STUDIES ON INTEGRABILITY OF SOME PERTURBED NONLINEAR EVOLUTION EQUATIONS

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विश्वं पश्यति कार्यकारणतया स्वस्वामि संबन्धतः
शिष्याचार्यतया तथैव पितृपुत्राधात्मना भेदतः।
स्वप्ने जाग्रती वा य एष पुरुषो मायापरीभ्रामित
स्तमै श्रीगुरुमूर्तये नमः

श्री शंकराचार्य।

DECLARATION

I hereby declare that the matter embodied in this thesis entitled "*Studies on Integrability of Some Perturbed Nonlinear Evolution Equations*" is the result of investigations carried out by me under the supervision of **Prof.K.Babu Joseph** in the Department of Physics, Cochin University of Science and Technology , Cochin –22 and that this work has not been included in any other thesis submitted previously for the award of any degree or diploma of any University.

Sreelatha. K.S.

CERTIFICATE

*Certified that the work contained in the thesis entitled " **Studies on Integrability of Some Perturbed Nonlinear Evolution Equations**" is the bona fide work carried out by **Ms. Sreelatha .K.S.** under my supervision in the Department of Physics, Cochin University of Science and Technology, Cochin – 22, in partial fulfillment of the requirements for the award of the Degree of Doctor of Philosophy and that the same has not been included in any other thesis submitted previously for the award of any degree or diploma of any University.*

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PREFACE

Nonlinear phenomena have become one of the most important subjects of research. This is because Nature often reveals her mysteries in terms of nonlinear processes. Nonlinear systems model almost all physical phenomena. In the nonlinear paradigm, an initial change in one variable does not produce a proportional change in the resultant variable. ie., cause and effect cannot be related by a mere proportionality constant. They are usually represented using nonlinear equations.

A dynamical system is defined by a collection of configurational coordinates and equations of motion obeyed by them. Given such equations of motion, one would like to solve them so that the dynamical variables at any time may be determined as a function of the initial variables and time. When a dynamical system has nonlinear equations of motion, the dynamic inertia of the system becomes dependent on the configuration. If it happens that this dynamic inertia tends to vanish, these are points of maximum fluctuation where even a small change in the configuration can cause a substantial change in the outcome. This type of dynamical systems are encountered in a large number of disciplines such as physical sciences, chemical sciences, engineering sciences, biological sciences etc., in the context of both fundamental and applied mechanics.

The discovery of nonlinear integrable field models continues to create immense excitement. One of the most fundamental, important and fascinating problems in the investigation of nonlinear dynamical system is to give a general criterion which decides the integrability. The link comes from the description by nonlinear evolution equations whose solutions represent the propagation of waves with a permanent profile; solitons.

Usually typical dynamical systems are non integrable. But few systems of practical interest are integrable. The soliton concept is a sophisticated mathematical construct based on the integrability of a class of nonlinear differential equations. An important feature in the development of the theory of solitons and of complete integrability has been the interplay between mathematics and physics. Every integrable system has a long list of special properties that hold for integrable equations and only for them. Actually there is no specific definition for integrability that is suitable for all cases.

There exist several integrable partial differential equations(pdes) which can be derived using physically meaningful asymptotic techniques from a very large class of pdes. It has been established that many nonlinear wave equations have solutions of the soliton type and the theory of solitons has found applications in many areas of science. Among these, well-known equations are Korteweg de-Vries(KdV), modified KdV, Nonlinear Schrödinger(NLS), sine Gordon(SG) etc..These are completely integrable systems. Since a small change in the governing nonlinear pde may cause the destruction of the integrability of the system, it is interesting to study the effect of small perturbations in these equations. This is the motivation of the present work.

The first chapter of the thesis gives a general introduction to the integrable systems and their importance. Various methods for the detection of integrability are also given. The second chapter deals with the integrability of a perturbed NLS equation. the main integrability detecting tools considered are Painleve analysis and generalized Lax method.

Nonlinear wave propagation through a 2D lattice is studied in the third chapter. For three cases, the Kadotsev-Petviashvili(KP) equation, modified KP equation, and an integro-differential equation are obtained using the reductive perturbation method. The integrability study of these three equations is done in the fourth chapter using Lax method and Painleve analysis. The fifth chapter deals with the integrability of another perturbed NLS equation.

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Contents

Chapter 1

General introduction

1.1 Introduction

The mathematical theory of nonlinear processes and their application has grown considerably during the past two decades. Most of the wave motions are represented mathematically by nonlinear differential equations. With the advent of new ideas and methods, new results and applications, studies are continually being added to the central subjects of fluid mechanics, plasma physics, solid mechanics, nonlinear optics and nonlinear systems and circuits; these are themselves developing remarkably and coalescing[1-4]. It is becoming more and more desirable for applied mathematicians,physicists and engineering scientists to study nonlinear phenomena as a whole.

Nonlinear ordinary and partial differential equations play a central role in almost all physical theories. The concept of solitons and the inverse scattering technique (IST) method for exact solutions of some nonlinear partial differential equations (pde), including some of physical interest, have had far reaching influence and consequences in various branches of mathematics, physics and engineering[5-7]. The most interesting and important development is the progress that has been made in the study of nonlinear pdes by means of the method developed by the group of scholars centered around Martin Kruskal in the 1960s[5]. In the recent years, these ideas have really taken hold and important further progress has been made.

Nonlinear integrable systems were discovered as early as the 18th century. At that time only a few were known and with no real understanding of their characteristics and solutions. Now, however, it is correct to say that it is impossible to over estimate their importance in the development of all areas of science. Problems with unexpected structure often turn out to be related to integrable systems.

The recent revival of the subject of complete integrability, starting with the solution of the KdV equation by inverse spectral methods had led to many new systems[1]. The standard soliton equations are considered to be completely integrable since they are highly idealised. But when we consider real situations, an arbitrarily small change in an integrable equation can destroy its integrability.

The first step towards the explanation of the relationship between the analytical structure of a system and its integrability is attributed to the Russian mathematician S.Kovalevskaya[8]. Her work focussed on the study of the motion of a rigid body with a fixed point from an analysis of the singularities of the solutions. Then, Korteveg and de Vries made an important contribution to water wave problems and discovered a nonlinear model equation for the unidirectional propagation of long surface waves in a uniform rectangular channel and the equation is now known as KdV equation[9]. Modern developments in the theory and application of KdV solitary waves began with the famous work by Fermi,Pasta and Ulam on their numerical study of discrete nonlinear mass strings[7]. They studied the behaviour of certain equations, which are primarily linear but in which nonlinearity is added as a perturbation. They believed that a smooth initial state would eventually relax to an equipartition of energy over all modes, the energy in each mode was shown to be almost periodic in time with no loss of energy to higher modes as time increases. This remarkable fact has become known as Fermi-Pasta -Ulam (FPU) recurrence phenomenon.

For a decade, the FPU problem remained as one unrelated to solitary waves. The word 'soliton' first appeared in the work of Zabusky and Kruskal[10]. They studied the KdV equation as a model FPU problem and reconfirmed the recurrence phenomenon. The most remarkable observation was that these pulses retain their identities even after their nonlinear interaction. Their preservation of shape and resemblance to particles led to the name 'soliton' However, these solitons were not to be regarded as new particles-like excitations that the system possessed by virtue of its nonlinear nature.

