 by introducing the concept of statistical metric space in which the distance between points is a probability distribution on the set of nonnegative real numbers rather than a mere nonnegative real number. Other attempts were made by M.J. Frank, U. Höhle, B. Schweizer, A. Sklar and others. An aspect in common to all these approaches is that they model impreciseness in a probabilistic manner. They are not able to deal with situations in which impreciseness is not apparently of a probabilistic nature.

L.A. Zadeh gave a convincing way out to this through his pioneering paper of 1965. This was a beginning of a new discipline in Mathematics - Fuzzy set theory. The characteristic function of a set assigns a value of either 1 or 0 to each individual in the universal set, thereby discriminating between members and nonmembers of the crisp set under consideration. This function can be generalized such that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set under consideration. Larger values denote higher degrees of set membership, such a function is called a membership function and the set defined by it a fuzzy set. More specifically, any fuzzy subset of a set X is a member belonging to the set I^X, the set of all functions from X to the unit interval I. An ordinary set thus becomes a special case of fuzzy set with a membership function which is reduced to the well known two valued (either 0 or 1) characteristic function.

The definitions, theorems, proofs and so on of fuzzy set theory always hold for nonfuzzy sets. Because of this generalization, the theory of fuzzy sets has a wider scope of applicability than classical set theory in solving problems that involve, to some degree, subjective evaluation. Fuzzy set theory has now become a major area of research and finds applications in various fields like artificial intelligence, image processing, biological and medical sciences, operation research, economics, geography, so on and so forth. Our interest of fuzzy set theory is in its application to the theory of functional analysis, especially to the theory of semi inner product spaces.

With the aim of carrying over Hilbert space type arguments to the theory of Banach spaces Lumer[LUM] introduced the concept of semi inner product space with a more general axiom system than that of inner product space. The importance of semi inner product space is that, whether the norm satisfies the parallelogram law or not, every normed space can be represented as a semi inner product so that the theory of operators on Banach spaces can be penetrated by Hilbert space type arguments. But Giles [GI] observed that the generality of the axiom system defining the semi inner product is a serious limitation on any extensive development of a theory of semi inner product space parallel to that of Hilbert spaces. He, therefore imposed further restrictions on the semi inner product to make further developments. He thus obtained an analogue of the Riesz representation theorem for semi inner product spaces. The theory of semi inner product has been studied by several mathematicians like Nath, Husain, Malviya, Torrance, etc.

To give a fuzzy analogue to this theory we require the concept of fuzzy real number. This was introduced by Hutton, B[HU] and Rodabaugh, S.E [ROD]. Our definition slightly differs from this with an additional minor restriction. The definition given by Clementina Felbin [CL_1] is entirely different.

The concept of fuzzy metric was introduced by Kaleva, O and Seikkala,s [KA;SE]. Morsi, N.N [MO_2] provided a method for introducing fuzzy pseudo-metric topologies on sets and fuzzy pseudo-normed topologies on vector spaces over R or C which will be fuzzy linear topologies. Katsaras,A.K and Liu, D.B [K;L] introduced the notion of fuzzy vector spaces and fuzzy topological vector spaces. Krishna, S.V and Sarma, K.K.M [KR_2;SA_2] studied about the fuzzy continuity of linear maps on vector spaces.

Clementina Felbin [CF_2] established the completion of a fuzzy normed linear space. Abdel wahab M. El-Abyad and Hassan M.El-Hamouly [A;H] succeeded in defining fuzzy inner product on an M(I) module. Parallel to this we are able to introduce the concept of fuzzy semi inner product.

To say it briefly this thesis is confined to introducing and developing a theory of fuzzy semi inner product spaces.

AN OVERVIEW OF THE MAIN RESULTS OF THIS THESIS

The thesis comprises six chapters and an introduction to the subject.

Chapter 1

In this chapter we give a brief summary of the arithmetic of fuzzy real numbers and the fuzzy normed algebra M(I). Also we explain a few preliminary definitions and results required in the later chapters. Fuzzy real numbers are introduced by Hutton, B [HU] and Rodabaugh, S.E [ROD]. Our definition slightly differs from this with an additional minor restriction. The definition of Clementina Felbin [CL_1] is entirely different. The notations of [HU] and [M;Y] are retained inspite of the slight difference in the concept.

Chapter 2

Kaleva, O [KA] introduced the notion of completion of fuzzy metric spaces. Mashhour, A.S & Morsi, N.N [M;M] defined M(I), a fuzzy normed algebra, whose underlying space is the smallest real vector space including all nonnegative fuzzy real numbers. Clementina Felbin [CF_2] established the completion of a fuzzy normed linear space. In this chapter we construct a real commutative algebra C(I) from M(I) analogous to the construction of the algebra of complex numbers from that of reals. We establish the existence of unique fuzzy completions M'(I) of M(I) and C'(I) of C(I). However, this is essentially different from the works of Felbin and Kaleva. For example our definition of R(I) and $R^*(I)$ are different from those of Felbin. Finally, we prove certain results about C'(I) - like that it is not an integral domain and it is a commutative algebra.

Chapter 3

In this chapter using the completion M'(I) of M(I) we give a fuzzy extension of real Hahn-Banach theorem. Some consequences of this extension are obtained. The idea of real fuzzy linear functional on fuzzy normed linear space is introduced. Some of its properties are studied. In the complex case we get only a slightly weaker analogue for the Hahn-Banach theorem, than the one in the crisp case.

Chapter 4

Lumer,G [LUM] introduced the notion of semi inner product space with a more general axiom system than that of inner product space. Parallel to this on a C'(I) module we are able to introduce the notion of fuzzy semi inner product. We prove that a fuzzy semi inner product generates a fuzzy norm and further that every fuzzy normed space can be made into a fuzzy semi inner product space. Also the notion of a fuzzy orthogonal set is introduced. Existence of a complete fuzzy orthogonal set is established. The concept of generalized fuzzy semi inner product is introduced.

Chapter 5

In this chapter we extend the idea of fuzzy semi inner product space of crisp points to that of fuzzy points. The notion of orthogonality on the fuzzy semi inner product of fuzzy points is introduced. Some of its properties are studied. Also the concepts like fuzzy numerical range of 'fuzzy linear maps' on the set of fuzzy points is introduced and some results are obtained.

Chapter 6

In this chapter the concept of the category of semi inner product spaces and that of fuzzy semi inner product spaces are introduced. Relation of the category of fuzzy semi inner product spaces with the categories of semi inner product spaces, fuzzy topological spaces and topological spaces are studied. We conclude with a more general approach to fuzzy semi inner product spaces by introducing the category of semi inner products in a given concrete category.

Chapter 1

PRELIMINARIES

1.0 INTRODUCTION

In this chapter we give a brief summary of the arithmetic of fuzzy real numbers and the fuzzy normed algebra M(I). Also we explain a few preliminary definitions and results required in the later chapters. Fuzzy real numbers are introduced by Hutton,B [HU] and Rodabaugh, S.E[ROD]. Our definition slightly differs from this with an additional minor restriction. The definition of Clementina Felbin [CL_1] is entirely different. The notations of [HU]and [M;Y] are retained inspite of the slight difference in the concept.

1.1 FUZZY REAL NUMBER

Definition 1.1.1

A fuzzy real number η is a nonincreasing, left continuous function from the set of real numbers R into I = [0,1] with $\eta(-\infty^{+}) = 1$ and $\eta(t) = 0$ for some $t \in R$. The set of all fuzzy real numbers will be denoted by R(I). The partial ordering $\geq$ on R(I) is just its natural ordering as a set of real functions. The set of all reals R is canonically embedded in R(I) in the following way.

A real number r is identified with the fuzzy real number $\bar{r} \in R(I)$ given by: for $t \in R$

$$\bar{r}(t) = \begin{cases} 1 & \text{if } t \leq r \\ 0 & \text{if } t > r \end{cases}$$

The set of all nonnegative fuzzy real numbers $R^{*}(I)$ is defined by

$$R^{*}(I) = \{\eta \in R(I) : \eta \geq \bar{0}\}.$$

Note 1.1.2

(i) The above definition differs from the standard definition given by Hutton, B[HU] and Rodabaugh, S.E[ROD] in the additional condition $\eta(t) = 0$ for some $t \in R$. We retain the same notation R(I) for our restricted set and later also in the further development, we ignore the difference in choosing our notations.

(ii) Clementina F [CL_1]has given a different definition for fuzzy real number as a fuzzy set on R.

Definition 1.1.3 [ROD]

Let η and β be two fuzzy real numbers in R(I), then

(i) Addition of fuzzy real numbers $\oplus$ is defined on R(I) by

$(\eta \oplus \beta)(s) = \text{Sup}\{\eta(t) \wedge \beta(s-t) : t \in R\}$

(ii) Scalar multiplication by a nonnegative $r \in R$ is defined on R(I) by $r\eta = \bar{0}$, if $r = 0$

$$(r\eta)(s) = \eta(s/r), \text{ if } r>0, \text{ where } s \in R.$$

Proposition 1.1.4 [M;Y]

(i) Addition and scalar multiplication preserve the order $\geq$ on R(I).

(ii) For η, β and $\alpha \in R(I)$ we have $\eta \oplus \beta \geq \alpha \oplus \beta$, iff $\eta \geq \alpha$.

Definition 1.1.5 [ROD]

Multiplication of two nonnegative fuzzy real numbers η and ξ is defined by

$$(\eta\ \xi)\ (s) = \begin{cases} 1 & \text{if } s \leq 0 \\ \sup\{\eta\ (b) \wedge \xi(s/b) : b > o\} & \text{if } s > 0 \end{cases}$$

where $s \in R$

Note 1.1.6

(i) η and ξ be two fuzzy real numbers such that $\eta\ (a) = 0$ and $\xi(b) = 0$ then $r\eta\ (ra) = 0$, $(\eta \oplus \xi)(a+b) = 0$ and $(\eta\ \xi)(ab) = 0$.

(ii) It may be noted that addition and scalar multiplication are well defined on R(I), and $R^*(I)$ is closed under multiplication.

Remark 1.1.7

If $\eta, \beta \in R^*(I)$ then

(i) $\eta \oplus \beta = \beta \oplus \eta$

(ii) $\eta\,\beta = \beta\,\eta$

(iii) $\eta\,\bar{0} = \bar{0}$

(iv) $\eta\,\bar{1} = \eta$

Definition 1.1.8 [HU]

For every $r \in R$, the fuzzy subset L_r of R(I) is defined by: for $\eta \in R(I)$, $L_r(\eta) = 1-\eta\,(r)$. It is obvious that the real function $L_r(\eta)$ is left continuous and nondecreasing in r, and is nonincreasing in η.

1.2 THE N-EUCLIDEAN ALGEBRA M(I)

Definition 1.2.1 [M;M]

The set M(I) is the cartesian product $R^*(I)xR^*(I)$ modulo the equivalence relation $\sim$ defined by $(\eta,\beta) \sim (\eta',\beta')$ iff $\eta \oplus \beta' = \eta' \oplus \beta$.

A member of M (I) is denoted simply by any one of its representative ordered pairs.

The partial order $\geq$ on M(I) is defined by $(\eta,\beta) \geq (\eta',\beta')$

iff $\eta \oplus \beta' \geq \eta' \oplus \beta$.

The set $M^*(I)$ is defined by

$$M^*(I) = \left\{(\eta,\beta) \in M(I):(\eta,\beta) \geq (\bar{0},\bar{0})\right\}$$

$$= \left\{(\eta,\beta) \in M(I):\eta \geq \beta\right\}$$

$R^*(I)$ is canonically embedded in M(I) by identifying each $\eta \in R^*(I)$ with $(\eta,\bar{0}) \in M(I)$, while R is canonically embedded in M(I) as follows

for $r \in R$, r is identified with $(\bar{r},\bar{0}) \in M(I)$ if $r \geq 0$ and with

$(\bar{0},(\overline{-r})) \in M(I)$ if $r < 0$.

Definition 1.2.2 [M;M]

Addition $\oplus$ and scalar multiplication are defined on M(I) by

(i) $(\eta,\beta) \oplus (\eta',\beta') = (\eta \oplus \eta',\beta \oplus \beta')$.

(ii) Let $t \in R$ then $t(\eta,\beta) = \begin{cases} (t\eta,t\beta) & \text{if } t \geq 0 \\ (|t|\beta,|t|\eta) & \text{if } t < 0 \end{cases}$

Definition 1.2.3 [KA_3]

A fuzzy pseudo-norm on a real or complex vector space X is a function $\|\ \|$ $X \longrightarrow R^*(I)$ which satisfies for $x, y \in X$ and S in the field

(i) $\|sx\| = |s|\ \|x\|$

(ii) $\|x+y\| \leq \|x\| \oplus \|y\|$ such $\|\ \|$ is called a fuzzy norm if in addition it satisfies

(iii) $\|x\| > \bar{0}$ for every nonzero $x \in X$.

Definition 1.2.4 [M;M]

The N-Euclidean norm on M(I) is the fuzzy norm $[\![\ ]\!]$ defined by for $(\eta,\beta) \in M(I)$

$$[\![(\eta,\beta)]\!] = \inf \left\{ \alpha \in R^*(I) : \alpha \geq (\eta,\beta) \ \&\ \alpha \geq (\beta,\eta) \right\}$$

$$= \inf \left\{ \alpha \in R^*(I) : \alpha \oplus \beta \geq \eta \ \&\ \alpha \oplus \eta \geq \beta \right\}$$

where $\alpha = (\alpha, \bar{0})$ according to the embedding of $R^*(I)$ in M(I).

Proposition 1.2.5 [M;Y]

(i) For $\eta,\beta \in R^*(I)$, $\eta^2<\beta^2$ iff $\eta < \beta$.

(ii) For every $\eta \in R^2(I)$, there exists a unique square root β in $R^*(I)$ such that $\beta^2 = \eta$.

(iii) The partial order $\geq$ on $M(I)$ is preserved under multiplication by elements of $M^*(I)$.

Remark 1.2.6

(i) If $(\alpha,\beta) \in M(I)$ then $[\![(\alpha,\beta)]\!] = [\![(\beta,\alpha)]\!]$

(ii) $[\![(\eta,\tilde{0})]\!] = \eta$ for $\eta \in R^*(I)$.

Definition 1.2.7 [M;Y]

Multiplication on $M(I)$ is defined by:

for (α,β), $(\alpha',\beta') \in M(I)$

$$(\alpha,\beta)\ (\alpha',\beta') = (\alpha\alpha' \oplus \beta\beta' \quad \alpha\beta' \oplus \beta\alpha').$$

Remark 1.2.8

With respect to the addition defined above $M(I)$ is an abelian group where (β,α) is the additive inverse of (α,β) ie $\ominus (\alpha,\beta) = (\beta,\alpha)$.

Theorem 1.2.9 [M;Y]

(i) Multiplication on M(I) is well defined.

(ii) The canonical embedding of R and $R^*(I)$ into M(I) preserve multiplication.

(iii) Under addition, scalar multiplication and multiplication M(I) is a real associative and commutative algebra with unit element $(\bar{1},\bar{0})$.

(iv) M(I) is not an integral domain.

Proposition 1.2.10 [M;Y]

(i) $M^*(I)$ is closed under multiplication.

(ii) The partial order $\geq$ on M(I) is preserved under multiplication by elements of $M^*(I)$.

(iii) For $\alpha,\beta,\gamma \in R^*(I)$, $\alpha^2 \geq (\beta,\gamma)^2$ iff $\alpha \geq (\beta,\gamma)$ and $\alpha \geq (\gamma,\beta)$.

(iv) For $(\alpha,\beta) \in M(I)$, We have $(\alpha,\beta)^2 \in M^*(I)$.