After the introduction of concept of solitons,Gardner, Greene, Kruskal and Miura[11,12] developed a method for exact solution of the initial value problem for the KdV equations resorting to the ideas of direct and inverse scattering. This method is one of the basic tools to study Nonlinear Evolution Equations(NLEE). It provides a procedure for explicitly obtaining the pure soliton solutions and qualitative information concerning the general solutions. A rigorous mathematical approach suitable for dealing with nonlinear problems was set up by Lax[13]. This method has been further developed, extended and applied by Zakharov and Shabat[14] to solve the so called nonlinear Schrodinger Equation (NLSE) which is of special importance in many branches of physics. The solution of the NLS equation possesses several remarkable properties which include the concept of envelope solitons, modulational instability and recurrence[15].

Additionally, it was discovered that there exist NLEEs in (2+1) dimensions and integro-differential equations which are solvable by inverse scattering[1]. Lakshmanan and Sahadevan[16] have given a succint exposition of nonlinear dynamics from the point of view of integrability and Painleve(P) analysis with many standard examples and applied the method to two,three and N-dimensional quadratic anharmonic oscillator.

1.2 Integrable systems and solitons

Much attention has been focussed on the classification of dynamical systems as integrable and nonintegrable ones. Toda lattice described by a Hamiltonian function with N degrees of freedom is a well-known example of a classical integrable system. In the 2N dimensional phase space, there are N first integrals and there is a Lax representation. In the field theory, the KdV equation in (1+1) dimension is an integrable system.The initial value problem of this equation can be solved with the help of the IST. Moreover, there is an infinite number of conservation laws. In the Quantum Field Theory, the best known example of integrable system is the quantum nonlinear Schrödinger equation[2]. This system can be solved with the help of IST. However, most dynamical systems are nonintegrable. In Classical Mechanics, we find among these nonintegrable systems those with chaotic behavior.

A most remarkable property of integrable systems is the existence of special type of solutions called "solitons" They are localised waves that travel without much change in shape. Actually the word "soliton" refers to solitary travelling waves which preserve their identities even after a collision.

Solitons exist everywhere. They are found in the sky as density waves in spiral galaxies, as red spots in the atmosphere of Jupiter and they exist in the ocean as waves bombarding oilwells[17]. They exist in smaller natural and laboratory systems such as plasmas, molecular systems, laser pulses propagating in solids, superfluid He, superconducting Josephson junction magnetic system, structural phase transitions, polymers, fluid flows, elementary particles and in liquid crystals[18]. Apart from the ubiquitous existence, the importance of solitary waves lies in their interesting properties as nonlinear waves.

By definition, solitons are special solutions of some nonlinear partial differential equation. Historically Scott-Russell[19] first observed and reported the phenomenon of the so-called solitary wave in the early 1840s. This was a wave of finite amplitude having a symmetrical form with a single hump which propagates at a uniform velocity without change of form. Russell discovered from his experiments, one of the most important relations between the speed c of the solitary wave and its maximum amplitude η above the free surface of water of finite depth h in the form

$$c^2 = g(h + \eta) \tag{1.1}$$

where g is the acceleration due to gravity

Later on, inspired by this work, Boussinesq and Rayleigh[20,21] independently proved the existence of the Russell solitary wave. It was Boussinesq who deduced the equation governing two dimensional irrotational flows if an inviscid liquid in a uniform rectangular channel into a nonlinear model equation of the form,

$$U_{tt} = c_0^2(U_{xx} + \frac{3}{2}(\frac{U_{xx}^2}{h} + \frac{1}{3}h^2U_{xxxx}) \tag{1.2}$$

where c_0 is the speed of the shallow water waves. This equation admits the solution,

$$U(x,t) = \eta sech^2(\frac{3\eta}{h^2})^{\frac{1}{2}}(x \pm ct) \tag{1.3}$$

for solitary waves travelling in positive or negative direction. Landmarks in the evolution of the subject were the proposal by Korteweg de Vries[9], of an equation incorporating both nonlinear and dispersive effects for the propagation of waves in shallow water. Later, Zabusky and Kruskal[10] revealed the existence of wave like excitations in shallow water which maintained a stable shape in the course of their propagation and emerged from collisions unchanged. Because of their particle like character, these solutions were named solitons by them.

There are other equations besides KdV whose solutions are isospectral potentials for the Schrödinger equation[22,23]. A rigorous mathematical approach suitable for dealing with such problems was set up by Lax. Using Lax's technique Zakharov and Shabat introduced a linear scattering problem to solve the so called NLSE, which is of special importance in many branches of Physics. Later, Ablowitz, Kaup, Newell and Segur[24], tried successfully the same ideas on the sine-Gordon equation.

The recent revival of the subject of complete integrability, starting with the resolution of the KdV equation by inverse spectral methods has led to many new systems which have the additional property of being solvable in terms of quadratures. Soliton solutions are considered to be the most remarkable property of the integrable systems. Nonlinear evolution equations (NLEE) having soliton solutions share many special properties such as an infinite sequence of conservation laws, Lie-Backlund symmetries, multisoliton solutions, Backlund transformations and reduction to ordinary differential equations of Painleve-type[25-28]. Furthermore, these equations may be obtained via compatibility of two associated linear operators, in other words, they can be put in the Lax's form[29]. All this suggests that the equations are exactly solvable.

There are both continuous and discrete versions of the theory of integrable systems[30]. In the continuous case one has to study either system of ordinary differential equations, or partial differential equations. Here the tools include finite dimensional differential geometry, Lie algebras and Painleve test(P-test) for ordinary differential equations and infinite dimensional differential geometry, loop algebras, and generalized P-test for partial differential equations. The typical examples are Korteweg de-Vries(KdV), Kadomtsev-Petviashvili(KP) and Nonlinear Schrodinger(NLS) equations. In the discrete case, there appear discretized operators which are either differential-difference operators or difference operators[31]. Our interest lies in the integrability study of continuous systems which are represented using partial differential equations.

At the centre of the theory of integrable systems lies the notion of Lax pair describing the isospectral deformation of a linear operator usually depending on a parameter. A Lax pair [L,M] is such that the time evolution of the Lax operator $\dot{L} = [L, M]$ is equivalent to the given nonlinear system. The study of the associated linear problem $L\psi = \lambda\psi$ can be carried out by various methods.

Another important approach to integrable systems is the Painleve analysis. The P-test, first used by Sophie Kovalewskia[8] in her classification of the integrable rigid body motion, is now recognised as a test for deciding the integrability of nonlinear systems.

Soliton bearing equations such as sine-Gordon, KdV, NLSE etc are familiar to mathematicians because of the remarkable complete integrability of the Hamiltonian systems from which they derive. Also these equations appear as approximate model descriptions of a vast and diverse array of physical phenomena[32,33]. However, realistic applications in various fields such as condensed matter physics, engineering etc demand the inclusion of various perturbations leading to problems beyond those of pure integrable systems. This is the motivation of the present work. In this work, we try to study the integrability of some nonlinear partial differential equations using mainly Lax method and Painleve analysis.

1.3 Methods for solving integrable equations

1.3.1 Inverse scattering method

This method was introduced by Gardner, Greene, Kruskal and Miura[11,12] to solve the KdV equation by proposing that the time evolution of the function U(x,t) could be studied through the properties of the quantum mechanical problem. i.e., For a given initial condition $U(x,0)$, find the bound state energy levels and wavefunctions of the Schrodinger equation in which $U(x,0)$ is the potential. This is the direct scattering problem. As U evolves or deforms as a function of t, the associated quantum mechanical properties termed the scattering data- will also evolve. Then the scattering at a later time t could be found and this can be used to reconstruct the potential $U(x,t)$. This is called inverse scattering method(ISM). Diagramatically the ISM can be represented as follows:

U(x,0) ⟶ Scattering data at t=0 ⟶ Scattering data at t ⟶ U(x,t)

First consider the differential equation

$$\psi_{xx} - (U - \lambda)\psi = 0 \tag{1.4}$$

This is the time-independent Schrödinger equation of quantum mechanics where ψ is the wave function, U is the potential and λ represents energy levels. The variable t only plays the role of a parameter. This equation then will admit a corresponding set of discrete eigenvalues, $\lambda_n = \kappa_n^2$ (n=1,2,...,n.) corresponding to negative energy bound states with associated eigenfunctions $\psi_n(x)$ given by

$$\psi_{n,xx} = (U_0(x) + \kappa_x^2)\psi_n \tag{1.5}$$

The bound state eigen functions are required to be square integrable and normalized to unity. i.e., $\int \mid \psi(x) \mid^2 dx = 1$

From the property $u \to 0$ as $\mid x \mid \to \infty$, the function belonging to $\lambda_n > 0$ takes the form,

$$\psi_n \sim C_n(t) exp(\kappa_n x)$$

for $x \to \infty$. At positive energy, the Schrödinger equation for $U_0(x)$ exhibits a continuous spectrum and we choose λ as a constant κ^2 For $\lambda = \kappa^2 > 0$,the solution for equation (1.4), for large values of $\mid x \mid$, is a linear combination of

$exp(\pm ikx)$, satisfying the given boundary condition:

$$\psi = exp(-ikx) + b(\kappa)exp(i\kappa x), \ x \rightarrow +\infty, \tag{1.6}$$

$$\psi = a(\kappa)exp(-i\kappa x), \ x \rightarrow -\infty, \tag{1.7}$$

The coefficients of transmission $a(\kappa)$ and reflection $b(\kappa)$ can be shown to satisfy $\mid a \mid^2 + \mid b \mid^2 = 1$.

Since the spectrum for $\lambda > 0$ is continous, λ can be chosen so that $\lambda_t = 0$. Then substituting (1.6) and (1.7) in (1.4) yield the integration constants $D = 0$, and $c = 4ik^3$ and two equations which easily give

$$a(\kappa, t) = a(\kappa, 0)$$

$$b(\kappa, t) = b(\kappa, 0)exp(\kappa^3 it)$$

Now, the scattering data a, b, c_n and λ_n are sufficient to allow the reconstruction of U at any time t. Let $K(x, y)$ for $y \geq 0$ be the solution of the Gel'fand-Levitan equation[34]

$$K(x, y) + B(x + y) + \int_x^\infty K(x, z)B(y + z)dz = 0 \tag{1.8}$$

with

$$B(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(\kappa)exp(i\kappa\zeta)d\kappa + \sum_n C_n^2 exp(\kappa_n \zeta) \tag{1.9}$$

Then

$$U(x,t) = 2\frac{\partial}{\partial x}K(x,x) \tag{1.10}$$

Thus the evolution of U(x,t) is obtained from the explicit dependence on time of $b(\kappa)$ and C_n.

The procedure for ISM can be summarised as follows:

(1)map the initial data $S = b(\kappa,0), C_n(0)$ and $\lambda_n(n = 1,2,...,N)$;

(2)compute the time evolution of the scattering data as indicated above;

(3)solve the Gel'fand -Levitan equation and calculate U(x,t).

Lax[13] stimulated an important development and added mathematical understanding to the inverse method. Subsequently, Zakharov and Shabat[14] showed that the NLSE which arises as a centrally important equation in fluid dynamics, nonlinear optics.,. etc could be solved by similar methods. Shortly after this, a procedure was developed by which KdV, NLS, mKdV, SG and indeed a class of nonlinear evolution equations could be solved[1].

1.3.2 Lax method and its generalization

Since 1967, when Gardner, Green, Kruskal and Miura (GGKM)[11] integrated the KdV equation thereby discovering the ISM, numerous attempts have been made to extend the range of application of this method. In 1968, Lax put forward a simplified argument for the basic result of GGKM and at the same time suggested the first method of searching for integrable equations. It was by this method that the many parts in the KdV family integrable by the IST were discovered.

Given the two differential operators

$$L = \frac{-\partial^2}{\partial x^2} + U$$

and

$$M = \frac{\partial^{2n+1}}{\partial x^{2n+1}} + lower \quad degree$$

(M is skew symmetric) such that there is a one parameter family of unitary operators U satisfying

$$U_t = MU$$

and $L$is unitary equivalent to U, ie,

$$U^{-1}(t)L(t)U(t)$$

is independent of t, then Lax suggests:

(1) the eigenvalues λ of L are integrals of motion, and

(2)the relation

$$\frac{\partial}{\partial t}(U^{-1}LU) = 0 \tag{1.11}$$

implies

$$L_t = [M, L] \tag{1.12}$$

with

$$U_t = MU$$

(3) Since $L_t = U_t$, the operator equation (1.12) is an evolution equation

$$U_t = K(U)$$

in which $K(U)$ is a functional of U.

Lax proved that the initial data U(x,0) determine the solution of such nonlinear evolution equations uniquely. The representation of nonlinear evolution equations in the 'Laxpair' form $L_t = [M, L]$ remains the most powerful technique for developing analytical solutions of such equations. Such a representation guarantees the constancy of the eigenvalue spectrum.

Following the work of Lax, the remarkable papers by Zakharov and Shabat created much of soliton theory as we now know it. Using

Lax's technique, they introduced a linear scattering problem and developed a method to obtain soliton solutions for the NLS equation. Another generalisation of the IST was also made by Ablowitz, Kaup, Newell and Segur(AKNS)[24]. Here the eigenvalue takes complex values in general, because, L and M are not self-adjoint. In the case of ZS scheme, the eigenvalue does not appear explicitly at any stage. In the first place both AKNS and ZS schemes incorporate the same form of matrix Marchenko equation. However, the definitions of the two F functions differ significantly. In fact the definition of F in the ZS scheme via the linear partial differential equation is often more useful in practice. This can be explained as follows.

Let F(x,z) and $K_{\pm}(x,z)$ be N×N matrices where

$$K_{+}(x,z) = 0, \quad z < x \tag{1.13}$$

$$K_{-}(x,z) = 0, \quad z > x \tag{1.14}$$

and let $\psi(x)$ be an N-vector. The integral operators J_F and $J_{\pm}$ on ψ are defined by

$$J_F(\psi) = \int_{-\infty}^{\infty} F(x,z)\psi(z)dz \tag{1.15}$$

$$J_{\pm}(\psi) = \int_{-\infty}^{\infty} K_{\pm}(x,z)\psi(z)dz \tag{1.16}$$

Here J_F and $J_\pm$ are related by the operator identity

$$(I + J_+)(I + J_F) = I + J_- \tag{1.17}$$

This identity operating on ψ, becomes

$$\int_{-\infty}^{\infty}(K_\pm(x,y) + F(x,y) + \int_x^\infty (K(x,z)F(y,z))\psi(z)dz = 0 \tag{1.18}$$