Proposition 1.2.11 [M;M]

(i) If $(\alpha,\beta) \in M^*(I)$ then

$$\llbracket(\alpha,\beta)\rrbracket = \inf \{\gamma \in R^*(I) : \gamma \geq (\alpha,\beta)\}.$$

(ii) For $(\alpha,\beta) \in M(I)$

$$\llbracket(\alpha,\beta)\rrbracket^2 = \llbracket(\alpha,\beta)^2\rrbracket = \inf \{\lambda \in R^*(I) : \lambda \geq (\alpha,\beta)^2\}.$$

(iii) For $(\alpha,\beta),(\gamma,\xi) \in M(I)$

$$\llbracket(\alpha,\beta)(\gamma,\xi)\rrbracket \leq \llbracket(\alpha,\beta)\rrbracket \; \llbracket(\gamma,\xi)\rrbracket$$

Definition 1.2.12 [M:M]

Let U be a fuzzy subset of a universe X and let $\alpha \in I_1 = [0,1)$. The α - cut of U is the crisp subset of X

$$U_{(\alpha)} = \{x \in X : U(x) > \alpha\}.$$

By considering a fuzzy real number $\eta \in R^*(I)$ as a fuzzy subset of R, we find immediately that each α - cut $\eta_{(\alpha)}$ of η is an interval in R of the form $[0,t]$ or $[0,t)$. Where $t = \vee\{x \in R : \eta(x) > \alpha\}$. Thus $\eta_{(\alpha)}$ can be identified with the number t. It is obvious that the α - cuts preserve the three operations on $R^*(I)$ in the following sense:

for every η, $\xi \in R^*(I)$, $\alpha \in I_1$ and $r \geq 0$ we have

(i) $(\eta \oplus \xi)_{(\alpha)} = \eta_{(\alpha)} + \xi_{(\alpha)}$

(ii) $(r\eta)_{(\alpha)} = r\eta_{(\alpha)}$

(iii) $(\eta\ \xi)_{(\alpha)} = \eta_{(\alpha)}\ \xi_{(\alpha)}$

(iv) $\eta \leq \xi$ iff $\eta_{(\alpha)} \leq \xi_{(\alpha)}, \forall\ \alpha \in I_1$.

Definition 1.2.13 [M;M]

Let $(\eta,\xi) \in M(I)$ and $\alpha \in I_1$. We define the α - cut of (η,ξ) to be the real number

$$(\eta,\xi)_{(\alpha)} = \eta_{(\alpha)} - \xi_{(\alpha)}.$$

Proposition 1.2.14 [M;M]

(i) The α - cut $(\eta,\xi)_{(\alpha)}$ is well defined on $M(I)$.

(ii) $(\eta,\xi) = (\alpha,\beta)$ in $M(I)$ iff they have same indexed family of α - cuts.

(iii) $(\eta,\xi) \in M^*(I)$ iff $\forall\ \alpha \in I_1$, $(\eta,\xi)_{(\alpha)} \geq 0$.

(iv) For each fixed $\alpha \in I_1$, α - cuts is an order preserving real algebra homomorphism from $M(I)$ onto R.

Proposition 1.2.15

Let η, $\beta \in R^{*}(I)$ then $\eta\,\beta \geq \eta^{2}$ iff $\beta \geq \eta$.

Proof:

$\eta\,\beta \geq \eta^{2}$ iff $(\eta\,\beta)_{(\alpha)} \geq (\eta^{2})_{(\alpha)}$, $\forall\ \alpha \in I_1$

iff $\eta_{(\alpha)}\beta_{(\alpha)} \geq \eta_{(\alpha)}\eta_{(\alpha)}$, $\forall\ \alpha \in I_1$

iff $(\eta)_{(\alpha)}(\beta_{(\alpha)} - \eta_{(\alpha)}) \geq 0$, $\forall\ \alpha \in I_1$

when $\eta_{(\alpha)} = 0$, $\beta_{(\alpha)} \geq \eta_{(\alpha)}$, $\forall\ \alpha \in I_1$

when $\eta_{(\alpha)} \neq 0$, $\beta_{(\alpha)} \geq \eta_{(\alpha)}$

ie $\eta\,\beta \geq \eta^{2}$ iff $\beta_{(\alpha)} \geq \eta_{(\alpha)}$, $\forall\ \alpha \in I_1$

ie iff $\beta \geq \eta$.

Definition 1.2.16

A sequence (η_n, ξ_n) in M(I) is said to be Cauchy

if $\lim\limits_{\substack{n \to \infty \\ m \to \infty}} [\![(\eta_n, \xi_n) \ominus (\eta_m, \xi_m)]\!] = \bar{0}$.

Definition 1.2.17 [LUM]

A real or complex vector space E is called semi inner product space, if to every pair of elements x,y in E, there corresponds a number [x,y], called semi inner product with the following properties

(i) $[x+y,z] = [x,z] + [y,z]$

$[\lambda x,y] = \lambda[x,y]$, x, y, z in E and λ scalar

(ii) $[x,x]>0$ for $x \neq 0$

(iii) $|[x,y]|^2 \leq [x,x][y,y]$.

CHAPTER 2

THE REAL COMMUTATIVE ALGEBRA C(I) AND ITS COMPLETION*

2.0 INTRODUCTION

Kaleva, O [KA] introduced the notion of completion of fuzzy metric spaces. Mashhour, A S & Morsi, N.N [M;M] defined M(I), a fuzzy normed algebra, whose underlying space is the smallest real vector space including all nonnegative fuzzy real numbers. Clementina Felbin [CL_2] established the completion of a fuzzy normed linear space.

In this chapter we construct, a real commutative algebra C(I) from M(I) analogous to the construction of the algebra of complex numbers from that of reals. We establish the existence of unique fuzzy completions M'(I) of M(I) and C'(I) of C(I). However, this is essentially different from the works of Felbin and Kaleva. For example, our definition of R(I) and $R^*(I)$ are different from those of Felbin.

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*Some results contained in this chapter have been included in a paper accepted for publication in The Journal of Fuzzy Mathematics.

Finally, we prove certain results about C'(I) - like that it is not an integral domain and that it is a commutative algebra.

2.1 FUZZY NORMED ALGEBRA C(I)

Definition 2.1.1

Define $C(I) = \left\{((\eta,\xi),(\eta',\xi')) \mid (\eta,\xi) \ \& \ (\eta',\xi') \in M(I)\right\}$

ie $C(I) = M(I) \times M(I)$

On C(I) addition, multiplication and scalar multiplication are defined by

$$((\eta_1,\xi_1),(\eta'_1,\xi'_1)) + ((\eta_2,\xi_2),(\eta'_2,\xi'_2))$$

$$= ((\eta_1,\xi_1) \oplus (\eta_2,\xi_2) \quad (\eta'_1,\xi'_1) \oplus (\eta'_2,\xi'_2))$$

$$((\eta_1,\xi_1),(\eta'_1,\xi'_1)) \times ((\eta_2,\xi_2),(\eta'_2,\xi'_2))$$

$$= ((\eta_1,\xi_1)\ (\eta_2,\xi_2) \ominus (\eta'_1,\xi'_1)\ (\eta'_2,\xi'_2),$$

$$(\eta_1,\xi_1)\ (\eta'_2,\xi'_2\ ) \oplus (\eta'_1,\xi'_1)\ (\eta_2,\xi_2))$$

Let $t \in R$ then

$t((\eta,\xi),(\eta',\xi')) = (t(\eta,\xi),t(\eta',\xi'))$.

Note 2.1.2

(i) It can be easily verified that C(I) is a real commutative algebra.

(ii) M(I) can be embedded in C(I) by representing each $(\eta,\xi) \in M(I)$ by $((\eta,\xi),(\bar{0},\bar{0})) \in C(I)$.

Definition 2.1.3

Define $[\![((\eta,\xi),(\eta',\xi'))]\!] = [(\eta,\xi)^2 \oplus (\eta',\xi')^2]^{1/2}$

Proposition 2.1.4

Treating C(I) as a real vector space $[\![\ ]\!]$ defined above is a fuzzy norm on C(I).

Proof:

(i) $[\![((\eta,\xi),(\eta',\xi'))]\!] = [(\eta,\xi)^2 \oplus (\eta',\xi')^2]^{1/2} = \bar{0}$

This is true iff $(\eta,\xi)^2 \oplus (\eta',\xi')^2 = (\bar{0},\bar{0})$

$$\text{iff } (\eta,\xi)^2 = (\bar{0},\bar{0}) \;\&\; (\eta',\xi')^2 = (\bar{0},\bar{0})$$

$$\text{iff } ((\eta,\xi),(\eta',\xi')) = ((\bar{0},\bar{0}),(\bar{0},\bar{0})).$$

(ii) Let $t \in R$, $t \neq 0$, consider

$$[\![t((\eta,\xi),(\eta',\xi'))]\!] = [\![t^2((\eta,\xi)^2 \oplus (\eta',\xi')^2)]\!]^{1/2}$$

$$\leq |t| \; [\![(\eta,\xi)^2 \oplus (\eta',\xi')^2]\!]^{1/2}$$

ie $\quad [\![t((\eta,\xi),(\eta',\xi'))]\!] \leq |t| \; [\![((\eta,\xi),(\eta',\xi'))]\!] \qquad$ (a)

also

$$[\![((\eta,\xi),(\eta',\xi'))]\!] = [\![\tfrac{1}{t} \times t((\eta,\xi),(\eta',\xi'))]\!]$$

$$\leq \frac{1}{|t|} \; [\![t((\eta,\xi),(\eta',\xi'))]\!]$$

ie $|t| [\![((\eta,\xi),(\eta',\xi'))]\!] \leq [\![t((\eta,\xi),(\eta',\xi'))]\!] \qquad$ (b)

by (a) & (b)

$$[\![t((\eta,\xi),(\eta',\xi'))]\!] = |t| \; [\![((\eta,\xi),(\eta',\xi'))]\!]$$

(iii) $[\![((\eta_1,\xi_1),(\eta_1',\xi_1')) + ((\eta_2,\xi_2),(\eta_2',\xi_2'))]\!]^2 =$

$$[\![((\eta_1,\xi_1) \oplus (\eta_2,\xi_2),(\eta_1',\xi_1') \oplus (\eta_2',\xi_2'))]\!]^2$$

$$
\begin{aligned}
&= \Box(((\eta_1,\xi_1) \oplus (\eta_2,\xi_2))^2 \oplus ((\eta_1',\xi_1') \oplus (\eta_2',\xi_2'))^2)\Box \\
&= \Box((\eta_1,\xi_1)^2 \oplus (\eta_2,\xi_2)^2 \oplus 2(\eta_1,\xi_1)(\eta_2,\xi_2) \oplus (\eta_1',\xi_1')^2 \\
&\qquad\qquad \oplus (\eta_2',\xi_2')^2 \oplus 2(\eta_1',\xi_1')(\eta_2',\xi_2'))\Box \\
&\le \Box((\eta_1,\xi_1)^2 \oplus (\eta_1',\xi_1')^2)\Box \oplus \Box((\eta_2,\xi_2)^2 \oplus (\eta_2',\xi_2')^2)\Box \\
&\qquad \oplus 2\Box((\eta_1,\xi_1)(\eta_2,\xi_2) \oplus (\eta_1',\xi_1')(\eta_2',\xi_2'))\Box \\
&\le \llbracket((\eta_1,\xi_1),(\eta_1',\xi_1'))\rrbracket^2 \oplus \llbracket((\eta_2,\xi_2),(\eta_2',\xi_2'))\rrbracket^2 \\
&\qquad \oplus 2\Box(((\eta_1,\xi_1)(\eta_2,\xi_2) \oplus (\eta_1',\xi_1')(\eta_2',\xi_2'))^2 \\
&\qquad\qquad \oplus ((\eta_1,\xi_1)(\eta_2,\xi_2) \ominus (\eta_1',\xi_1')(\eta_2',\xi_2'))^2)\Box^{1/2} \\
&\quad = \llbracket((\eta_1,\xi_1),(\eta_1',\xi_1'))\rrbracket^2 \oplus \llbracket((\eta_2,\xi_2),(\eta_2',\xi_2'))\rrbracket^2 \\
&\qquad\qquad \oplus 2\llbracket((\eta_1,\xi_1)(\eta_2,\xi_2) \oplus (\eta_1',\xi_1')(\eta_2',\xi_2'), \\
&\qquad\qquad\qquad (\eta_1,\xi_1)(\eta_2,\xi_2) \ominus (\eta_1',\xi_1')(\eta_2',\xi_2'))\rrbracket \\
&\quad = \llbracket((\eta_1,\xi_1),(\eta_1',\xi_1'))\rrbracket^2 \oplus \llbracket((\eta_2,\xi_2),(\eta_2',\xi_2'))\rrbracket^2 \\
&\qquad\qquad \oplus 2\llbracket((\eta_1,\xi_1),\ominus(\eta_1',\xi_1')) \times ((\eta_2,\xi_2),(\eta_2',\xi_2'))\rrbracket
\end{aligned}
$$

$$\leq [\![((\eta_1,\xi_1),(\eta_1',\xi_1'))]\!]^2 \oplus [\![((\eta_2,\xi_2),(\eta_2',\xi_2'))]\!]^2$$

$$\oplus\ 2[\![((\eta_1,\xi_1),(\eta_1',\xi_1'))]\!]\ [\![((\eta_2,\xi_2),(\eta_2',\xi_2'))]\!]$$

ie, $[\![((\eta_1,\xi_1),(\eta_1',\xi_1')) + ((\eta_2,\xi_2),(\eta_2',\xi_2'))]\!]^2$

$$\leq \left([\![((\eta_1,\xi_1),(\eta_1',\xi_1'))]\!] \oplus [\![((\eta_2,\xi_2),(\eta_2',\xi_2'))]\!]\right)^2$$

ie, $[\![((\eta_1,\xi_1),(\eta_1',\xi_1')) + ((\eta_2,\xi_2),(\eta_2',\xi_2'))]\!]$

$$\leq [\![((\eta_1,\xi_1),(\eta_1',\xi_1'))]\!] \oplus [\![((\eta_2,\xi_2),(\eta_2',\xi_2'))]\!].$$

Note 2.1.5

The subset $C_1(I) = \left\{((\eta,\xi),(\bar{0},\bar{0})) \mid \eta,\xi \in R^*(I)\right\}$ of $C(I)$ is a partially ordered set with the partial order defined by

$((\eta_1,\xi_1),(\bar{0},\bar{0})) \geq ((\eta_2,\xi_2),(\bar{0},\bar{0}))$

iff $(\eta_1,\xi_1) \geq (\eta_2,\xi_2)$.

2.2 FUZZY COMPLETION OF M(I)

Notation 2.2.1

The α level set of $[\![(\eta_n,\xi_n) \ominus (\eta,\xi)]\!]$ denoted by

$$\lceil\!\lceil (\eta_n,\xi_n) \ominus (\eta,\xi) \rfloor\!\rfloor_{(\alpha)} = \left\{ t \in R : \lceil\!\lceil (\eta_n,\xi_n) \ominus (\eta,\xi) \rfloor\!\rfloor (t) \geq \alpha \right\}$$

This is identified by $\lambda_\alpha((\eta_n,\xi_n) \ominus (\eta,\xi))$, the maximal element of the above set.

Theorem 2.2.2

There is a complete fuzzy normed space $\langle M'(I), \lceil\!\lceil\ \rfloor\!\rfloor'\rangle$ such that M(I) is congruent to a dense subset of M'(I), say $M_0(I)$ and the fuzzy norm on M'(I) extends the fuzzy norm on M(I).

Proof:

On the class of all Cauchy sequences in $\langle M(I), \lceil\!\lceil\ \rfloor\!\rfloor\rangle$ consider the relation $\leftrightarrow$ defined by $\{(\eta_n,\xi_n)\} \leftrightarrow \{(\eta_n',\xi_n')\}$ iff $\lim_{n\to\infty} \lceil\!\lceil (\eta_n,\xi_n) \ominus (\eta_n',\xi_n') \rfloor\!\rfloor = \bar{0}$.

That is iff $\lim_{n\to\infty} \lambda_\alpha((\eta_n,\xi_n) \ominus (\eta_n',\xi_n')) = 0$, $\forall\, \alpha \in (0,1]$. It can be easily verified that $\leftrightarrow$ is an equivalence relation. The collection of all equivalence classes is denoted by M'(I). Let $[\eta,\xi]$ & $[\alpha,\beta] \in M'(I)$ and $\{(\eta_n,\xi_n)\} \in [\eta,\xi]$ and $\{(\alpha_n,\beta_n)\} \in [\alpha,\beta]$. Then $\{(\eta_n,\xi_n) \oplus (\alpha_n,\beta_n)\}$ is a Cauchy sequence. Also if $\{(\eta_n',\xi_n')\} \in [\eta,\xi]$ $\{(\alpha_n',\beta_n')\} \in [\alpha,\beta]$ so

that $\{(\eta_n,\xi_n)\} \leftrightarrow \{(\eta_n',\xi_n')\}$, $\{(\alpha_n,\beta_n)\} \leftrightarrow \{(\alpha_n',\beta_n')\}$, then $\{(\eta_n,\xi_n) \oplus (\alpha_n,\beta_n)\} \leftrightarrow \{(\eta_n',\xi_n') \oplus (\alpha_n',\beta_n')\}$.
Define $[\eta,\xi] \oplus [\alpha,\beta]$ to be the class to which $\{(\eta_n,\xi_n) \oplus (\alpha_n,\beta_n)\}$ belongs. If $r \in R$ and $\{(\eta_n,\xi_n)\} \in [\eta,\xi]$, define $r[\eta,\xi]$ as the class containing $\{r(\eta_n,\xi_n)\}$. M'(I) together with these operations is a linear space.

On M'(I) define $\|\;\|'$ as follows.
Let $[\eta,\xi] \in M'(I)$ and $\{(\eta_n,\xi_n)\} \in [\eta,\xi]$.

Define $\|[\eta,\xi]\|'_{(\alpha)} = [0,\lambda_\alpha[\eta,\xi]]$

$$= [0, \lim_{n\to\infty} \lambda_\alpha(\eta_n,\xi_n)].$$

Here after we denote $\|[\eta,\xi]\|'_{(\alpha)}$ by $\lambda_\alpha[\eta,\xi]$.

Proceeding as in the proof given by Clementina Felbin [CL_2] we can prove that $\|\;\|'$ is a fuzzy norm.

Next to show M'(I) contains an every where dense subspace $M_0(I)$ congruent to M(I). Define ϕ: M(I) $\longrightarrow$ M'(I) by setting $\phi\,(\eta,\xi)$ as the equivalence class to which the repeated sequence $\{(\eta,\xi),(\eta,\xi)\ldots\}$ belongs.

Then for all $\alpha \in (0,1]$

$$[\![\phi(\eta,\xi) \ominus \phi(\eta',\xi')]\!]'_{(\alpha)} = \lambda_\alpha((\eta,\xi) \ominus (\eta',\xi'))$$

$$= [\![(\eta,\xi) \ominus (\eta',\xi')]\!]_{(\alpha)}$$

ie. $[\![\phi(\eta,\xi) \ominus \phi(\eta',\xi')]\!]' = [\![(\eta,\xi) \ominus (\eta',\xi')]\!]$

ie. ϕ is an isometry.

Let $M_0(I) = \phi(M(I))$

To show that $\overline{M_0(I)} = M'(I)$

Let $[\eta,\xi] \in M'(I)$ and $\{(\eta_n,\xi_n)\} \in [\eta,\xi]$

since $\{(\eta_n,\xi_n)\}$ is Cauchy, given $\varepsilon > 0$ & $\alpha \in (0,1]$ there exists N such that $\forall$ $m,n \geq N$

$\lambda_\alpha((\eta_n,\xi_n) \ominus (\eta_m,\xi_m)) < \varepsilon$

Consider $\{\phi(\eta_n,\xi_n)\} \in M_0(I) \subset M'(I)$, then

$$[\![[\eta,\xi] \ominus \phi(\eta_n,\xi_n)]\!]'_{(\alpha)} = \lim_{m\to\infty} \lambda_\alpha((\eta_m,\xi_m) \ominus (\eta_n,\xi_n))$$

for $n \geq N$ R.H.S is less than ε

$\longrightarrow [\![[\eta,\xi] \ominus \phi(\eta_n,\xi_n)]\!]' \rightarrow \bar{0}$ as $n \rightarrow \infty$.

ie, given $[\eta,\xi] \in M'(I)$ it is possible to construct a sequence of points in $M_0(I)$ converging to $[\eta,\xi]$. Hence $\overline{M_0(I)} = M'(I)$.

ϕ is a 1-1 mapping of M(I) onto $M_0(I)$. It is easy to prove ϕ is linear.

To prove M'(I) is complete with respect to $[\!]\ ]\!]'$ consider first a special type of Cauchy sequence

$\{\phi\,(\eta_1,\xi_1),\phi\,(\eta_2,\xi_2),\ldots,\phi\,(\eta_n,\xi_n),\ldots\}$

where $\{(\eta_i,\xi_i),(\eta_i,\xi_i),\ldots\} \in \phi\,(\eta_i,\xi_i)$, i = 1,2,...

consider the sequence $\{(\eta_n,\xi_n)\}$ obtained by taking the isometric pre-images of $\{\phi\ (\eta_n,\xi_n)\}$.

Since $\{\phi\,(\eta_n,\xi_n)\}$ is Cauchy in M'(I) and ϕ is an isometry we have

$[\!]\phi\,(\eta_n,\xi_n) \ominus \phi\,(\eta_m,\xi_m)]\!]' = [\!](\eta_n,\xi_n) \ominus (\eta_m,\xi_m)]\!]$ this implies that $\{(\eta_n,\xi_n)\}$ is Cauchy and belongs to some class $[\eta,\xi]$, say. Now for α in (0,1]

$$[\!]\phi\,(\eta_n,\xi_n) \ominus [\eta,\xi]]\!]'_{(\alpha)} = \lim_{m\to\infty} \lambda_\alpha((\eta_n,\xi_n) \ominus (\eta_m,\xi_m))$$

As $n\to\infty$ we get $\lim_{n\to\infty} [\!]\phi\,(\eta_n,\xi_n) \ominus [\eta,\xi]]\!]'_{(\alpha)} = 0$

ie, $\phi\,(\eta_n,\xi_n)$ converges to $[\eta,\xi]$

For the general case, let $\{[\eta_n,\xi_n]\} \in$ M'(I) be an arbitrary Cauchy sequence. Since $\overline{M_0(I)}$ = M'(I), there exist points in $M_0(I)$, $\phi\,(\eta_1',\xi_1'),\phi\,(\eta_2',\xi_2'),\ldots,\phi\,(\eta_n',\xi_n')$, such that

$$\|[\eta_n,\xi_n] \ominus \phi(\eta_n',\xi_n')\|' \to 0 \text{ as } n \to \infty$$

ie, $\{[\eta_n,\xi_n]\} \leftrightarrow \{\phi(\eta_n,\xi_n)\}$.

Consider

$$\begin{aligned}\|\phi(\eta_n',\xi_n') \ominus \phi(\eta_m',\xi_m')\|' \leq\ & \|\phi(\eta_n',\xi_n') \ominus [\eta_n,\xi_n]\|' \\ & \oplus \|[\eta_m,\xi_m] \ominus \phi(\eta_m',\xi_m')\|' \\ & \oplus \|[\eta_n,\xi_n] \ominus [\eta_m,\xi_m]\|'\end{aligned}$$

This gives that $\{\phi(\eta_n',\xi_n')\}$ is Cauchy hence as in the above case it converges to some $[\eta,\xi] \in M'(I)$. From the inequality

$$\begin{aligned}\|[\eta_n,\xi_n] \ominus [\eta,\xi]\|' \leq\ & \|[\eta_n,\xi_n] \ominus \phi(\eta_n',\xi_n')\|' \\ & \oplus \|\phi(\eta_n',\xi_n') \ominus [\eta,\xi]\|'\end{aligned}$$

it follows that $\{[\eta_n,\xi_n]\}$ converges to $[\eta,\xi]$ ie, $M'(I)$ is complete.

To show that if there exist two completions of $\langle M(I),\|\ \|\rangle$, then they are congruent. Let $\langle M''(I),\|\ \|''\rangle$ be another completion. we show that $\langle M'(I),\|\ \|'\rangle$ is congruent to

$\langle M''(I), [\![\ ]\!]'' \rangle$. Let ϕ'' be the linear isometric imbedding of $M(I)$ into $M''(I)$. Let $\phi''(M(I)) = M_0''(I)$. Since ϕ $(M(I))$ is dense in $M'(I)$, if $[\eta,\xi] \in M'(I)$ then there exists $\{(\eta_n,\xi_n)\}$ in $M(I)$ such that $\{\phi\ (\eta_n,\xi_n)\}$, (we denote it by $\{[\eta_n',\xi_n']\}$) converges to $[\eta,\xi]$. Each of this points has an isometric image in $M''(I)$. Thus the Cauchy sequence, $\{[\eta_n',\xi_n']\}$ in $M'(I)$ gives rise to another Cauchy sequence in $M''(I)$, $\{[\eta_n'',\xi_n'']\}$. Since $M''(I)$ is complete, this sequence must have a limit $[\eta'',\xi'']$ in $M''(I)$.

Define $W: M'(I) \longrightarrow M''(I)$ by

$$W\ ([\eta,\xi]) = [\eta'',\xi''].$$

It can be easily proved that w is 1-1, onto and linear. To show that W is an isometry,

let $\phi\ (\eta_n,\xi_n) = [\eta_n',\xi_n'] \longrightarrow [\eta,\xi] \in M'(I)$ and

$$\phi\ (\alpha_n,\beta_n) = [\alpha_n',\beta_n'] \longrightarrow [\alpha,\beta] \in M'(I)$$

$$\text{as } n \longrightarrow \infty$$

we have $[\eta_n',\xi_n']$ & $[\alpha_n',\beta_n'] \in M_0(I)$ also

$$\| [\eta,\xi] \ominus [\alpha,\beta] \|' \leq \| [\eta,\xi] \ominus [\eta'_n,\xi'_n] \|' \oplus$$

$$\| [\eta'_n,\xi'_n] \ominus [\alpha'_n,\beta'_n] \|' \oplus \| [\alpha'_n,\beta'_n] \ominus [\alpha,\beta] \|'$$

$$= \| [\eta,\xi] \ominus [\eta'_n,\xi'_n] \|' \oplus \| (\eta_n,\xi_n) \ominus (\alpha_n,\beta_n) \|$$

$$\oplus \| [\alpha'_n,\beta'_n] \ominus [\alpha,\beta] \|', \forall\, n \in N$$

(since ϕ is an isometry)

where $[\eta'_n,\xi'_n] = \{(\eta_n,\xi_n),(\eta_n,\xi_n),\ldots\}$ and

$$[\alpha'_n,\beta'_n] = \{(\alpha_n,\beta_n),(\alpha_n,\beta_n),\ldots\}$$

Thus $\| [\eta,\xi] \ominus [\alpha,\beta] \|' \leq \lim\limits_{n\to\infty} \| (\eta_n,\xi_n) \ominus (\alpha_n,\beta_n) \|$ (1)

Also $\| (\eta_n,\xi_n) \ominus (\alpha_n,\beta_n) \| = \| \phi\,(\eta_n,\xi_n) \ominus \phi\,(\alpha_n,\beta_n) \|'$

$$= \| [\eta'_n,\xi'_n] \ominus [\alpha'_n,\beta'_n] \|'$$

$$\leq \| [\eta'_n,\xi'_n] \ominus [\eta,\xi] \|'$$

$$\oplus \| [\eta,\xi] \ominus [\alpha,\beta] \|'$$

$$\oplus \| [\alpha,\beta] \ominus [\alpha'_n,\beta'_n] \|'$$

ie, $\lim_{n\to\infty} [\![(\eta_n,\xi_n) \ominus (\alpha_n,\beta_n)]\!] \le [\![[\eta,\xi] \ominus [\alpha,\beta]]\!]'$ (2)

by (1) & (2)

$[\![[\eta,\xi] \ominus [\alpha,\beta]]\!]' = \lim_{n\to\infty} [\![(\eta_n,\xi_n) \ominus (\alpha_n,\beta_n)]\!]$

Let $[\eta',\xi']$, $[\alpha',\beta'] \in M''(I)$. There exist Cauchy sequences $\{[\eta''_n,\xi''_n]\}$, $\{[\alpha''_n,\beta''_n]\}$ in $M''_0(I)$ such that $\{[\eta''_n,\xi''_n]\}$ converges to $[\eta',\xi']$ and $\{[\alpha''_n,\beta''_n]\}$ converges to $[\alpha',\beta']$.

ϕ'' being a linear isometry $\{\phi''^{-1}[\eta''_n,\xi''_n]\}$ and $\{\phi''^{-1}[\alpha''_n,\beta''_n]\}$ are Cauchy sequences in M(I). So $\{\phi\ (\phi''^{-1}[\eta''_n,\xi''_n])\}$ and $\{\phi\ (\phi''^{-1}[\alpha''_n,\beta''_n])\}$ are Cauchy sequences in $M_0(I)$. But

$\phi\ (\phi''^{-1}[\eta''_n,\xi''_n]) = W^{-1}[\eta''_n,\xi''_n] = [\eta'_n,\xi'_n]$ and

$\phi\ (\phi''^{-1}[\alpha''_n,\beta''_n]) = W^{-1}[\alpha''_n,\beta''_n] = [\alpha'_n,\beta'_n]$.

Being Cauchy sequences these will converge respectively to $[\eta,\xi]$ and $[\alpha,\beta] \in M'(I)$. Thus

$$[\![[\eta',\xi'] \ominus [\alpha',\beta']]\!]'' = \lim_{n\to\infty} [\![[\eta''_n,\xi''_n] \ominus [\alpha''_n,\beta''_n]]\!]''$$

$$= \lim_{n\to\infty} [\![\phi''(\eta_n,\xi_n) \ominus \phi''(\alpha_n,\beta_n)]\!]''$$

$$= \lim_{n\to\infty} [\![(\eta_n,\xi_n) \ominus (\alpha_n,\beta_n)]\!]$$

$$= \lim_{n\to\infty} \lVert \phi\,(\eta_n,\xi_n) \ominus \phi\,(\alpha_n,\beta_n) \rVert'$$

$$= \lVert [\eta,\xi] \ominus [\alpha,\beta] \rVert'$$

So that $\lVert [\eta,\xi] \ominus [\alpha,\beta] \rVert' = \lVert [\eta',\xi'] \ominus [\alpha',\beta'] \rVert''$

$$= \lVert W([\eta,\xi]) \ominus W([\alpha,\beta]) \rVert''$$

Hence W is a linear isometry of M'(I) on to M"(I)

ie, M'(I) is the fuzzy completion of M(I).

Proposition 2.2.3

M'(I) is a partially ordered set with the partial order defined by $[\eta,\xi] \leq [\alpha,\beta]$, iff $(\eta_n,\xi_n) \leq (\alpha_n,\beta_n)$ for large n, for every $\{(\eta_n,\xi_n)\} \in [\eta,\xi]$ and $\{(\alpha_n,\beta_n)\} \in [\alpha,\beta]$.

Proof:

$\leq$ is a partial order for,

(i) $[\eta,\xi] \leq [\eta,\xi]$ since $\{(\eta_n,\xi_n)\}$ and $\{(\eta'_n,\xi'_n)\} \in [\eta,\xi]$

then $\lim_{n\to\infty} \lVert (\eta_n,\xi_n) \ominus (\eta'_n,\xi'_n) \rVert = \bar{0}$

ie, $\lim_{n\to\infty} ((\eta_n,\xi_n) \ominus (\eta'_n,\xi'_n)) = \bar{0}$

ie, $(\eta_n,\xi_n) \le (\eta'_n,\xi'_n)$ for large n.