That is,

$$K(x,y) + F(x,y) + \int_x^\infty K(x,z)F(y,z)dz = 0 \tag{1.19}$$

which is the Gel'fand-Levitan-Merchenko equation in the ISM

We shall now relate F and $K_\pm$ (in t and y) by introducing appropriate (linear) differential operator Δ_0 on $\psi(x,t,y)$ which has only constant coefficients and which commutes with integral operator J_F. i.e.,

$$[\Delta_0, J_F] = \Delta_0 J_F - J_F \Delta_0 = 0 \tag{1.20}$$

Further, introducing an associated differential operator, Δ which is defined by the operator identity

$$\Delta(I + J_+) = (I + J_+)\Delta_0 \tag{1.21}$$

We choose two pairs of operators

$$\Delta_0^{(1)} = I\alpha\frac{\partial}{\partial t} - M_0, \quad \Delta_0^{(2)} = I\beta\frac{\partial}{\partial y} + L_0 \tag{1.22}$$

and

$$\Delta^{(1)} = I\alpha\frac{\partial}{\partial t} - M, \quad \Delta^{(2)} = I\beta\frac{\partial}{\partial y} + L \tag{1.23}$$

where α and β are constants and L_0, M_0, L, and M are differential operators in x only. Here L_0 and M_0 are comprised of constant coefficients and so $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$ commute. Also, both $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$ are to commute with the same operator J_+, i.e.,

$$[\Delta_0^{(1)}, J_F] = 0$$

and

$$[\Delta_0^{(2)}, J_F] = 0$$

The operators $\Delta^{(i)}$are defined according to the equation(1.23) with the same J_+, i.e.,

$$\Delta^{(i)}(I + J_+) = (I + J_+)\Delta_0^{(i)} \tag{1.24}$$

Now , let us examine the operation,

$$\begin{aligned} [\Delta^{(1)}, \Delta^{(2)}](I + J_+) &= \Delta^{(1)}\Delta^{(2)}(I + J_+) - (I + J_+)\Delta^{(2)}\Delta^{(1)} \\ &= \Delta^{(1)}(I + J_+)\Delta_0^{(2)} - \Delta^{(2)}(I + J_+)\Delta_0^{(1)} \\ &= (I + J_+)\Delta_0^{(1)}\Delta_0^{(2)} - \Delta_0^{(2)}\Delta_0^{(1)}(I + J_+) \\ &= (I + J_+)[\Delta_0^{(1)}, \Delta_0^{(2)}] \end{aligned} \tag{1.25}$$

However, $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$ are so chosen that they commute with one another, hence we arrive at

$$[\Delta^{(1)}, \Delta^{(2)}] = 0 \tag{1.26}$$

Now, introducing the choice given in equation (1.23), we get

$$(I\alpha\frac{\partial}{\partial t} - M)(I\beta\frac{\partial}{\partial y} + L) - (I\beta\frac{\partial}{\partial y} + L)(I\alpha\frac{\partial}{\partial t} - M) = 0 \qquad (1.27)$$

which simplifies to

$$\alpha\frac{\partial L}{\partial t} + \beta\frac{\partial M}{\partial y} + [L, M] = 0 \qquad (1.28)$$

This is the generalisation of the Lax pairs of two auxiliary variables, the Lax equation is recovered if $\beta = 0$, and $\alpha = 1$. This equation represents the system of nonlinear evolution equations that can be solved by ZS scheme. The variable coefficients which arise in the operators L and M constitute the functions which satisfy the system of evolution equations.

Accordingly if it is possible to find the linear operators L and M, satisfying for a given equation $u_t = K(u)$, the initial value problem for u may be solved as follows.

(1)Direct problem: Solve the eigenvalue problem $L\phi = \lambda\phi$ for a given U(x,0) and obtain the scattering data at t=0,

(2)Time evolution of scattering data; in terms of equation $i\phi_t = M\phi$, and the asymptotic form of M as $| x | \rightarrow \infty$, calculate the time evolution of the scattering data.

(3) Inverse problem: Determine U(x,t) from a knowledge of the scattering data for L.

Although this formulation is quite general, it is rather difficult to find appropriate operators L and M and to solve explicitly the inverse problem for L.

1.3.3 Painleve analysis

Another important approach to find integrable equations is the so called singularity analysis or P-analysis of the solutions in the complex plane[8]. With the recent developements in soliton equations, this analysis has received much attention and now many of the integrable dynamical systems are associated with the so called P-property, in that they are free from movable critical points/manifolds.

The clasification of first order and second order nonlinear ordinary differential equations(ODE), which are free from movable critical points, was achieved through the works of Fuchs, Painleve and his co-workers[35] in the last century.

S.Kovalewskaya investigated the integrable cases of rigid body motion around a fixed point under the influence of gravity through the singularity structure analysis[8]. Kovalewskaya's work was completely new and also addressed to uniquely determining the parameter values for which the only movable singularies of the solutions on the complex plane were poles.

It was the French mathematician Paul Painleve[36] who, following the ideas of Fuchs, Kovalewskaya and others, completely classified first order equations and studied second order equations. He found 50 types of second order equations whose only movable singularities were ordinary poles. This special analytical property now carries his name and is known as Painleve Property (PP). Of these 50 types of equations, 44 can be integrated in terms of known functions such as trignometric functions, Elliptic functions etc. The other six, inspite of having meromorphic solutions, do not have algebraic integrals that allow one to reduce the equation to quadratures. These are now known as Painleve transcendents. It has been found that P-transcendents often appear in similarity reduction of equations with solutions. Also, a certain relationship seems to exist between equation with the PP and Isomonodromy Transformation of certain linear equations.

The validity of P-analysis as a suitable procedure for detecting the integrability of an equation may lie here eventhough there is no definitive proof of why the singularity analysis for an equation turns out to be a test of integrability.

Although the P-equations are integrable in principle, their integration could not be performed with the methods available at that time.

Several attempts have been made to extend P-analysis results to higher order equations. After the introduction of IST, Ablowitz, Ramani and Segur (ARS)[37] developed an algorithm to determine whether an ordinary differential equation had the PP. An ODE is said to have the PP if the only movable singularities of its solutions are poles. The ARS algorithm is a method for determining the nature of the singularities of the solutions of an ODE on the basis of an analysis of their local properties. The study of similarity reductions of PDEs that can be solved by IST, led Ablowitz Ramani and Segur to formulate the ARS conjecture: "Every ordinary differential equation that can be obtained as the similarity reduction of a PDE solvable by IST has the PP up to a smooth change of variable" This conjecture provides a necessary condition for checking whether a PDE is integrable or not. To check that a PDE has PP using the ARS conjecture, one must find all the possible similarity reductions and check that all the resulting ODEs do have the PP, eventhough one has to make transformations of variables.Owing to the huge number of reductions to ODEs shown by some equations, the ARS conjecture becomes tedious. Also the number of symmetries shown by the original equation decreases during these reductions. Again, it is not clear which transformations of variables are permitted while checking whether the corresponding ODE is of P-type.