(ii) Let

$[\eta,\xi] \le [\alpha,\beta]$ & $[\alpha,\beta] \le [\eta,\xi]$ then $(\eta_n,\xi_n) \le (\alpha_n,\beta_n)$

& $(\alpha_n,\beta_n) \le (\eta_n,\xi_n)$ for large n.

Where $\{(\eta_n,\xi_n)\} \in [\eta,\xi]$ and $\{(\alpha_n,\beta_n)\} \in [\alpha,\beta]$

ie, $(\eta_n,\xi_n) \ominus (\alpha_n,\beta_n) \to \bar{0}$, as $n \to \infty$

ie, $\Box[(\eta_n,\xi_n) \ominus (\alpha_n,\beta_n)] \to \bar{0}$, as $n \to \infty$

ie, $\{(\eta_n,\xi_n)\} \leftrightarrow \{(\alpha_n,\beta_n)\}$

ie, $[\eta,\xi] = [\alpha,\beta]$.

(iii) Let $[\eta,\xi] \le [\alpha,\beta]$ & $[\alpha,\beta] \le [\gamma,\delta]$

ie, $(\eta_n,\xi_n) \le (\alpha_n,\beta_n)$ & $(\alpha_n,\beta_n) \le (\gamma_n,\delta_n)$ for large n

$\to (\eta_n,\xi_n) \le (\gamma_n,\delta_n)$ for large n

ie, $[\eta,\xi] \le [\gamma,\delta]$.

Note 2.2.4

With proper understanding of notations we denote $[\![\ ]\!]'$ by $[\![\ ]\!]$

2.3 FUZZY COMPLETION OF C(I)

Proposition 2.3.1

There is a complete fuzzy normed space $\langle C'(I), [\![\ ]\!]' \rangle$ such that C(I) is congruent to a dense subset of C'(I), say $C_0(I)$ and the fuzzy norm on C'(I) extends the fuzzy norm on C(I).

Proof:

Similar to the proof of 2.2.2.

Proposition 2.3.2

$C'(I) = M'(I) \times M'(I)$.

Proof:

We have $C'(I) = \left\{ [(\eta,\xi),(\eta',\xi')] \,\middle|\, (\eta,\xi) \ \& \ (\eta',\xi') \in M(I) \right\}$

let $\{((\eta_{1n},\xi_{1n}),(\eta'_{1n},\xi'_{1n}))\} \in [(\eta_1,\xi_1),(\eta'_1,\xi'_1)]$

ie, $\{((\eta_{1n},\xi_{1n}),(\eta'_{1n},\xi'_{1n}))\}$ is a Cauchy sequence in C(I)

then $\lim_{\substack{n\to\infty\\ m\to\infty}} [\![((\eta_{1n},\xi_{1n}),(\eta'_{1n},\xi'_{1n})) \ominus ((\eta_{1m},\xi_{1m}),(\eta'_{1m},\xi'_{1m}))]\!] = \bar{0}$

then

$$\lim_{\substack{n\to\infty\\ m\to\infty}} [\![((\eta_{1n},\xi_{1n})\ominus(\eta_{1m},\xi_{1m}))^2 \oplus ((\eta'_{1n},\xi'_{1n})\ominus(\eta'_{1m},\xi'_{1m}))^2]\!]^{1/2} = \bar{0}.$$

this is possible only if

$$\lim_{\substack{n\to\infty\\ m\to\infty}} [\![(\eta_{1n},\xi_{1n}) \ominus (\eta_{1m},\xi_{1m})]\!] = \bar{0} \ \&$$

$$\lim_{\substack{n\to\infty\\ m\to\infty}} [\![(\eta'_{1n},\xi'_{1n}) \ominus (\eta'_{1m},\xi'_{1m})]\!] = \bar{0}$$

$\Rightarrow \{(\eta_{1n},\xi_{1n})\} \in [\eta_1,\xi_1]$ &

$\{(\eta'_{1n},\xi'_{1n})\} \in [\eta'_1,\xi'_1]$

ie, $\{((\eta_{1n},\xi_{1n}),(\eta'_{1n},\xi'_{1n}))\} \in [\eta_1,\xi_1]\text{x}[\eta'_1,\xi'_1]$

ie, $[(\eta_1,\xi_1),(\eta'_1,\xi'_1)] \subset [\eta_1,\xi_1]\text{x}[\eta'_1,\xi'_1]$. (3)

Conversely let $\{(\eta_{1n},\xi_{1n})\} \in [\eta_1,\xi_1]$ & $\{(\eta'_{1n},\xi'_{1n})\} \in [\eta'_1,\xi'_1]$

then $\{(\eta_{1n},\xi_{1n})\} \times \{(\eta'_{1n},\xi'_{1n})\}$

$$= \{((\eta_{1n},\xi_{1n}),(\eta'_{1n},\xi'_{1n}))\} \in [(\eta_1,\xi_1),(\eta'_1,\xi'_1)]$$

ie, $[\eta_1,\xi_1] \times [\eta'_1,\xi'_1] \subset [(\eta_1,\xi_1),(\eta'_1,\xi'_1)]$ (4)

by (3) & (4)

$[(\eta_1,\xi_1),(\eta'_1,\xi'_1)] = [\eta_1,\xi_1] \times [\eta'_1,\xi'_1]$

ie, $C'(I) = M'(I) \times M'(I)$

Definition 2.3.3

On $C'(I)$ the product $[(\eta_1,\xi_1),(\eta_2,\xi_2)] \times [(\eta_3,\xi_3),(\eta_4,\xi_4)]$ is defined as the class containing

$\left\{((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n}))\times((\eta_{3n},\xi_{3n}),(\eta_{4n},\xi_{4n}))\right\}$ where

$\{((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n}))\} \in [(\eta_1,\xi_1),(\eta_2,\xi_2)]$ and

$\{((\eta_{3n},\xi_{3n}),(\eta_{4n},\xi_{4n}))\} \in [(\eta_3,\xi_3),(\eta_4,\xi_4)]$.

Note 2.3.4

The above definition is well done for,

let $\{((\eta'_{1n},\xi'_{1n}),(\eta'_{2n},\xi'_{2n}))\} \in [(\eta_1,\xi_1),(\eta_2,\xi_2)]$ and

$\{((\eta'_{3n},\xi'_{3n}),(\eta'_{4n},\xi'_{4n}))\} \in [(\eta_3,\xi_3),(\eta_4,\xi_4)]$ then

$\lim_{n\to\infty} [\![((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n})) \ominus ((\eta'_{1n},\xi'_{1n}),(\eta'_{2n},\xi'_{2n}))]\!] = \bar{0}$ and

$\lim_{n\to\infty} [\![((\eta_{3n},\xi_{3n}),(\eta_{4n},\xi_{4n})) \ominus ((\eta'_{3n},\xi'_{3n}),(\eta'_{4n},\xi'_{4n}))]\!] = \bar{0}$.

To Prove

$$\lim_{n\to\infty} [\![((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n}))((\eta_{3n},\xi_{3n}),(\eta_{4n},\xi_{4n})) - ((\eta'_{1n},\xi'_{1n}),(\eta'_{2n},\xi'_{2n}))((\eta'_{3n},\xi'_{3n}),(\eta'_{4n},\xi'_{4n}))]\!] = \bar{0} \quad (5)$$

Consider

$$\left[\!\left[((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n}))((\eta_{3n},\xi_{3n}),(\eta_{4n},\xi_{4n})) - ((\eta'_{1n},\xi'_{1n}),(\eta'_{2n},\xi'_{2n}))((\eta'_{3n},\xi'_{3n}),(\eta'_{4n},\xi'_{4n}))\right]\!\right]_{(\alpha)}$$

$$= \left[\!\left[((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n}))\left[((\eta_{3n},\xi_{3n}),(\eta_{4n},\xi_{4n})) - ((\eta'_{3n},\xi'_{3n}),(\eta'_{4n},\xi'_{4n}))\right] + ((\eta'_{3n},\xi'_{3n}),(\eta'_{4n},\xi'_{4n}))\left[((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n})) - ((\eta'_{1n},\xi'_{1n}),(\eta'_{2n},\xi'_{2n}))\right]\right]\!\right]_{(\alpha)}$$

$$\leq \left[\!\left[((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n}))\right]\!\right]_{(\alpha)} \times \left[\!\left[((\eta_{3n},\xi_{3n}),(\eta_{4n},\xi_{4n})) - ((\eta'_{3n},\xi'_{3n}),(\eta'_{4n},\xi'_{4n}))\right]\!\right]_{(\alpha)} + \left[\!\left[((\eta'_{3n},\xi'_{3n}),(\eta'_{4n},\xi'_{4n}))\right]\!\right]_{(\alpha)} \times \left[\!\left[((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n})) - ((\eta'_{1n},\xi'_{1n}),(\eta'_{2n},\xi'_{2n}))\right]\!\right]_{(\alpha)}$$

$\longrightarrow 0$ as $n \longrightarrow \infty$. This proves (5).

Proposition 2.3.5

On $C(I)$ define $*$ by $((\eta_1,\xi_1),(\eta_2,\xi_2))*((\eta_3,\xi_3),(\eta_4,\xi_4))$

$= ((\eta_1,\xi_1),(\eta_2,\xi_2)) \times ((\eta_3,\xi_3),(\xi_4,\eta_4))$ and on $C'(I)$

define *' by $[(\eta_1,\xi_1),(\eta_2,\xi_2)]*'[(\eta_3,\xi_3),(\eta_4,\xi_4)]$

$= [(\eta_1,\xi_1),(\eta_2,\xi_2)]\mathrm{x}[(\eta_3,\xi_3),(\xi_4,\eta_4)]$, then

$$[(\eta_1,\xi_1),(\eta_2,\xi_2)]*'[(\eta_3,\xi_3),(\eta_4,\xi_4)]$$

$$= \lim_{n\to\infty}\Big[((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n}))*((\eta_{3n},\xi_{3n}),(\eta_{4n},\xi_{4n}))\Big]$$

where

$\{((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n}))\} \in [(\eta_1,\xi_1),(\eta_2,\xi_2)]$ &

$\{((\eta_{3n},\xi_{3n}),(\eta_{4n},\xi_{4n}))\} \in [(\eta_3,\xi_3),(\eta_4,\xi_4)]$

Proof:

$$[(\eta_1,\xi_1),(\eta_2,\xi_2)]*'[(\eta_3,\xi_3),(\eta_4,\xi_4)] =$$

$$[(\eta_1,\xi_1),(\eta_2,\xi_2)]\mathrm{x}[(\eta_3,\xi_3),(\xi_4,\eta_4)]$$

$$= \lim_{n\to\infty}((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n})) \;\mathrm{x}\; \lim_{n\to\infty}((\eta_{3n},\xi_{3n}),(\xi_{4n},\eta_{4n}))$$

$$= \lim_{n\to\infty}\Big[((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n})) \;\mathrm{x}\; ((\eta_{3n},\xi_{3n}),(\xi_{4n},\eta_{4n}))\Big]$$

$$= \lim_{n\to\infty}\Big[((\eta_{1n},\xi_{1n}),(\eta_{2n},\xi_{2n})) * ((\eta_{3n},\xi_{3n}),(\eta_{4n},\xi_{4n}))\Big]$$

Proposition 2.3.6

$C''(I) = \{[(\eta,\xi),(\bar{0},\bar{0})] \mid \eta,\xi \in R^*(I)\}$ is a partially ordered set with partial order defined by

$[(\eta_1,\xi_1),(\bar{0},\bar{0})] \geq [(\eta_2,\xi_2),(\bar{0},\bar{0})]$ iff $[\eta_1,\xi_1] \geq [\eta_2,\xi_2]$.

Proof:

Follows from 2.2.3·

Proposition 2.3.7

$C'(I)$ is not an integral domain·

Example:

$$\text{Define } \eta(t) = 1, \; t \leq 0$$
$$= .5, \; 0 < t \leq 1$$
$$= 0, \; t > 1$$

then $\eta \neq \bar{0}$, $\eta \neq \bar{1}$ and $\eta^2 = \eta$

consider $[(\eta,\bar{0}),(\bar{0},\bar{0})]$ & $[(\bar{1},\eta),(\bar{0},\bar{0})]$

then $[(\eta,\bar{0}),(\bar{0},\bar{0})] \times [(\bar{1},\eta),(\bar{0},\bar{0})]$

= the class containing $((\eta,\bar{0}),(\bar{0},\bar{0}))\times((\bar{1},\bar{\eta}),(\bar{0},\bar{0}))$

$= \text{the class containing } ((\bar{0},\bar{0}),(\bar{0},\bar{0}))$

$= [(\bar{0},\bar{0}),(\bar{0},\bar{0})].$

Proposition 2.3.8

$\llbracket [\eta,\xi] \rrbracket = \lim_{n\to\infty} \llbracket (\eta_n,\xi_n) \rrbracket$ where $\{(\eta_n,\xi_n)\} \in [\eta,\xi]$

Proof:

$$\llbracket [\eta,\xi] \rrbracket_{(\alpha)} = [0,\lambda_\alpha[\eta,\xi]]$$

$$= [0,\lim_{n\to\infty} \lambda_\alpha(\eta_n,\xi_n)]$$

$$= \lim_{n\to\infty} \llbracket (\eta_n,\xi_n) \rrbracket_{(\alpha)}$$

Hence $\llbracket [\eta,\xi] \rrbracket = \lim_{n\to\infty} \llbracket (\eta_n,\xi_n) \rrbracket$.

Note 2.3.9

(i) It can be easily verified that $C'(I)$ is a real commutative algebra.

(ii) Here after with proper understanding of notations we denote $[\![\ ]\!]'$ by $[\![\ ]\!]$ and $[(\bar{0},\bar{0}),(\bar{0},\bar{0})]$ by $\bar{0}$.

Chapter 3

A FUZZY EXTENSION OF HAHN-BANACH THEOREM*

3.0 INTRODUCTION

In this chapter using the completion M'(I) of M(I) we give a fuzzy extension of real Hahn-Banch theorem. Some consequences of this extension are obtained. The idea of real fuzzy linear functional on fuzzy normed linear space is introduced. Some of its properties are studied. In the complex case we get only a slightly weaker analogue for the Hahn-Banch theorem, than the one [B;N] in the crisp case.

3.1 FUZZY (REAL) HAHN-BANACH THEOREM

Definition 3.1.1

Let X be a real vector space. A real fuzzy linear functional on X is a function $f: X \longrightarrow M'(I)$ satisfying the

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* Some of the results contained in this chapter have been included in a paper communicated for publication in the Tamkang Journal of Mathematics.

following conditions

(i) $f(x+y) = f(x) \oplus f(y)$

(ii) $f(tx) = t\ f(x)$, $\forall\ x,y \in x$ & $t \in R$.

Theorem 3.1.2

Suppose (i) Y is a subspace of a real vector space X

(ii) $p: X \longrightarrow M'(I)$ satisfies $p(x+y) \le p(x) \oplus p(y)$ and

$p(tx) = t\ p(x), \forall\ x,y \in X$ and $t \in R$ such that $t \ge 0$

(iii) $f: Y \longrightarrow M'(I)$ is a real fuzzy linear functional and $f(x) \le p(x)$ on Y, then there exists a real fuzzy linear functional $\Lambda: X \longrightarrow M'(I)$ such that $\Lambda(x) = f(x)$, $\forall\ x \in Y$ and $-p(-x) \le \Lambda\ (x) \le p(x)$, $\forall\ x \in X$.

Proof:

If $Y \ne X$, choose $x_1 \in X$ & $x_1 \notin Y$, define

$Y_1 = \{x+tx_1 : x \in Y \text{ \& } t \in R\}$. Then Y_1 is a subspace of X.

If $x,y \in Y$ we have

$$f(x) \oplus f(y) = f(x+y) \le p(x+y) = p(x-x_1+x_1-y)$$

$$\text{ie } f(x) \oplus f(y) \le p(x+y) = p(x-x_1) \oplus p(x_1+y)$$

ie $f(x) \oplus f(y) \le p(x-x_1) \oplus p(x_1+y)$

hence $f(x) \ominus p(x-x_1) \le p(x_1+y) \ominus f(y)$ (1)

let α be the limit of the left side of (1) as x ranges over Y [such an α exists since M'(I) is complete].

Then $f(x) \ominus p(x-x_1) \le \alpha$

$\Longrightarrow f(x) \ominus \alpha \le p(x-x_1), \forall\ x \in y$ (2)

also $\alpha \le p(y+x_1) \ominus f(y)$

ie $f(y) \oplus \alpha \le p(y+x_1)$ (3)

define f_1 on Y_1 by

$f_1(x+tx_1) = f(x)+t\alpha$, where $x \in Y$ & $t \in R$.

Let $t \in R$, $t>0$, then $t^{-1}x \in Y$

replacing x by $t^{-1}x$ in (2) and multiplying by t we get

$f(tt^{-1}x) \ominus \alpha\ t \le p(x-tx_1)$

ie $f(x) \ominus \alpha\ t \le p(x-tx_1), \forall\ x \in Y$

ie $f_1(x-tx_1) \le p(x-tx_1), \forall\ x \in Y$ (4)

replacing y by $t^{-1}y$ in (3) and multiplying by t we get

$f_1(y+tx_1) \le p(y+tx_1), \forall\ y \in Y$ (5)

by (4) & (5) we get

$f_1 \le p$ on Y_1 and $f_1 = f$ on Y.

Let $\mathscr{A}$ be the collection of all ordered pairs (Y',f') where Y' is a subspace of X that contains Y and f' a fuzzy linear functional on Y' that extends f and satisfies $f' \leq p$ on Y'. $\mathscr{A}$ is partially ordered by the order $\leq$ defined by $(Y',f') \leq (Y'',f'')$ if $Y' \subset Y''$ & $f' = f''$ on Y' by Hausdorff's maximality theorem there exists a maximal totally ordered sub collection Ω of $\mathscr{A}$. Let K be the collection of all Y' such that $(Y',f') \in \Omega$, then K is totally ordered by set inclusion and $\bar{Y}$, the union of all members of K is then a subspace of X. If $x \in \bar{Y}$ then $x \in Y'$ for some $Y' \in K$. Define $\Lambda(x) = f'(x)$, where f' is the function which occurs in the pair $(y',f') \in \Omega$. Hence Λ is linear, $\Lambda \leq p$ & $\bar{Y} = X$.

Thus there exists a fuzzy linear functional such that $\Lambda(x) = f(x)$ on Y and $\Lambda : X \longrightarrow M'(I)$ & $\Lambda \leq p$.

$\Lambda \leq p \longrightarrow \Lambda(x) \leq p(x)$ on X

$\longrightarrow -\Lambda(x) \geq -p(x)$

ie $-p(-x) \leq -\Lambda(-x) = -\bar{1}.-\bar{1}\,\Lambda(x) = \Lambda(x)$

ie $-p(-x) \leq \Lambda(x) \leq p(x)$, $\forall\, x \in X$.

Theorem 3.1.3

Suppose Y is a subspace of a real vector space X, p a fuzzy norm on X and f a real fuzzy linear functional on Y such that $|f(x)| \leq p(x), \forall\, x \in Y$. Then f extends to a real fuzzy linear functional Λ on X such that

$|\Lambda\,(x)| \leq p(x),\ \forall\, x \in X.$

Proof:

$|f(x)| \leq p(x),\ \forall\, x \in Y$

$\Longrightarrow f(x) \leq p(x)\ \&\ \ p(x) \geq \bar{0},\ \forall\, x \in Y$

by 3.1.2, there exists a fuzzy linear functional Λ such that

$\Lambda \leq p$ and $-p(-x) \leq \Lambda\,(x) \leq p(x),\ \forall\, x \in X.$

Since p is a fuzzy norm $p(-x) = p(x) \geq \bar{0}$, we get

$-p(x) \leq \Lambda\,(x) \leq p(x),\ \forall\, x \in X$

ie $|\Lambda\,(x)| \leq p(x),\ \forall\, x \in X.$

Corollary 3.1.4

Let X be a fuzzy normed space and $x_0 \in X$, then there exists a real fuzzy linear functional Λ such that $\Lambda(x_0) = \|x_0\|$ and $[\!(\Lambda(x))\!] \leq \|x\|, \forall x \in X$.

Proof:

In 3.1.3 take $p(x) = \|x\|$, $Y =$ linear span of x_0 and $f(tx_0) = t\|x_0\|$ in Y.

Definition 3.1.5

Let X be a fuzzy normed space. A fuzzy linear functional f on X is said to be bounded if there exists a $k \in R^*(I)$ such that $[\!(f(x))\!] \leq k\|x\|, \forall x \in X$.

Definition 3.1.6

Let f be a bounded fuzzy linear functional on a fuzzy normed space X. Then $\|f\|$ is defined as

$$\|f\| = \inf \left\{ k \in R^*(I) \mid [\!(f(x))\!] \leq k\|x\|, \forall x \in X \right\}.$$

Proposition 3.1.7

$\| \ \|$ defined above is a fuzzy norm on the fuzzy dual space X_L of X.

Proof:

(i) If f = 0, the zero functional then $\|f\| = \bar{0}$.

If $f \neq 0$, then $f(x) \neq [\bar{0},\bar{0}]$ for some $x \neq 0$

ie $[\![f(x)]\!] \neq 0$, for some $x \neq 0$

ie $\|f\| \neq 0$

(ii) Let $f,g \in X_L$, then

$$\|f+g\| = \inf \left\{k \in R^*(I) \mid [\![f(x) \oplus g(x)]\!] \leq k \|x\|\right\}$$

$$\leq \inf \left\{k_1+k_2 \in R^*(I) \mid [\![f(x)]\!] \oplus [\![g(x)]\!] \leq (k_1+k_2)\|x\|\right\}$$

$$\leq \inf \left\{k_1 \in R^*(I) \mid [\![f(x)]\!] \leq k_1\|x\|\right\}$$

$$\oplus \inf \left\{k_2 \in R^*(I) \mid [\![g(x)]\!] \leq k_2\|x\|\right\}$$

ie $\|f+g\| \leq \|f\| \oplus \|g\|$

(iii) Let $\ell \in R$

$$\text{then } \|\ell f\| = \inf \{k \in R^*(I) \mid [\![\ell f(x)]\!] \leq k \|x\|\}$$

$$= \inf \{k \in R^*(I) \mid |\ell| [\![f(x)]\!] \leq k \|x\|\}$$

$$\leq \inf \{|\ell| k_1 \in R^*(I) \mid |\ell| [\![f(x)]\!] \leq |\ell| k_1 \|x\|\}$$

$$= |\ell| \inf \{k_1 \in R^*(I) \mid [\![f(x)]\!] \leq k_1 \|x\|\}$$

$$\text{ie } \|\ell f\| \leq |\ell| \|f\| \tag{6}$$

$$\text{also } \|f\| = \|\frac{1}{\ell} \ell f\| \leq \frac{1}{|\ell|} \|\ell f\|$$

$$\text{ie } |\ell| \|f\| \leq \|\ell f\| \tag{7}$$

by (6) and (7)

$$\|\ell f\| = |\ell| \|f\|.$$

Remark 3.1.8

If f is a bounded fuzzy linear map then $[\![f(x)]\!] \leq \|f\| \|x\|$.

Corollary 3.1.9

Let X be a fuzzy normed space. Then corresponding to every $x_0 \in X$, there exists a bounded real fuzzy linear map f_{x_0} on

X such that $f_{x_0}(x_0) = \|x_0\|^2$ and $\|f_{x_0}\| \leq \|x_0\|$.

Proof:

Take Y = linear span of x_0. Then Y will be a subspace of X. Define $f:Y \longrightarrow M'(I)$ by $f(tx_0) = t\|x_0\|^2$, then f is a real fuzzy linear map on Y.

Take $p(x) = \|x_0\| \ \|x\|$

then $f(x) \leq p(x)$ on Y.

Also $p(x+y) \leq p(x) \oplus p(y)$ and $p(\alpha x) = \alpha p(x)$.

Hence by 3.1.2 there exists a real fuzzy linear functional

$f_{x_0} \ X \longrightarrow M'(I)$ such that $f_{x_0}(x) = f(x)$ on Y and

$f_{x_0}(x) \leq p(x), \ \forall x \in X$.

Also $f_{x_0}(x_0) = f(x_0) = \|x_0\|^2$ and

$$[\![f_{x_0}(x)]\!] \leq p(x) = \|x_0\| \ \|x\|$$

ie $\|fx_0\| \leq \|x_0\|$

Theorem 3.1.10

Suppose f be a bounded real fuzzy linear functional on a fuzzy normed subspace Y of a fuzzy normed space X. Then there exists a bounded real fuzzy linear functional F, extending f, defined on the whole space having the same fuzzy norm as F.

Proof:

We have $[\![f(x)]\!] \leq \|f\| \, \|x\|$

define $p(x) = \|f\| \, \|x\| \quad \forall \; x \in X$

then $p(x+y) \leq p(x) \oplus p(y)$

$$p(tx) = |t| \; p(x).$$

Also $[\![f(x)]\!] \leq p(x), \; \forall \; x \in Y.$

By 3.1.2 we can extend f to a new fuzzy linear functional F, defined on all of X such that

$$[\![F(x)]\!] \leq p(x) = \|f\| \, \|x\|.$$

In view of this result it is clear that F is a bounded fuzzy linear functional and also that

$$\|F\| \leq \|f\| \qquad (8)$$

Also we have

$$\|F\| = \inf\{ \, k \in R^*(I) \mid [\![F(x)]\!] \leq k \, \|x\| \, \}$$

when $x \in Y$

$$[\![f(x)]\!] = [\![F(x)]\!] \leq \|F\| \; \|x\|$$

ie $\|f\| \leq \|F\|$ (9)

by (8) and (9) $\|f\| = \|F\|$.

Theorem 3.1.11

Let x_0 be a nonzero vector in the fuzzy normed linear space X. Then there exists a bounded real fuzzy linear functional F, defined on the whole space, such that

$\|F\| = \bar{1}$ and $F(x_0) = \|x_0\|$.

Proof:

Let $Y = \text{span}\,\{x_0\}$. Consider f on Y defined by

$f(\alpha\, x_0) = \alpha \,\|x_0\|$

clearly f is a real fuzzy linear functional with

$f(x_0) = \|x_0\|$

further for any $x \in Y$

$[\![f(x)]\!] = |\alpha| \; \|x_0\| = \|\alpha\, x_0\| = \|x\|$ (10)

ie f is a bounded fuzzy linear functional on Y also $\|f\| \leq \bar{1}$.

If k be a real number such that $k<1$ and

$[\!(f(x))\!] \leq \bar{k} \, \|x\|, \ \forall \ x \in Y$

this would contradict the equality of (10)

hence $\|f\| = \bar{1}$

ie f is a bounded fuzzy linear map on Y, by 3.1.10 there exists a bounded fuzzy linear functional F on X extending f, and having the same norm as F, that is $\|F\| = \bar{1}$ and $F(x_0) = \|x_0\|$.

Remark 3.1.12

(i) If x is not a trivial space, the fuzzy dual space is not trivial. That is nonzero bounded fuzzy linear functional must exist on any nontrivial fuzzy normed space.

(ii) If all the bounded fuzzy linear functionals vanish on a given vector, the vector must be zero. Since one of the bounded fuzzy linear functionals, when applied to the vector, must assume the norm of the vector as its value, the norm must be zero. That is the vector is zero.

3.2 FUZZY (COMPLEX) HAHN-BANACH THEOREM

Note 3.2.1

We do not get an exact analogue for the Hahn-Banach theorem in the complex case. However, we get a slightly weaker form which is given in the theorem below.

Note 3.2.2

Let f be a function from a real or complex vector space X to C'(I).

Suppose $f(x) = [(\eta_1,\xi_1),(\eta_2,\xi_2)] \in C'(I)$

write $f(x) = [(\eta_1,\xi_1),(\bar{0},\bar{0})]+[(\bar{0},\bar{0}),(\eta_2,\xi_2)]$

$= [(\eta_1,\xi_1),(\bar{0},\bar{0})]+i[(\eta_2,\xi_2),(\bar{0},\bar{0})]$

where

$i = [(\bar{0},\bar{0}),(\bar{1},\bar{0})]$

ie $f(x) = f_1(x)+if_2(x)$

Theorem 3.2.3

Suppose

(i) Y is a subspace of a complex Vector space X

(ii) p: $X \longrightarrow M'(I)$ satisfies $p(x+y) \leq p(x) \oplus p(y)$ and $p(\alpha x) = |\alpha| p(x)$ for every complex number α and $x,y \in X$.

(iii) f: $Y \longrightarrow C'(I)$ is a fuzzy linear functional and $[\![f(x)]\!] \leq p(x)$ on Y, then there exists a fuzzy linear functional Λ: $X \longrightarrow M'(I)$ such that $\Lambda(x) = f(x)$ on Y and $[\![\Lambda(x)]\!] \leq 2p(x)$ for all $x \in X$.

Proof:

We have $f(x) = f_1(x)+if_2(x)$ we claim that $f_1(x)$ & $f_2(x)$ are real fuzzy linear functionals. By a real fuzzy linear function we mean the following: g is a real fuzzy linear functional on the complex vector space V. If α is a real number implies $g(\alpha x) = \alpha g(x)$ and $g(x+y) = g(x) \oplus g(y)$ for every $x,y \in V$.

To prove f_1 and f_2 have this property, let α be a real number and consider

$$\alpha\, f(x) = \alpha\, f_1(x)+i\alpha\, f_2(x)$$

since f is a fuzzy linear functional, this must equal to

$$f(\alpha\, x) = f_1(\alpha\, x)+if_2(\alpha\, x)$$

ie $f_1(\alpha\, x) = \alpha\, f_1(x)$ and $f_2(\alpha\, x) = \alpha\, f_2(x)$.

In a similar way we can show that sums are also preserved.

Now consider

$$i(f_1(x)+if_2(x)) = if(x) = f(ix) = f_1(ix)+if_2(ix)$$

$$\text{ie } f_1(ix) = -f_2(x)$$

$$\text{ie } f(x) = f_1(x) - if_1(ix)$$

also $$[\![f(x)]\!] \leq p(x) \quad \forall\, x \in Y$$

$$\text{ie } f_1(x) \leq p(x), \quad \forall\, x \in Y$$

by 3.1.2. there exists a real fuzzy linear functional Λ_1 defined on X extending f_1 and satisfying

$$\Lambda_1(x) \leq p(x)$$

For every $x \in X$ we define

$$\Lambda\, (x) = \Lambda_1(x) - i\Lambda_1(ix)$$

Λ_1 extends f_1, so

$\Lambda_1(x) = f_1(x)$ and $\Lambda_1(ix) = f_1(ix) = -f_2(x)$

thus, $\Lambda(x) = f_1(x)+if_2(x) = f(x)$

ie Λ is an extension of f. Since, Λ is clearly a real fuzzy linear functional it only remains to show that

$$\Lambda(ix) = i\Lambda(x)$$

Consider $\Lambda(ix) = \Lambda_1(ix) - i\Lambda_1(-x)= \Lambda_1(ix)+i\Lambda_1(x)$

comparing this to $i\Lambda(x) = i\Lambda_1(x)+\Lambda_1(ix)$

we get $\Lambda(ix) = i\Lambda(x)$

ie Λ is a complex fuzzy linear functional on X which extends f.