These limitations of this method suggest that it would be interesting to have a direct method that would allow one to study whether the PDEs under study are integrable. Weiss, Tabor and Carnevale[38] introduced the P-property for PDEs, or Painleve PDE test, as a method of applying the Painleve ODE test directly to a given PDE without having to reduce it to an ODE. A PDE is said to possess the P-property if solutions of the PDE are "single-valued" in the neighbourhood of the non-characteristic, movable singularity manifolds. WTC proposed this method by seeking a solution of a given PDE in the form of a Laurent series.

$$U(z_1, z_2, ..., z_n) = U(\mathbf{z}) = \phi^{-p}(z) \sum_{j=0}^{\alpha} U_j(z)\phi^j(z) \tag{1.29}$$

where U_j(z), j=0,1,2,... are analytic functions of $\mathbf{z} = (z_1,z_2,...,z_n)$ with $U_0(\mathbf{z}) \neq 0$ in the neighbourhood of a non-characteristic, movable singularity manifold defined by ϕ(z) $= 0$ where ϕ(z) is an analytic function of $z_1, z_2, ..., z_n$ Substituting (1.29) into the given equation and equating coefficients of like powers of ϕ determines p and defines reccursion relations for U_n, for n$\geq$1, of the function,

$$(n - \beta_1)(n - \beta_2)...(n - \beta_N)U_n = F_n(U_0, U_1, ..., U_{n-1}, \phi(z)), \tag{1.30}$$

where N is the order of the equation, for some function F_n. This defines U_n unless n = β_j for some j ,1 $\leq$ j $\leq$ N. n = β_1,β_2,... β_N are the resonances. For each positive integer resonance there is a compatibility condition (ie, F_β = 0) which must be identically satisfied for the pole to have a solution of the form (1) and then U_β(z) is an arbitrary function(commonly n = -1, is a resonance and it is usually associated to the singularity manifold defined by ϕ = 0, being arbitrary). Recently there have been studies into the role of negative resonances, suggesting that they are important[39]. The main three steps involving in the P-analysis of PDEs are (1) determination of the leading order behaviours (2) identification of the powers at which arbitrary functions can enter into the Laurent series called resonance and (3) verifying that at the resonance values, sufficient number of arbitrary functions exist without the introduction of movable critical manifolds. The remarkable feature of P-analysis is that a natural connection exists between the P-property and linearization properties, Lax pairs, Backlund transformations, integrability[36] etc.

1.3.4 Hirota method

In 1971 Hirota[40] introduced a new direct method for constructing multisoliton solution to integrable nonlinear evolution equations. The idea was to make a transformation into new variables, so that in these new variables multisoliton solutions appear in a particularly simple form. Hirota's method is actually a summation technique, effectively based on the Pade approximation. In this method, a dependent variable is replaced by a fractional form say, G/F and this equation to be satisfied by G and F are obtained in bilinear form. Power series solutions for G and F are then sought to provide a Pade approximation to the original quantity.

The Hirota's bilinear formalism has played a crucial role in the study of integrable systems[41-43]. The integrable PDEs that appear in some particle physics problems are not usually in the best form for further analysis. For constructing soliton solutions, Hirota's bilinear form is the best form and soliton solutions appear as polynomials of simple exponentials only in the corresponding new variables.

In order to write given nonlinear equation in the bilinear form, the first step is to transform this equation into a form that is quadratic in the dependent variables.In doing this, one should note that the leading derivative should go together with the nonlinear term, and in particular, have the same number of derivatives. Usually, the transformation to a new dependent variable is in the form[44],

$$U = \partial_x^2 w$$

where $U_t = K(u)$ represents the nonlinear PDE.Equations of this form can usually be bilinearized by introducing a new dependent variable whose natural degree would be zero, eg: $logF$ or f/g. The common form of this transformation is $U = 2\partial_x^2 logF$. Then substituting in the given equation results in an equation in F

In addition to being quadratic in the dependent variables, an equation in the bilinear form must satisfy a condition that the derivatives should only appear in combinations that can be expressed using Hirota's D-operator, which is defined by,

$$D_x^n f.g = (\partial_{x1} - \partial_{x2})^n f(x_1)g(x_2)$$

The D-operator, operates on a product of two functions such that

$$D_x f.g = f_x g - f g_x$$

$$D_x D_t f.g = f g_{xt} - f_x g_t - f_t g_x + f g_{xt}$$

The D-operator has some useful properties which help to find out the solution of the equation, expressed in the bilinear form, easily.

In order to find the solution of this equation, consider a whole class of bilinear equations of the form

$$P(D_x, D_y, ...)F.F = 0 \tag{1.31}$$

where P is a polynomial in the Hirota partial derivative D. We may assume that P is even, because the odd terms cancel due to the antisymmetry of the D operator.

The multisoliton solutions are obtained by finite perturbation expansion around the vaccuum $F = 1$,

$$F = 1 + \epsilon f_1 + \epsilon f_2 + \epsilon^3 f_3 + \tag{1.32}$$

j with ϵ as an expansion parameter.

For 1 soliton solution(1SS), only one term is needed. i.e., If we substitute $F = 1 + \epsilon f_1$ in equation (1.31), we get

$$P(D_{x,...})(1.1 + \epsilon 1.f_1 + \epsilon f_1.1 + \epsilon^2 f_1.f_1) = 0 \tag{1.33}$$

Collecting the terms of order ϵ^1, we get

$$P(\partial_x, \partial_y, ...)f_1 = 0 \tag{1.34}$$

The soliton solution corresponds to the exponential solution of (1.34). Usually we take

$$f_1 = e^\eta, \tag{1.35}$$

$$\eta = px + gy + \quad + constant \tag{1.36}$$

and then (1.34) becomes the dispersion relation on the parameters p,q,...

$$P(p, q, ...) = 0 \tag{1.37}$$

The order ϵ^2 term vanishes because

$$P(D)e^{\eta}.e^{\eta} = e^{2\eta}P(p - p) = 0 \tag{1.38}$$

by the vacuum condition $P(0, 0, ...) = 0$.

The two soliton solution (2SS) is built from two 1SS. One chooses the combination

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1 + \eta_2} \tag{1.39}$$

Substituting this in the given equation, we get the 2SS and also

$$A_{12} = \frac{-P(p_1 - p_2)}{P(p_1 + p_2)} \tag{1.40}$$

Thus, we were able to construct a 2SS for a huge class of equations, whose bilinear form is of the type(1.31). In particular, this includes many non-integrable systems also. Here the only condition is that the parameter p_i are only required to satisfy the dispersion relation. This extends to N soliton solution.

This method turned out to be very effective and was quickly shown to give N soliton solutions to the KdV, mKdV, SG and NLS equations. It is also suitable for obtaining several types of special solutions of many nonlinear evolution equations. Moreover,it has been used for the study of algebraic structure of evolution equations and extension of integrable systems.

The disadvantage of this method is that the process of bilinearization is far from being algorithmic. It is also difficult to find beforehand how many new independent or dependent variables are needed for bilinearization. Recently there have been some indications that singularity analysis can be used to find the transformation since the number of dependent variables seems to be related to the number of singular manifolds[45].

The IST method is more powerful since it can handle general initial conditions and at the same time more complicated. If one just wants to find soliton solutions, Hirota's method is the fastest in producing results.

1.3.5 Backlünd transformations

Backlünd transformations were organised in 1880s from the work of Lie and Backlünd for the study of surfaces in differential geometry[46,47]. They arose as a generalization of contact transformations. That is, transformations that take surfaces with a common tangent at a point in one space into surfaces in another space, which also have a common tangent at the corresponding point. A Backlund transformation is essentially defined as being a system of equations relating the solutions of a given equation, either to another solution of the same equation or to a solution of another equation.

This can be illustrated as follows. Consider two uncoupled partial differential equations in two input variables x and t, for two functions u and v; such that $P(u) = 0$ and $Q(v) = 0$ where P and Q are two operators, which are in general nonlinear. Let $R_i = 0$ be a pair of relations,

$$R_i(u, v, u_x, v_x, u_t, v_t, ...; x, t) = 0$$

between the two functions u and v. Then $R_i = 0$ is a Backlünd transformation, if it is integrable for v when $P(u) = 0$ and if the resulting v is a solution of $Q(v) = 0$, and vice versa. If $P = Q$ so that u and v satisfy the same equation, then $R_i = 0$ is called an auto-Backlünd transformation.

One of the simplest auto-Backlünd transformations is the pair,

$$U_x = V_t, \;\; U_t = -V_x$$

the Cauchy-Riemann relation for Laplace's equation,

$$U_{xx} + U_{yy} = 0, \;\; V_{xx} + V_{yy} = 0.$$

Thus, if $V(x, y) = xy$, then $U(x, y)$ can be determined from $U_x = x$, $U_y = -y$ and so $U(x, y) = \frac{1}{2}\,(x^2 \;\; y^2)$ is another solution of Laplace's equation.

Another example is the SG equation which can be written as

$$\theta_{\zeta\eta} = sin\theta \tag{1.41}$$

with the transformation $\zeta = \frac{1}{2}(\mathrm{x} + \mathrm{t})$, $\eta = \frac{1}{2}(\mathrm{x} \;\; \mathrm{t})$. The BT for this case is found to be

$$\theta_{1,\zeta} = 2a\ sin[\frac{1}{2}(\theta + \theta_0)] + \theta_{0,\zeta} \tag{1.42}$$

$$\theta_{1,\eta} = \frac{2}{a} sin[\frac{1}{2}(\theta - \theta_0)] - \theta_{0,\eta} \tag{1.43}$$

where a is a constant. Assuming $\theta_0 = 0$, one obtains

$$\theta_1 = 4\ tan^{-1}[exp(a\zeta + \frac{\eta}{a})] \tag{1.44}$$

By considering

$$c = \frac{(1 - a^2)}{(1 + a^2)}$$

we get the soliton solutions,

$$\theta = 4tan^{-1}(exp\frac{[x - ct - x_0]}{[1 - c^2]^{\frac{1}{2}}}) \tag{1.45}$$

$$\theta = 4tan^{-1}(exp\frac{[-(x-ct-x_0)]}{[1-c^2]^{\frac{1}{2}}}) \tag{1.46}$$

In short, the BT method enables one to find a new solution from a given one. When applied repeatedly, the BT equation gives the breather and the N-solutions of the SG equation. It can also be applied to the KdV and other equations[46]. The difficulty is in finding P and Q.

1.4 Complete integrability

The word "integrability" actually comes from "integral" which is closely connected with differential equations. The integrable systems with finite degrees of freedom could be best described by integrable nonlinear ordinary differential equations. Systems exhibiting regular and periodic behaviour having analytic solution with infinite degrees of freedom could be expressed in terms of partial differential equations (pdes). The nonlinearity inherent in most classical equations of motion makes the question of stability and the prediction of long term behaviour difficult[48].

According to Poincare, integrating a differential equation means finding the general solution in terms of finite expression, possibly multivalued in a finite numbers of functions. The word finite indicate that integrability is related to a global rather than local knowledge of the solution. The properties and behaviors of integrable dynamical systems are typical. But in generic families of dynamical systems, the integrable ones are rare. Also an arbitrary small change in an integrable system can destroy its integrability still, some structure of the integrable systems persists under perturbations. In fact, at the beginning of this century, only a few integrable dynamical systems were discovered and their importance was mainly owing to their mathematical beauty. Segur[49] points out that "if a given problem can be approximated by an integrable model, then it is likely that it can also be approximated to the same accuracy by a model that is not integrable" But, Calegero[50] argues that a limiting procedure applied to a large class of nonlinear pdes leads to a universal equation which is integrable. If this limiting procedure is physically reasonable, this guarantees the wide applicability of the integrable equations. One expects integrable equations to play a non-negligible role in the description of realistic physical systems, eventhough they are expected to describe some limiting asymptotic situation.

The term integrability indicates the existence of integrals of motion. Complete integrability means that these integrals exist in sufficient number. For the complete integrability of a Hamiltonian system with n degrees of freedom[48], there exists n first integrals in involution i.e., the vanishing Poisson bracket allows the construction of n integrals $F_1, F_2, ..., F_n$ including Hamiltonian itself.

$$[F_i, F_j] = 0, \quad i = 1, ..., n, \quad j = 1, ..., n \tag{1.47}$$

In the case PDEs, Mussette summerises the meaning of complete integrability as[51]

(1) either the nonlinear pde can be related to a linear pde by an explicit linear transformation

(2) or the equation passes the P-test and possesses P-property for pdes

(3) or the equation possesses solitary waves, N-soliton solutions for arbitrary N, an infinite number of conservation laws, etc.

(4) or the equation satisfies the ARS conjecture on the relationship of all its relations to ODEs without movable critical points.

Partial integrability means that some of the above listed properties are not satisfied but the equation possesses explicit analytic solutions such as degenerate solitary waves, N soliton solutions with N bounded, etc.

Another approach to the detection of integrability is through the numerical study of the behaviour of the solutions in real time. The numerically detected chaos is a clear indication of the nonintegrability of the given system. But it is not true that nonintegrable systems do not necessarily exhibit large scale chaos. Hence chaos may appear even in the simulation of an integrable system; if one is careless with numerical implimentation. Hence we can conclude that there is no particular definition, that is universally accepted, for complete integrability.

Chapter 2

Integrability studies on a perturbed NLSE

2.1 Introduction

Among the important class of nonlinear integrable systems, the well known nonlinear Schrödinger equation plays a significant role in the theory of envelope of wave trains in which no dissipation occurs[52,4]. The wide applicability is due to the presence of the special type of stable solitary wave solutions, called envelope solitons. For example, in optics, these solitons are expected to be suitable information carriers in optical fiber communication systems[53-54]. Solitons themselves can form a nonlinear superposition but do not mix their energies; i.e., the interaction of solitons in any integrable system is elastic. This may be due to the fact that the associated equations possess an infinite number of conserved quantities. The simple form of NLSE is

$$\beta U_{xx} + \gamma U | U |^2 = iU_t$$

where β and γ are constants. Here $| U |^2$ represents the potential which has the effect of trapping the wave energy which otherwise tends to spread due to dispersion. At some values of the pulse width, the spreading effect due to nonlinearity balances and a stationary pulse can be formed.

NLSE has two types of soliton solutions, namely *bright solitons* which arise when the dispersion and the cubic nonlinear coefficients have identical signs, and *dark solitons*, which occur when the two coefficients take opposite signs.

The simple cubic NLSE belongs to the class of integrable systems and can be solved by IST[1-3]. Grimshaw has investigated the slowly varying solitary wave solution of the variable coefficient NLSE[55]. Since a small variation in the guiding equation can destroy the integrability of the given system and can affect the solitary wave solution, it is important to study the NLSE with different perturbations[56-68]. It has been shown that the interaction of solitons described by nonintegrable equation leads to additional radiation being emitted from the impact area of the soliton radiation. However, the rate of radiation and the details of interaction between the soliton like solutions in the nonintegrable systems is still an open question.

An important field of research today is the propagation of solutions of NLSE in fiber, which has experimentally proved to be an efficient way of pulse compression. The theoretical model assumed lossless fibers. Since the real fibers have finite losses, it is difficult to apply soliton propagation in practical long distance systems. The first experimental observation of solitons in optical fibers was made in 1980[69,70]. Now, of the research laboratories around the world, solitons are proving the key to repeaterless transoceanic optical fiber cables. An array of nonlinear wave guides seems to be a unique system for observation of the competition between nonlinearity and disorder[71]. Taking into account these facts, in this work we perform an analytical investgation of the influence of nonlinearity on the process of envelope soliton propagation.