Also we have $\Lambda(x) = \Lambda_1(x) - i\Lambda_1(ix)$

$$[\![\Lambda(x)]\!] = [\![\Lambda_1(x)-i\Lambda_1(ix)]\!] \leq [\![\Lambda_1(x)]\!]+[\![\Lambda_1(ix)]\!]$$

$$= [\![\Lambda_1(x)]\!]+[\![\Lambda_1(x)]\!]$$

ie $[\![\Lambda(x)]\!] \leq [\![\Lambda_1(x)]\!]+[\![\Lambda_1(x)]\!] \leq p(x) \oplus p(x) = 2p(x).$

Chapter 4

FUZZY SEMI INNER PRODUCT SPACES*

4.0 INTRODUCTION

Lumer, G [LUM] introduced the idea of semi inner product space with a more general axiom system than that of inner product space. The importance of semi inner product is that whether the norm satisfies the parallelogram law or not, every normed space can be represented as a semi inner product space, so that the theory of operators can be extended further by Hilbert space type arguments. Parallel to this on a C'(I) module we are able to introduce the notion of fuzzy semi inner product. We prove that a fuzzy semi inner product generates a fuzzy norm and further that every fuzzy normed space can be made into a fuzzy semi inner product space.

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*Some results contained in this chapter have been included in a paper accepted for publication in The Journal of Fuzzy Mathematics.

Also the notion of fuzzy orthogonal set is introduced. Existence of a complete fuzzy orthogonal set is established. The concept of generalized fuzzy semi inner product is introduced.

4.1 FUZZY SEMI INNER PRODUCT

Definition 4.1.1

A fuzzy semi inner product on a C'(I) module x is a function $* \quad X x X \longrightarrow C'(I)$ which satisfies the following conditions

(i) $(x+y)*z = x*z+y*z$

$(\lambda x)*y = \lambda\ (x*y)$

ie, * is linear in the first argument where $x,y,z \in X$ and $\lambda \in C'(I)$

(ii) $x*x > \bar{0}$ for every nonzero $x \in X$

(iii) $[\![x*y]\!]^2 \leq [\![x*x]\!] [\![y*y]\!]$

then <X,*> is called a fuzzy semi inner product space.

Note 4.1.2

(i) If * is linear in first and conjugate linear in the second arguments also satisfies the above conditions (i) & (ii) then <x,*> will be called a fuzzy inner product space. Clearly a fuzzy inner product space is a fuzzy semi inner product space.

(ii) The conjugate of $[(\eta,\xi),(\eta',\xi')]$ is $[(\eta,\xi),(\xi',\eta')]$.

Theorem 4.1.3

Let <X,*> be a fuzzy semi inner product space. Considering X as a real vector space the function $\| \ \| : X \longrightarrow R^*(I)$ defined by $\|x\| = [\![x*x]\!]^{1/2}$ is a fuzzy norm on X.

Proof:

(i) $x*x > \bar{0}$ for every nonzero x

ie, $[\![x*x]\!] > \bar{0}$

$$\|x\|^2 > \bar{0}$$

$$\|x\| > \bar{0}$$

(ii) $\|x+y\|^2 = [\![(x+y)*(x+y)]\!]$

$$= [\![x*(x+y)+y*(x+y)]\!]$$

$$\leq [\![x^*(x+y)]\!] \oplus [\![y^*(x+y)]\!]$$

$$\leq [\![x^*x]\!]^{1/2}[\![(x+y)^*(x+y)]\!]^{1/2} \oplus$$

$$[\![y^*y]\!]^{1/2}[\![(x+y)^*(x+y]\!]^{1/2}$$

ie, $\|x+y\|^2 \leq \|x\| \; \|x+y\| \oplus \|y\| \; \|x+y\|$

$$\|x+y\|^2 \leq (\|x\| \oplus \|y\|) \; \|x+y\|$$

ie, $\|x+y\| \leq \|x\| \oplus \|y\|$

(iii) Let $t \in R$ & $t \neq 0$

Consider $\|tx\|^2 = [\![tx^*tx]\!]$

$$\|tx\|^2 \leq |t| \; [\![x^*tx]\!]$$

$$\|tx\|^2 \leq |t| \; \|x\| \; \|tx\|$$

ie $\|tx\| \leq |t| \; \|x\|$ (1)

also $\|x\| = \|\frac{1}{t} tx\| \leq \frac{1}{|t|} \|tx\|$

ie, $|t| \; \|x\| \leq \|tx\|$ (2)

by (1) & (2) $\|tx\| = |t|\|x\|$.

When $t = 0$, $tx = 0$, $|t| = 0$

hence $\|tx\| = \bar{0} = |\bar{0}|\|x\|$

ie, $\| \ \|$ is a fuzzy norm on X.

Note 4.1.4

Let $\langle X,*\rangle$ be a fuzzy semi inner product space. If $\| \ \|$ is the fuzzy norm generated from the fuzzy semi inner product *, then the fuzzy semi inner product space is denoted by $\langle X,*,\| \ \|\rangle$.

Theorem 4.1.5

On C'(I) define * by

$$[(\eta_1,\xi_1),(\eta_1',\xi_1')] * [(\eta_2,\xi_2),(\eta_2',\xi_2')] =$$

$$[(\eta_1,\xi_1),(\eta_1',\xi_1')] \times [(\eta_2,\xi_2),(\xi_2',\eta_2')] \text{ then}$$

$\langle C'(I),*,[\![\ ]\!]\rangle$ is a fuzzy semi inner product space.

Proof:

(i) $$\left\{[(\eta_1,\xi_1),(\eta_1',\xi_1')]+[(\eta_2,\xi_2),(\eta_2',\xi_2')]\right\} * [(\eta_3,\xi_3),(\eta_3',\xi_3')]$$

$$= [(\eta_1,\xi_1),(\eta_1',\xi_1')] * [(\eta_3,\xi_3),(\eta_3',\xi_3')]+ [(\eta_2,\xi_2),(\eta_2',\xi_2')] * [(\eta_3,\xi_3),(\eta_3',\xi_3')]$$

also if $t \in R$ then

$$\left\{t[(\eta_1,\xi_1),(\eta_1',\xi_1')]\right\} * [(\eta_2,\xi_2),(\eta_2',\xi_2')] = t\left\{[(\eta_1,\xi_1),(\eta_1',\xi_1')] * [(\eta_2,\xi_2),(\eta_2',\xi_2')]\right\}$$

(ii) $$[(\eta_1,\xi_1),(\eta_1',\xi_1')] * [(\eta_1,\xi_1),(\eta_1',\xi_1')] = [(\eta_1,\xi_1),(\eta_1',\xi_1')]\times[(\eta_1,\xi_1),(\xi_1',\eta_1')]$$

$$= \lim_{n\to\infty} ((\eta_{1n},\xi_{1n}),(\eta_{1n}',\xi_{1n}'))((\eta_{1n},\xi_{1n}),(\xi_{1n}',\eta_{1n}'))$$

$$= \lim_{n\to\infty} ((\eta_{1n},\xi_{1n})^2 \oplus (\eta_{1n}',\xi_{1n}')^2,\ (\bar{0},\bar{0}))$$

where $\left\{((\eta_{1n},\xi_{1n}),(\eta'_{1n},\xi'_{1n}))\right\} \in [(\eta_1,\xi_1),(\eta'_1,\xi'_1)]$

let $[(\eta_1,\xi_1),(\eta'_1,\xi'_1)] \neq \bar{0}$

then $[(\eta_1,\xi_1),(\eta'_1,\xi'_1)] * [(\eta_1,\xi_1),(\eta'_1,\xi'_1)] = \bar{0}$

iff $\lim_{n\to\infty} ((\eta_{1n},\xi_{1n})^2 \oplus (\eta'_{1n},\xi'_{1n})^2 = \bar{0}$

iff $\lim_{n\to\infty} ((\eta_{1n},\xi_{1n})^2 = \bar{0}$ & $\lim_{n\to\infty} (\eta'_{1n},\xi'_{1n})^2 = \bar{0}$

iff $\lim_{n\to\infty} [\![((\eta_{1n},\xi_{1n}),(\eta'_{1n},\xi'_{1n})) - ((\eta_1,\xi_1),(\eta'_1,\xi'_1))]\!] \neq \bar{0}$

iff $\left\{((\eta_{1n},\xi_{1n}),(\eta'_{1n},\xi'_{1n}))\right\} \notin [(\eta_1,\xi_1),(\eta'_1,\xi'_1)]$

but this is not the case

hence $[(\eta_1,\xi_1),(\eta'_1,\xi'_1)] * [(\eta_1,\xi_1),(\eta'_1,\xi'_1)] > \bar{0}$

when $[(\eta_1,\xi_1),(\eta'_1,\xi'_1)] \neq \bar{0}$

(iii) Consider $[\![[(\eta_1,\xi_1),(\eta'_1,\xi'_1)] * [(\eta_2,\xi_2),(\eta'_2,\xi'_2)]]\!]^2$

$= [\![[(\eta_1,\xi_1),(\eta'_1,\xi'_1)] x [(\eta_2,\xi_2),(\xi'_2,\eta'_2)]]\!]^2$

$= \lim_{n\to\infty} [\![((\eta_{1n},\xi_{1n}),(\eta'_{1n},\xi'_{1n})) x ((\eta_{2n},\xi_{2n}),(\xi'_{2n},\eta'_{2n}))]\!]^2$

where $\left\{((\eta_{1n},\xi_{1n}),(\eta'_{1n},\xi'_{1n}))\right\} \in [(\eta_1,\xi_1),(\eta'_1,\xi'_1)]$ and

$\left\{((\eta_{2n},\xi_{2n}),(\eta'_{2n},\xi'_{2n}))\right\} \in [(\eta_2,\xi_2),(\eta'_2,\xi'_2)]$

ie, $[\![[(\eta_1,\xi_1),(\eta'_1,\xi'_1)] * [(\eta_2,\xi_2),(\eta'_2,\xi'_2)]]\!]^2 =$

$$\lim_{n\to\infty} [\![((\eta_{1n},\xi_{1n})(\eta_{2n},\xi_{2n}) \oplus (\eta'_{1n},\xi'_{1n})(\eta'_{2n},\xi'_{2n}),$$

$$(\eta'_{1n},\xi'_{1n})(\eta_{2n},\xi_{2n}) \ominus (\eta_{1n},\xi_{1n})(\eta'_{2n},\xi'_{2n}))]\!]^2$$

$$= \lim_{n\to\infty} \Box((\eta_{1n},\xi_{1n})(\eta_{2n},\xi_{2n}) \oplus (\eta'_{1n},\xi'_{1n})\ (\eta'_{2n},\xi'_{2n}))^2 \oplus$$

$$((\eta'_{1n},\xi'_{1n})(\eta_{2n},\xi_{2n}) \ominus (\eta_{1n},\xi_{1n})(\eta'_{2n},\xi'_{2n}))^2 \Box$$

$$= \lim_{n\to\infty} \Box((\eta_{1n},\xi_{1n})^2 \oplus (\eta'_{1n},\xi'_{1n})^2)\ ((\eta_{2n},\xi_{2n})^2 \oplus (\eta'_{2n},\xi'_{2n})^2) \Box$$

$$\le \lim_{n\to\infty} \left\{ \Box(\eta_{1n},\xi_{1n})^2 \oplus (\eta'_{1n},\xi'_{1n})^2 \Box\ \Box(\eta_{2n},\xi_{2n})^2 \oplus (\eta'_{2n},\xi'_{2n})^2 \Box \right\}$$

$$= \lim_{n\to\infty} \left\{ [\![((\eta_{1n},\xi_{1n}),(\eta'_{1n},\xi'_{1n})) x ((\eta_{1n},\xi_{1n}),(\xi'_{1n},\eta'_{1n}))]\!] x \right.$$

$$\left. [\![((\eta_{2n},\xi_{2n}),(\eta'_{2n},\xi'_{2n})) x ((\eta_{2n},\xi_{2n}),(\xi'_{2n},\eta'_{2n}))]\!] \right\}$$

ie, $[\![[(\eta_1,\xi_1),(\eta_1',\xi_1')] * [(\eta_2,\xi_2),(\eta_2',\xi_2')]]\!]^2 \le$

$$[\![[(\eta_1,\xi_1),(\eta_1',\xi_1')] * [(\eta_1,\xi_1),(\eta_1',\xi_1')]]\!] \times$$

$$[\![[(\eta_2,\xi_2),(\eta_2',\xi_2')] * [(\eta_2,\xi_2),(\eta_2',\xi_2')]]\!] \cdot$$

Theorem 4.1.6

Every fuzzy normed space can be made into a fuzzy semi inner product space.

Proof:

Let X be a fuzzy normed space. By fuzzy Hahn-Banach theorem corresponding to every $x_0 \in X$ there exists a bounded fuzzy linear map f_{x_0} On X such that $f_{x_0}(x_0) = \|x_0\|^2$ and

$\|f_{x_0}\| \le \|x_0\|$

define $f_y(x) = x*y$

then

(i) $f_y(x_1+x_2) = (x_1+x_2)*y = f_y(x_1)+f_y(x_2)$

$$= x_1 * y + x_2 * y$$

$$f_y(\alpha x) = (\alpha x) * y = \alpha f_y(x) = \alpha (x * y)$$

(ii) $f_y(y) = \|y\|^2 > \bar{0}$ ie, $y * y > \bar{0}$, when $y \neq 0$

(iii) Consider $|f_y(x)| \leq \|f_y\| \; \|x\| \leq \|y\| \; \|x\|$

ie, $|f_y(x)|^2 \leq \|y\|^2 \|x\|^2$

$$= f_y(y) \; f_x(x) = |f_y(y)| \; |f_x(x)|.$$

4.2 FUZZY ORTHOGONAL SET

Definition 4.2.1

A subset A of a fuzzy semi inner product space $\langle X, *, \| \; \| \rangle$ is said to be fuzzy orthogonal in X if $x * y = \bar{0}$ for every $x, y \in A$.

Definition 4.2.2

A fuzzy orthogonal set A in a fuzzy semi inner product space is said to be complete if there exist no other fuzzy orthogonal set properly containing A.

Proposition 4.2.3

A fuzzy orthogonal set A in <X,*> is complete iff for any x such that $x \perp A$, x must be zero.

Proof:

Suppose A is complete and x is a nonzero element of X such that $x \perp A$, clearly this is contradictory because the fuzzy orthogonal set A U {x} contains A, properly and contradicts the maximality of A.

Conversely suppose the above condition is satisfied. That is $x \perp A$ implies $x = 0$. If A is not complete, there exist some fuzzy orthogonal set B such that B properly contains A. In such case there exists an $x \in B-A$, where $x \perp A$ and $x \neq 0$, this is a contradiction. ie, A is complete.

Theorem 4.2.4

Let <X,*> be a fuzzy semi inner product space.

(i) There exists a complete fuzzy orthogonal set in X.

(ii) Any fuzzy orthogonal set can be extended to a complete fuzzy othogonal set.

Proof:

It is clear if (ii) can be proved, this will imply (i). By virtue of the fact that in any fuzzy semi inner product space, fuzzy orthogonal sets must exist for, any nonzero vector x, {x} is a fuzzy orthogonal set. Hence we shall prove (ii).