The perturbed NLSE under study is of the form,

$$ir_t + r_{xx} + 2|\, r\,|^2 r = iF \tag{2.1}$$

where r is the complex field envelope. This equation is assumed to represent a small perturbing influence on the propagating soliton through a non-ideal anomalously dispersive single mode optical fiber. With an appropriate choice of F, the complex term iF can represent this perturbing influence. The perturbation is responsible for the generation of a background field, which is superimposed on the soliton pulses. Depending on the nature of perturbation, this can exhibit quite complicated features. We take a general perturbation in nonlinearity; i e, $F = i\varepsilon f(x,t)| r |^2 r$ where f is a real valued function and carry out the integrability studies using Painleve and Lax methods. We also try to find the nature of the solution of this equation when the function f depends only on time. Over the large transoceanic distances with signal levels large enough to dominate over the amplified spontaneous emission, it is impossible to avoid Kerr nonlinearity, which can lead to serious signal distortions. Recently, Burtsev etal [72] studied the interaction of the fundamental soliton with a localized inhomogeneity in nonlinearity represented by δ-function. They found that when the input is a fundamental soliton, the amplitude of the output soliton rapidly decreases with increase in inhomogeneity strength and at a critical point, it bifurcates. The present study also reveals similar results and can be considered to be a generalization of the δ function perturbation.

In this chapter, we study the integrability property of the perturbed NLSE

$$ir_t + r_{xx} + 2| r |^2 r = -\varepsilon f(x,t)| r |^2 r \tag{2.2}$$

In the first section,we use the Painleve method. Here we assume a Laurent series solution and carry out the analysis. This method is actually a natural generalization of the exploration of critical singular points associated with solution of an ordinary differential equation in the complex plane With the perturbing term $\varepsilon f | r |^2 r$, the NLSE is found to pass the Painleve test irrespective of whether f depends on x and t or only on t. We obtain a Backlünd transformation also for the equation. The next section deals with the Lax integrability of the nonlinear system. It is found to have Lax pairs when f is both x and t dependent subject to a certain condition. But when f is time dependent only, Lax integrability fails. In the third section, we try to find the solution of the equation using direct integration This can be done only if f is independent of x and depends only on t We also study its variation with perturbation strength graphically. Discussions and conclusions are presented in the final section.

2.2 Painleve analysis

Painleve analysis is considered to be the most powerful method for identifying integrable systems This method can be applied to systems of ordinary and partial differential equations alike. Here we use the WTC method[38] for pdes Painleve pde test-in which there is no need to reduce the pde to an ode. As explained in the first chapter, a pde is said to possess the Painleve property if the solutions of the pde are singlevalued in the neighbourhood of a non-charecteristic movable singularity manifold. Before applying the P-test, we seek a solution of the given equation in the form of a Laürent series:

$$U(x,t) = \phi^{\alpha} \sum_{j=0}^{\infty} U_j \phi^j \tag{2.3}$$

where ϕ and U are analytic functions of x and t.

The major difference between the Painleve analysis of odes and pdes is that the singularities of the latter, in general, are not isolated as the solutions are functions of several complex variables, but rather lie on manifolds determined by the condition;

$$\phi(x,t) = 0$$

Thus if $U(x,t)$ is a solution of the pde,

$$U_t + K(U) = 0 \tag{2.4}$$

then we require that, in the neighbourhood of the manifold

$$\phi(x,t) = 0, \tag{2.5}$$

$U_0 \neq 0 and$,$U_j = U(x,t)$ and $\phi = \phi(x,t)$ are analytic functions of x and t and α is an integer.

The WTC formalism is important because it can lead to connection with solitons and other integrability properties. Also there exists a natural connection between the P-property and the linearization property, Lax pairs , Backlünd transformations, etc.

In order to apply the P-test to equation (2.2), we rewrite it in terms of the two complex valued functions U and V defined by $U = r$ and $V = r^*$ Then we have,

$$iU_t + U_{xx} + 2U^2V(1 + \varepsilon f) = 0 \tag{2.6}$$

and

$$-iV_t + V_{xx} + 2V^2U(1 + \varepsilon f) = 0 \tag{2.7}$$

Now we seek solutions of equations (2.6) and (2.7) in the Laurent series form:

$$U(x,t) = \phi^\alpha \sum_{j=0}^{\infty} U_j \phi^j \tag{2.8}$$

$$V(x,t) = \phi^\beta \sum_{j=0}^{\infty} V_j \phi^j \tag{2.9}$$

To simplify the calculations, we use the Kruskal ansatz[30]: $\phi(x,t) = x + \psi(t)$, where $\psi(t)$ is an arbitrary analytic function and U_j and V_j are analytic functions such that $U_0 = 0$ and $V_0 = 0$ in the neighbourhood of the non-characteristic movable singularity manifold $\phi(x,t) = 0$.

Assuming leading orders to be of the form $U = U_0\,\phi^\alpha$ and $V = V_0\phi^\beta$ we find , from equations (2.6) and (2.7), by balancing the dominant terms, that $\alpha = \beta = -1$ and

$$U_0 V_0 = \frac{-1}{(1+\varepsilon f)} \tag{2.10}$$

In order to find the resonances, i e, the power at which the arbitrary functions enter into the generalized Laurent expansion, we expand

$$U(x,t) = \sum_{j=0}^{\infty} U_j \phi^{j-1} \tag{2.11}$$

$$V(x,t) = \sum_{j=0}^{\infty} V_j \phi^{j-1} \tag{2.12}$$

and use them in equations (2.6) and (2.7), retaining leading order terms alone. Detailed calculations give the following resonance equation:

$$j(j+1)(j-3)(j-4) = 0 \tag{2.13}$$

and so the resonant values are j = -1,0,3 and 4.

To probe the existence of sufficient number of arbitrary functions, we substitute the Laürent series solutions given by the equations (2.11) and (2.12) into equations (2.6) and (2.7) and find the following by collecting the coefficients of different powers of ϕ:

ϕ^{-3} We get equation (2.10) which means that U_0 or V_0 is arbitrary for the arbitrary selection of functions.

ϕ^{-2}:

$$U_1 = \frac{-1}{2} U_0 \ \psi_t \tag{2.14}$$

$$V_1 = \frac{1}{2} V_0 \ \psi_t \tag{2.15}$$

ϕ^{-1}:

$$U_2 = \frac{1}{12} U_0 \ \psi_t^2 + \frac{i}{\varepsilon V_0} + (U_0 V_{0t} + 2V_0 U_{0t}) \tag{2.16}$$

$$V_2 = \frac{1}{12} V_0 \ \psi_t^2 - \frac{i}{\varepsilon U_0} + (V_0 U_{0t} + 2U_0 V_{0t}) \tag{2.17}$$

Similarly,collecting the coefficients of ϕ^0, we obtain

$$U_0 V_3 + V_0 U_3 = \frac{1}{2}(iV_0 U_{1t} + U_2 V_0 \psi_t - \frac{\psi_t^3 U_0 V_0}{4} - 2V_2 U_0 \psi_t) \tag{2.18}$$

$$U_0 V_3 + V_0 U_3 = \frac{1}{2}(-iU_0 V_{1t} + V_2 U_0 \psi_t - \frac{\psi_t^3 U_0 V_0}{4} - 2U_2 V_0 \psi_t) \tag{2.19}$$

Substituting the values of U_1, V_1, U_2 and V_2 in equations (2.18) and (2.19), we can show that the rhs are also equal which means that U_3 or V_3 is arbitrary. Proceeding further to coefficient of ϕ^1, we found that U_4 or V_4 is arbitrary.This procedure admits a straightforward extension to higher orders.