Let A be a fuzzy orthogonal set and $\mathscr{A}$ be the collection of all fuzzy orthogonal sets containing A. Then $\mathscr{A}$ is partially ordered by set inclusion. Let T be a totally ordered subset of $\mathscr{A}$ Let

$T = \{A_\alpha\}$ $\alpha \in \Lambda$, for any α, $A_\alpha \subset \bigcup_\alpha A_\alpha$ also $A \subset \bigcup_\alpha A_\alpha$

Let $x,y \in \bigcup_{\alpha} A_{\alpha} \Rightarrow$ there exist A_{α} & A_{β} such that $x \in A_{\alpha}$ &

$y \in A_{\beta}$

Since T is totally ordered either $A_{\alpha} \subset A_{\beta}$ or $A_{\beta} \subset A_{\alpha}$ suppose the former inclusion holds, then we can say $x,y \in A_{\beta}$

then $x \perp y$, hence $\bigcup_{\alpha} A_{\alpha} \in \mathscr{A}$.

Then $\bigcup_{\alpha} A_{\alpha}$ is an upper bond for T in $\mathscr{A}$.

Hence by Zorn's lemma there must exist a maximal element in $\mathscr{A}$. Because of the maximality no other fuzzy orthogonal set containing this maximal element.

4.3. GENERALIZED FUZZY SEMI INNER PRODUCT

Definition 4.3.1

A C'(I) module E is called a generalized fuzzy semi inner product space if

(i) There is a submodule M of E which is a fuzzy semi inner product space, and

(ii) there is a nonempty set α of fuzzy linear operators on E which has the following properties.

(a) $\alpha E \subset M$

(b) if $Tx = 0$, $\forall\ T \in \alpha$ then $x = 0$

A generalized fuzzy semi inner product space is represented by the triple (E,α,M).

Remark 4.3.2

Every fuzzy semi inner product space is a generalized fuzzy semi inner product space.

Proposition 4.3.3

Let (E,α,M) be a generalized fuzzy semi inner product space and $x \in E$

(a) if $Tx*y = \bar{0}$, $\forall\ y \in M$, T in α, then $x = 0$

(b) if $Tx*Tx = \bar{0}$, $\forall$ T in α, then $x = 0$.

Proof:

(a) $Tx*y = \bar{0}$, $\forall\ y \in M$

in particular $Tx*Tx = \bar{0}$

$\Rightarrow Tx = 0, \forall T \in \alpha$

ie, $x = 0$.

(b) $Tx^*Tx = \bar{0} \Rightarrow Tx = 0, \forall T \in \alpha \Rightarrow x = 0$.

Chapter 5

FUZZY SEMI INNER PRODUCT OF FUZZY POINTS*

5.0 INTRODUCTION

In this chapter we extend the idea of fuzzy semi inner product space of crisp points to that of fuzzy points. The notion of orthogonality on the fuzzy semi inner product of fuzzy points is introduced. Some of its properties are studied. Also the concepts like fuzzy numerical range of 'fuzzy linear maps' on the set of fuzzy points is introduced and some results are obtained.

5.1 FUZZY SEMI INNER PRODUCT OF FUZZY POINTS

Definition 5.1.1 [WO_3]

Let X be a set, then a fuzzy subset x_λ, where $\lambda \in (0,1]$ is

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* Some results contained in this chapter have been included in a paper communicated for publication in the International Journal for Fuzzy Sets and Systems.

called a fuzzy point on X

if

$$x_\lambda(y) = \lambda, \text{ if } y = x$$
$$= 0, \text{ otherwise}$$

Note 5.1.2

Let X be a C'(I) module. If $\alpha \in C'(I)$ and x_λ & y_μ be two fuzzy points on X then

(a) $\alpha\, x_\lambda$ is defined as the fuzzy point $(\alpha\, x)_\lambda$.

(b) $x_\lambda \oplus y_\mu$ is the fuzzy point which takes the value $\lambda \wedge \mu$ at x+y

(c) The set of all fuzzy points on X is denoted by $\hat{X}$

Definition 5.1.3

Let X be a C'(I) module. A fuzzy semi inner product of fuzzy points on X is a function

$* : \hat{X} \times \hat{X} \longrightarrow C'(I)$ satisfying the following conditions

(i) * is a linear in first argument

ie $(x_\lambda \oplus y_\mu) * z_\psi = x_\lambda * z_\psi + y_\mu * z_\psi$ and

$(\alpha\, x_\lambda) * z_\psi = \alpha\, (x_\lambda * z_\psi)$

where x_λ, y_μ and $z_\psi \in \hat{X}$ & $\alpha \in C'(I)$.

(ii) $x_\lambda * x_\lambda > \bar{0}$, $\forall\, x_\lambda \neq 0_\beta$, where $\beta \in (0,1]$

(iii) $[\![x_\lambda * y_\mu]\!]^2 \leq [\![x_\lambda * x_\lambda]\!]\, [\![y_\mu * y_\mu]\!]$, where x_λ, y_μ and $z_\psi \in \hat{X}$,

then $\langle \hat{X}, * \rangle$ is called a fuzzy semi inner product space of fuzzy points.

Note 5.1.4

As $\hat{X}$ is not a fuzzy linear space the term `fuzzy semi inner product' used above is not in the usual sense.

Theorem 5.1.5

Let $\langle \hat{X}, * \rangle$ be a fuzzy semi inner product space of fuzzy points. Then treating X as a real vector space, the function $\|\ \| : \hat{X} \to R^*(I)$ defined by $\|x_\lambda\| = [\![x_\lambda * x_\lambda]\!]^{1/2}$ is a fuzzy norm on $\hat{X}$.

Proof:

Similar to the proof of 4.1.3.

Note 5.1.6

Let $<\hat{X},*>$ be a fuzzy semi inner product space of fuzzy points. If $\| \ \|$ is the fuzzy norm generated from the fuzzy semi inner product * of fuzzy points. Then the fuzzy semi inner product space of fuzzy points is denoted by $<\hat{X},*,\| \ \|>$.

Definition 5.1.7

Let $<\hat{X},*,\| \ \|>$ be a fuzzy semi inner product space of fuzzy points. Let $x_\alpha, y_\beta \in \hat{X}$, then x_α is said to be orthogonal to y_β (denoted by $x_\alpha \perp y_\beta$) or y_β is transversal to x_α if $y_\beta * x_\alpha = \bar{0}$.

Proposition 5.1.8

Let x_α, y_β and z_r be three fuzzy points in $<\hat{X},*,\| \ \|>$ such that $x_\alpha \perp y_\beta$ and $x_\alpha \perp z_\gamma$ then $x_\alpha \perp (ay_\beta \oplus bz_\gamma)$ for every $a,b \in C'(I)$.

Proof:

Given $y_\beta * x_\alpha = \bar{0}$ & $z_\gamma * x_\alpha = \bar{0}$

$$\begin{aligned} \text{then} \quad (ay_\beta \oplus bz_\gamma) * x_\alpha &= (ay_\beta) * x_\alpha + (bz_\gamma) * x_\alpha \\ &= a(y_\beta * x_\alpha) + b(z_\gamma * x_\alpha) \\ &= a\bar{0} + b\bar{0} = \bar{0} \end{aligned}$$

ie $x_\alpha \perp (ay_\beta \oplus bz_\gamma)$.

Proposition 5.1.9

Let $<\hat{X}, *, \| \|>$ be a fuzzy semi inner product space of fuzzy points. If $x_\alpha \perp y_\beta$ then

$$\|x_\alpha \oplus ay_\beta\| \geq \|x_\alpha\| \text{ for every } a \in C'(I).$$

Proof:

Given $x_\alpha \perp y_\beta$ ie $y_\beta * x_\alpha = \bar{0}$.

Consider

$$[\![(x_\alpha \oplus ay_\beta) * x_\alpha]\!] \le [\![(x_\alpha \oplus ay_\beta)*(x_\alpha \oplus ay_\beta)]\!]^{1/2} x\ [\![x_\alpha * x_\alpha]\!]^{1/2}$$

$$[\![x_\alpha * x_\alpha \oplus ay_\beta * x_\alpha]\!] \le \|x_\alpha \oplus ay_\beta\|\ \|x_\alpha\|$$

ie $\|x_\alpha\|^2 \le \|x_\alpha \oplus ay_\beta\|\ \|x_\alpha\|$

ie $\|x_\alpha\| \le \|x_\alpha \oplus ay_\beta\|$ for every $a \in C'(I)$.

Definition 5.1.10

Let $<\hat{X},*_1,\|\ \|_1>$ and $<\hat{Y},*_2,\|\ \|_2>$ be two fuzzy semi inner product spaces of fuzzy points. Then the function $f: \hat{X} \longrightarrow \hat{Y}$ is called a 'fuzzy linear map' if $f(x_\lambda \oplus z_\gamma) = f(x_\lambda) \oplus f(z_\gamma)$ and $f(\alpha\, x_\lambda) = \alpha\, f(x_\lambda)$ for every x_λ, $z_\gamma \in \hat{X}$ and $\alpha \in C'(I)$.

Definition 5.1.11

Let $<\hat{X},*,\|\ \|>$ be a fuzzy semi inner product space of fuzzy points. The function $f: \hat{X} \longrightarrow C'(I)$ is called a 'fuzzy linear functional' if $f(x_\lambda \oplus y_\mu) = f(x_\lambda) + f(y_\mu)$ and $f(\alpha\, x_\lambda) = \alpha\, f(x_\lambda)$ for every $x_\lambda, y_\mu \in \hat{X}$ and $\alpha \in C'(I)$.

Definition 5.1.12

Let T be a `fuzzy linear map' on $<\hat{X},*,\| \|>$. Then T is said to be bounded if there exists a , $k \in R^*(I)$ such that $\|Tx_\lambda\| \leq k\|x_\lambda\|$ for every $x_\lambda \in \hat{X}$ in this case we define

$$\|T\| = \inf \{k \in R^*(I) \mid \|Tx_\lambda\| \leq k\|x_\lambda\|\}.$$

5.2 FUZZY NUMERICAL RANGE, WEAK LIMITS AND `FUZZY LINEAR FUNCTIONALS'

Definition 5.2.1

Let T be a `fuzzy linear map' on $<\hat{X},*,\| \|>$ then by the fuzzy numerical range of T, denoted by w(T) we mean the set

$$w(T) = \left\{Tx_\lambda * x_\lambda \mid \|x_\lambda\| = \bar{1}\right\} \text{ and}$$

$$[\![w(T)]\!] = \sup_{x_\lambda \in \hat{X}} \left\{[\![Tx_\lambda * x_\lambda]\!] \mid \|x_\lambda\| = \bar{1}\right\}.$$

Proposition 5.2.2

Let $<\hat{X},*,\| \|>$ be a fuzzy semi inner product space of fuzzy points. Let T & T' be two fuzzy bounded linear maps on $\hat{X}$ then

(i) $[\![w(T)]\!] \leq \|T\|$

(ii) $[\![w(T+T')]\!] \leq [\![w(T)]\!] \oplus [\![w(T')]\!]$

Proof:

(i) $w(T) = \left\{Tx_\lambda * \; x_\lambda \mid \|x_\lambda\| = \bar{1}\right\}$

$$[\![w(T)]\!] = \sup_{x_\lambda \in \hat{X}} \left\{[\![Tx_\lambda * \; x_\lambda]\!] \mid \|x_\lambda\| = \bar{1}\right\}$$

$$= \sup_{\|x_\lambda\|=\bar{1}} \left\{\|Tx_\lambda\| \; \|x_\lambda\|\right\}$$

$$= \sup_{\|x_\lambda\|=\bar{1}} \|Tx_\lambda\| \leq \|T\|$$

ie $[\![w(T)]\!] \leq \|T\|$

$$\text{(ii)}\ [\![w(T+T')]\!] = \sup_{x_\lambda \in \hat{X}} \left\{ [\![(T+T')x_\lambda * x_\lambda]\!] \mid \|x_\lambda\| = \bar{1} \right\}$$

$$= \sup_{x_\lambda \in \hat{X}} \left\{ [\![(Tx_\lambda + T'x_\lambda) * x_\lambda]\!] \mid \|x_\lambda\| = \bar{1} \right\}$$

$$\leq \sup_{x_\lambda \in \hat{X}} \left\{ [\![Tx_\lambda * x_\lambda]\!] \mid \|x_\lambda\| = \bar{1} \right\} \oplus$$

$$\sup_{x_\lambda \in \hat{X}} \left\{ [\![T'x_\lambda * x_\lambda]\!] \mid \|x_\lambda\| = \bar{1} \right\}$$

ie $[\![w(T+T')]\!] \leq [\![w(T)]\!] \oplus [\![w(T')]\!]$

Definition 5.2.3

A fuzzy semi inner product space of fuzzy points $\langle \hat{X}, *, \|\ \| \rangle$ is said to be strictly convex if $[\![x_\lambda * y_\mu]\!] = \|x_\lambda\| \|y_\mu\|$ then $y_\mu = [\![\alpha]\!] x_\lambda$, where $x_\lambda \neq 0_\beta \neq y_\mu$ and $\alpha \in C'(I)$.

Definition 5.2.4

A sequence $y_{n_{\mu_n}}$ in $\langle \hat{X}, *, \|\ \| \rangle$ is said to converge weakly in the second component to y_ψ if $x_\lambda * y_{n_{\mu_n}}$ converges to $x_\lambda * y_\psi$ for all $x_\lambda \in \hat{X}$.

Proposition 5.2.5

Let $<\hat{X}, *, \| \ \|>$ be a strictly convex fuzzy semi inner product of fuzzy points. Then the weak limit in the case of weak convergence with respect to the second component of the fuzzy semi inner product in $\hat{X}$ is unique.

Proof:

Let y_μ and y'_ψ be two weak limits of the sequence $y_{n_{\mu_n}}$ in $<\hat{X}, *, \| \ \|>$. Then

$$x_\lambda * y_\mu = x_\lambda * y'_\psi \qquad \forall \ x_\lambda \in \hat{X}.$$

Let $x_\lambda = y_\mu$ then $y_\mu * y_\mu = y_\mu * y'_\psi$

$$\text{ie } [\![y_\mu * y_\mu]\!] = [\![y_\mu * y'_\psi]\!]$$

$$\text{ie } \|y_\mu\|^2 \le \|y_\mu\| \ \|y'_\psi\|$$

$$\text{ie } \|y_\mu\| \le \|y'_\psi\|$$

similarly taking $x_\lambda = y'_\psi$ we get

$$\|y'_\psi\| \le \|y_\mu\|$$

$$\text{ie } \|y_\mu\| = \|y'_\psi\| \qquad (1)$$

$$\text{also} \quad \|y_\mu\|^2 = \|y_\mu\| \; \|y'_\psi\|$$

$$\text{ie } [\![y_\mu * y'_\psi]\!] = \|y_\mu\| \; \|y'_\psi\|$$

$$\longrightarrow y_\mu = [\![\alpha]\!] y'_\psi$$

$$\text{ie } \quad y_\mu = y'_\psi \quad \text{by (1).}$$

Proposition 5.2.6

Let $\langle \hat{X}, *, \| \, \| \rangle$ be a fuzzy semi inner product space of fuzzy points. Consider the map $f_{x_\lambda} \; \hat{X} \longrightarrow C'(I)$ defined by $f_{x_\lambda}(y_\mu) = y_\mu * x_\lambda$, then f_{x_λ} is a `fuzzy linear functional' on $\hat{X}$.

Proof:

$$f_{x_\lambda}(y_\mu \oplus z_\psi) = (y_\mu \oplus z_\psi) * x_\lambda = y_\mu * x_\lambda + z_\psi * x_\lambda$$
$$= f_{x_\lambda}(y_\mu) + f_{x_\lambda}(z_\psi)$$

also $f_{x_\lambda}(\alpha\, y_\mu) = (\alpha\, y_\mu) * x_\lambda = \alpha\,(y_\mu * x_\lambda)$
$$= \alpha\, f_{x_\lambda}(y_\mu).$$

Notation 5.2.7

$\hat{X}_L$ denotes the set of all `fuzzy linear functionals' on $\hat{X}$ of the form f_{x_λ}

Proposition 5.2.8

Let $<\hat{X},*,\|\ \|>$ be a fuzzy semi inner product space of fuzzy points. Then the map $*'$ defined on $\hat{X}_L x\, \hat{X}_L$ by $f_{x_\lambda} *' f_{y_\mu} = y_\mu * x_\lambda$ satisfies the following conditions

(i) $f_{x_\lambda} *' f_{x_\lambda} > \bar{0}$, when $x_\lambda \neq 0_\beta$

(ii) $[\![f_{x_\lambda} *' f_{y_\mu}]\!]^2 \leq [\![f_{x_\lambda} *' f_{x_\lambda}]\!][\![f_{y_\mu} *' f_{y_\mu}]\!]$.

Proof:

(i) $f_{x_\lambda} *' f_{x_\lambda} = x_\lambda * x_\lambda > \bar{0}$ when $x_\lambda \neq 0_\beta$

(ii) $[\![f_{x_\lambda} *' f_{y_\mu}]\!]^2 = [\![y_\mu * x_\lambda]\!]^2$

$$\leq [\![y_\mu * y_\mu]\!][\![x_\lambda * x_\lambda]\!]$$

$$= [\![f_{x_\lambda} *' f_{x_\lambda}]\!][\![f_{y_\mu} *' f_{y_\mu}]\!].$$

Remark 5.2.9

If $y_\mu \perp x_\lambda$ in $\hat{X}$ then $f_{x_\lambda} \perp f_{y_\mu}$ in $\hat{X}_L$

Chapter 6

CATEGORY OF FUZZY SEMI INNER PRODUCT SPACES

6.0 INTRODUCTION

In this chapter the concept of the category of semi inner product spaces and that of fuzzy semi inner product spaces are introduced. Relation of the category of fuzzy semi inner product spaces with the categories of semi inner product spaces, fuzzy topological spaces & topological spaces are studied. We conclude with a more general approach to fuzzy semi inner product spaces by introducing the category of semi inner products in a given concrete category.

6.1 THE CATEGORIES SIP, FSIP, FTOP AND TOP

Definition 6.1.1

Let SIP be the category whose objects are semi inner product spaces and $hom_{SIP}(A,B)$ for any two semi inner product spaces

A&B is the set of maps $f:A \longrightarrow B$ satisfying

$f(x+y) = f(x)+f(y)$, $f(\alpha x) = \alpha f(x)$ and

$|[x,y]| \geq |[f(x),f(y)]|$, for every $x,y \in A$ and $\alpha \in C$, the set of complex numbers.

Remark 6.1.2

In the above definition if $f \in hom_{SIP}(A,B)$ then f will be a continuous map from A to B.

Definition 6.1.3

Let FSIP be the category whose objects are fuzzy semi inner product spaces and $hom_{FSIP}(M,N)$ for any two fuzzy semi inner product spaces $\langle M,*_1,\| \|_1\rangle$ and $\langle N,*_2,\| \|_2\rangle$ is the set of maps $f: M \longrightarrow N$, satisfying $f(x+y) = f(x)+f(y)$, $f(\alpha x) = \alpha f(x)$ and $[\![x *_1 y]\!] \geq [\![f(x) *_2 f(y)]\!]$ for every $x,y \in M$ and $\alpha \in C'(I)$.

Proposition 6.1.4

Consider the categories SIP and FSIP. Then there is a faithful functor $\mathcal{F}$ from SIP to FSIP.

Proof:

Let $\langle X,[\,]_1,|\,|_1\rangle$ be any object in SIP. We define

$\mathcal{F}(\langle X,[\,]_1,|\,|_1\rangle) = \langle X,*_1,\|\ \|_1\rangle$ where $x_1 *_1 x_2 = [x_1,x_2]_1$

under the identification (a,b) with $((\bar{a},\bar{0}),(\bar{b},\bar{0}))$

then $[\![x_1 *_1 x_2]\!] = |\,[x_1,x_2]_1|$

since, $|(a,b)| = (a^2+b^2)^{1/2}$ & $[\![((\bar{a},\bar{0}),(\bar{b},\bar{0}))]\!] = \left(\bar{a}^2 + \bar{b}^2\right)^{1/2}$

$$= \left(\overline{a^2+b^2}\right)^{1/2}$$

Also for each $f \in \mathrm{Hom}_{SIP}(X,Y)$ define $\mathcal{F}(f) = f$, then

$f \in \mathrm{hom}_{FSIP}(\mathcal{F}(X), \mathcal{F}(y))$, since

$|[x_1,x_2]_1| \geq |[f(x_1),f(x_2)]_2|$

$$[\![x_1 *_1 x_2]\!] = |\,[x_1,x_2]_1| \geq |\,[f(x_1),f(x_2)]_2|$$

$$= [\![f(x_1) *_2 f(x_2)]\!]$$

ie $\quad [\![x_1 *_1 x_2]\!] \geq [\![f(x_1) *_2 f(x_2)]\!]$.

Proposition 6.1.5

Let FTOP be the category of fuzzy topological spaces. Then there exists a faithful functor Λ from FSIP to FTOP which maps $\langle X,*,\| \ \|\rangle$ to $(X,\mathcal{T}_{\| \|})$, where $\mathcal{T}_{\| \|}$ is the fuzzy topology on X having basis $\{B(x,\bar{r})/x \in X \ \& \ r \in R^+\}$.

Proof:

Let $\langle X,*,\| \ \|\rangle$ be any object in FSIP we define

$$\Lambda\ (\langle X,*,\| \ \|\rangle) = (X,\mathcal{T}_{\| \|})$$

let $f \in hom_{FSIP}(\langle X,*_1,\| \ \|_1\rangle,\langle Y,*_2,\| \ \|_2\rangle)$

then $f(x+y) = f(x)+f(y)$

$f(\alpha\ x) = \alpha\ f(x)$ and

$[\![x *_1 y]\!] \geq [\![f(x) *_2 f(y)]\!]$

$\|x\|_1^2 \geq \|f(x)\|_2^2$

ie $\|x\|_1 \geq \|f(x)\|_2$

If $g \in hom_{FTOP}((X,\mathcal{T}_{\| \|_1}),(Y,\mathcal{T}_{\| \|_2}))$, then g will be a fuzzy continuous map from X to Y

$$\text{ie } B(x,\bar{r}) \leq g^{-1}(B(g(x),\bar{r}))$$

claim $f \in hom_{FTOP}((X,\mathcal{T}_{\|\|_1}),(Y,\mathcal{T}_{\|\|_2}))$

for $B(x,\bar{r})(x') = L_r\|x-x'\| = 1-\|x-x'\|(r)$

$$f^{-1}(B(f(x),\bar{r}))(x') = B(f(x),\bar{r})(f(x'))$$

$$= L_r\|f(x)-f(x')\|$$

$$= 1-\|f(x-x')\|(r) \geq 1-\|x-x'\|(r)$$

ie $B(x,\bar{r}) \leq f^{-1}(B(f(x),\bar{r}))$

ie $f \in hom_{FTOP}((X,\mathcal{T}_{\|\|_1}),(Y,\mathcal{T}_{\|\|_2}))$

define Λ (f) = f

then Λ will be a faithful functor from FSIP to FTOP.

Proposition 6.1.6

Let TOP be the category of topological spaces. Then there exists a functor $\hbar$: SIP $\longrightarrow$ TOP which maps the semi inner product spaces $\langle X,[\,],|\,|\rangle$ into the topological spaces $(X,\mathcal{T}_{|\,|})$.

Proof:

Let $f \in hom_{SIP}(\langle X,[\,]_1,|\,|_1\rangle,\langle Y,[\,]_2,|\,|_2\rangle)$ then

$f(x+y) = f(x)+f(y)$, $f(\alpha\, x) = \alpha\, f(x)$

$|\,[x,y]_1| \geq |\,[f(x),f(y)]_2|$, here f is a continuous map from X to Y.

Hence $f \in hom_{TOP}((X,\mathcal{T}_{|\,|_1}),(Y,\mathcal{T}_{|\,|_2}))$

Let $\hbar$ be the map from SIP to TOP which maps $\langle X,[\,],|\,|\rangle$ to $(X,\mathcal{T}_{|\,|_1})$ and $\hbar(f) = f$, then $\hbar$ is a functor from SIP to TOP.

Definition 6.1.7

Let $(X,\mathcal{T})$ be a topological space. Then the related fuzzy topological space consists of the collection of all lower semi continuous functions from X to [0,1].

Proposition 6.1.8

There exists a functor $\mathcal{G}$: FTOP $\longrightarrow$ TOP which associates the fuzzy topological spaces $(X,\mathcal{T}_F)$ to the related topological space $(X,\mathcal{T})$ and $f \in hom_{FTOP}((X_1,\mathcal{T}_{F_1}),(X_2,\mathcal{T}_{F_2}))$ to the same $f \in hom_{TOP}((X_1,\mathcal{T}_1),(X_2,\mathcal{T}_2))$.

Proposition 6.1.9

There exists a functor $\mathcal{G}'$ from TOP to FTOP which maps the topological space $(X,\mathcal{T})$ to the related fuzzy topological space $(X,\mathcal{T}_F)$ and $f \in hom_{TOP}((X_1,\mathcal{T}_1),(X_2,\mathcal{T}_2))$ to the same $f \in hom_{FTOP}((X_1,\mathcal{T}_{F_1}),(X_2,\mathcal{T}_{F_2}))$.

Remark 6.1.10

$$\mathcal{G} \Lambda \mathcal{F} = \hbar.$$

Proposition 6.1.11

Consider $\mathcal{G}'\hbar$ & $\Lambda \mathcal{F}$ from SIP to FTOP then

$$\Lambda \mathcal{F} (SIP) \subsetneq \mathcal{G}'\hbar (SIP).$$

Proof:

$\Lambda \mathcal{F}$ maps $<X,[\,],|\,|>$ to $(X,\mathcal{T}_{\|\,\|})$ and

$f \in \hom_{SIP}(\langle X,[\,]_1\rangle,\langle y,[\,]_2\rangle)$ to the same

$f \in \hom_{FTOP}((X,\mathcal{T}_{\|\|_1}),(Y,\mathcal{T}_{\|\|_2}))$

$\mathcal{G}'h$ maps $\langle X,[\,],|\,|\rangle$ to $\mathcal{G}'(X,\mathcal{T}_{|\,|}) = \mathcal{T}_F$ the related fuzzy topology of $\mathcal{T}_{|\,|}$ and

$f \in \hom_{SIP}(\langle X,[\,]_1\rangle,\langle Y,[\,]_2\rangle)$ to the same

$f \in \hom_{FTOP}((X,\mathcal{T}_{F_1}),(Y,\mathcal{T}_{F_2}))$

claim $\mathcal{T}_{\|\|} \subsetneqq \mathcal{T}_F$

let $U \in \mathcal{T}_{\|\|}$, then $U = \bigcup_{x,r} B(x,\bar{r})$

$$= \bigcup_{x,r} \chi_{B(x,r)}$$

$$= \chi_{\bigcup_{x,r} B(x,r)}$$

ie U is the characteristic function of an open set in $\mathcal{T}_{|\,|}$ and hence U is open in the related fuzzy topological space $\mathcal{T}_F$

ie $U \in \mathcal{T}_F$

also constant fuzzy sets being lower semi continuous on X, are open in $\mathcal{T}_F$. Constant functions are not necessarily open in $\mathcal{T}_{\|\|}$

Hence $\mathcal{T}_{\|\|} \subsetneq \mathcal{T}_F$

$\Lambda\ \mathcal{F}$ (SIP) $\subsetneq \mathcal{G}'\hbar$ (SIP).

Proposition 6.1.12

There exists a natural transformation $\eta:\mathcal{G}''$ to $\mathcal{G}'$, where $\mathcal{G}''$ & $\mathcal{G}'$ are functors from TOP to FTOP. $\mathcal{G}''$ maps $(X,\mathcal{T})$ to $(X,\mathcal{T}_{\chi_F})$ where $\mathcal{T}_{\chi_F}$ is the fuzzy topology on X obtained by identifying open sets of $\mathcal{T}$ with its characteristic function.

Proof:

If $f \in \hom_{TOP}((X,\mathcal{T}_1),(Y,\mathcal{T}_2))$ then f is a continuous map from X to Y, also f is a fuzzy continuous map from $(X,\mathcal{T}_{\chi_{F_1}})$ to $(Y,\mathcal{T}_{\chi_{F_2}})$. Hence $\mathcal{G}''(f) = f$. Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be two objects in TOP and $f \in \hom_{TOP}((X,\mathcal{T}_1),(Y,\mathcal{T}_2))$. Consider the identity maps

$I(X,\mathcal{T}_1) \in \hom_{FTOP}((\mathcal{G}''(X,\mathcal{T}_1),\mathcal{G}'(X,\mathcal{T}_1))$ and

$I(Y,\mathcal{T}_2) \in \hom_{FTOP}(\mathcal{G}''(Y,\mathcal{T}_2),\mathcal{G}'(Y,\mathcal{T}_2))$ then the diagram

$$\begin{array}{ccc}
\mathcal{G}''(X,\mathcal{T}_1) & \xrightarrow{\quad I(X,\mathcal{T}_1) \quad} & (\mathcal{G}'(X,\mathcal{T}_1) \\
\Big\downarrow {\scriptstyle \mathcal{G}''(f)} & & \Big\downarrow {\scriptstyle \mathcal{G}'(f)} \\
\mathcal{G}''(Y,\mathcal{T}_2) & \xrightarrow[\quad I(Y,\mathcal{T}_2) \quad]{} & (\mathcal{G}'(Y,\mathcal{T}_2)
\end{array}$$

commutes. Hence I is the natural transformation from $\mathcal{G}''$ to $\mathcal{G}'$

6.2 THE CATEGORY $\mathcal{C}_{SIP}$ OF SEMI INNER PRODUCTS IN A CATEGORY $\mathcal{C}$

We conclude with a more general approach to fuzzy semi inner product spaces.

Definition 6.2.1

Let $\mathcal{C}$ be a concrete category of sets with finite products containing zero object and the set C'(I). An ordered pair (X,m) is called a semi inner product in $\mathcal{C}$ if

(i) X is an object in $\mathscr{C}$

(ii) $m: X \times X \to C'(I)$ be such that for each $x_0 \in X$,

$m_{x_0}(x) = m(x, x_0) \in \hom_{\mathscr{C}}(X, C'(I))$

(iii) $m(x, x_0) > \bar{0}$, if $x \neq 0$, where 0 is the image of zero object under the unique morphism

(iv) $[\![m(x,y)]\!]^2 \leq [\![m(x,x)]\!]\ [\![m(y,y)]\!]$.

Remark 6.2.2

(i) In the above definition we take $\mathscr{C} = \mathscr{A}$, the category of linear spaces and the range of m equal to the set of all complex numbers, we will get (X,m) as the semi inner product space defined by Lumer.

(ii) When $\mathscr{C}$ = the category of C'(I) modules, we will get (X,m) as the fuzzy semi inner product space.

Definition 6.2.3

Let $\mathscr{C}_{SIP}$ be the class of semi inner products in $\mathscr{C}$. If (X,m) and $(Y,m') \in \mathscr{C}_{SIP}$ a function $f: X \to Y$ is called a morphism if

(i) f is a morphism in $\mathscr{C}$ and

(ii) $[\![m(x,y)]\!] \geq [\![m'(f(x),f(y))]\!]$ $\forall$ $x,y \in X$

then $\mathscr{C}_{SIP}$ becomes a category called the category of semi inner product spaces. When $\mathscr{C}$ = the category of linear spaces and the range of m's = the set of all complex numbers then $\mathscr{C}_{SIP}$ = SIP, the category of semi inner product spaces. When $\mathscr{C}$ = the category of C'(I) modules and range of m's = C'(I), then $\mathscr{C}_{SIP}$ = FSIP, the category of fuzzy semi inner product spaces.

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