It is possible to construct the Backlünd transformation(BT) and Hirota bilinear form from the singular expansion obtained for the pde. That is, by truncating the Laürent expansions (2.11) and (2.12) up to the constant level term, we can formally write the BT as

$$\begin{aligned} r = U = U_0\phi^{-1} + U_1, \\ r^* = V = V_0\phi^{-1} + V_1 \end{aligned} \quad (2.20)$$

In order to derive the Hirota bilinear form, we consider the vacuum solution

$$U_1 = V_1 = 0$$

in the above equations. Then we have

$$\begin{aligned} r = U_0\phi^{-1}, \\ r^* = V_0\phi^{-1} \end{aligned}$$

This suggests us to take the bilinear transformation in the form,

$$r = \frac{g}{f}$$

and

$$rr^* = \mid r \mid^2 = (\frac{-1}{1+\varepsilon f})\frac{\partial^2}{\partial x^2} ln\phi$$

Here g is a real function and ϕ is a complex function. Under these transformations,equation (2.2) can be linearized as,

$$(iD_t + D_x{}^2)f.g = 0 \quad (2.21)$$

where the operator D is defined as

$$D_x^m D_t^n f(x,t).g(x',t') \mid_{x=x',t=t'} = (\frac{\partial}{\partial x}-\frac{\partial}{\partial x'})^m(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'})^n f(x,t).g(x',t') \mid_{x=x',t=t'} \tag{2.22}$$

Equation (2.21) is the bilinear form of the cubic NLSE. The only difference is in the transformation.

Thus the solution (U V) of equation (2.2) admits the required number of arbitrary functions without the introduction of movable critical manifolds, and hence the (1+1) dimensional NLSE (2.2) is found to pass the P-test and it is expected to be integrable.

2.3 The Lax method

In the Lax method we consider two operators L and M, where L is the operator of the spectral problem and M, the operator governing the associated time evolution of the eigenfunctions such that

$$L\psi = \lambda\psi \tag{2.23}$$

$$\psi_t = M\psi, \tag{2.24}$$

where the subscript t denotes the differentiation with respect to time. Then the equation

$$L_t + [L, M] = 0 \tag{2.25}$$

with $\lambda_t = 0$ is called Lax's equation. Here $[L, M] = LM - ML$

If a nonlinear pde arises as the compatibility condition of two such operators L and M , then equation (2.25) is called Lax's representation and L and M constitute a Lax pair

The more general version of the Lax representation is given by the equation (1.28).

Here we take $\alpha = 1$ and $\beta = 1$ and we get

$$L_t - M_x + [L, M] = 0 \tag{2.26}$$

Here we shall confine ourselves to the case where L and M are 2×2 matrices and L is a linear function of λ. By the proper chice of: L and M, we can construct different integrable equations. Here, we shall assume that $L(x, t)$ is of the form,

$$L = i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \tag{2.27}$$

where q and r are complex valued functions of x and t. We shall so choose the matrix M(λ) that Equation (2.24) is reduced to certain pde in q and r.

M is chosen as follows ,

$$M = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & q_x \\ -r_x & 0 \end{pmatrix} - i \begin{pmatrix} rq & 0 \\ 0 & -rq \end{pmatrix} - i\varepsilon \begin{pmatrix} frq & 0 \\ 0 & -frq \end{pmatrix} \tag{2.28}$$

where $f(x,t)$ is a real function. Substituting Equations (2.27) and (2.28) in Equation (2.26) and solving, we find that Equation (2.26) is equivalent to the system of equations;

$$ir_t + r_{xx} + 2r^2 q(1 + \varepsilon f) = 0 \tag{2.29}$$

$$iq_t + q_{xx} - 2q^2 r(1 + \varepsilon f) = 0 \tag{2.30}$$

setting q $= r^*$ we get,

$$ir_t + r_{xx} + 2 \mid r \mid^2 r(1 + \varepsilon f) = 0 \tag{2.31}$$

and for r $= q^*$,

$$iq_t + q_{xx} - 2 \mid q \mid^2 q(1 + \varepsilon f) = 0 \tag{2.32}$$

provided

$$\frac{\partial f(x,t) \mid q \mid^2}{\partial x} = 0 \tag{2.33}$$

The existence of Lax pairs indicates that the system given by Equations (2.31) and (2.32) may be integrable. When $\varepsilon = 0$, these equations reduce to simple cubic NLSE without any perturbation.

2.4 Solutions of perturbed NLSE

Let

$$r = \phi(x) \quad exp \quad (ikt) \tag{2.34}$$

be a solution of equation(2.31). The direct integration method will be now to find the solution of the equation with the perturbing coefficient depending only on time.

Substituting equation(2.34) in equation(2.31) we get,

$$\phi_{xx} = k\phi - \phi^3(1 + \varepsilon f) \tag{2.35}$$

On integration, this gives

$$\frac{1}{4}\phi_x^2 = k\phi^2 - \phi^4(1 + \varepsilon f) + constant. \tag{2.36}$$

Applying the boundary condition that ϕ vanishes at infinity, the constant in (2.33) can be dropped. On solving this we find,

$$\phi = \sqrt{\frac{2k}{1 + \varepsilon f}} sech \sqrt{2k}(x - x_0) \tag{2.37}$$

Therefore,

$$r(x,t) = \sqrt{\frac{2k}{1 + \varepsilon f}} exp(ikt) sech \sqrt{2k}(x - x_0) \tag{2.38}$$

Comparing this with the solution of simple cubic NLSE:

$$q(x,t) = \sqrt{2k} exp(ikt) sech \sqrt{2k}(x - x_0) \tag{2.39}$$

we can see that the amplitude of the wave given in equation (2.38) depends on the strength of the perturbation, i e, as the value of f(t) increases, the amplitude decreases [fig.2(a)] an effect which is expected because the existence of soliton solution in a nonlinear dynamical system is sensitive to perturbations. The variation of the soliton wave form with different values of the strength of the perturbations is shown in the graphs. Fig. 2(b) represents the soliton wave form when $\varepsilon = 0.005$. This remains unchanged upto $\varepsilon \approx 0.1$. If we again increase the value of ε, slight changes begin to occur in the profile and at $\varepsilon \approx 0.6$, the splitting of the wave becomes pronounced, as shown in fig.2(c). Fig.2(d) clearly shows the effect of perturbation at $\varepsilon \approx 1$. Beyond this value , small radiations arise on either side of the wave.

2.5 Conclusion

In this work, we considered a cubic nonlinear Schrodinger equation with a perturbation in nonlinearity. When the perturbation strength is zero, this equation reduces to simple cubic NLSE. With the perturbation, this equation represents the propagation of an optical soliton through a fiber with negative group velocity dispersion and positive Kerr coefficient. The perturbing term may represent a change in refractive index due to some external disturbance.

Since the simple NLSE is a completely integrable system, it is interesting to check whether it retains its integrability property under the perturbation. This equation is found to pass the Painleve criteria, irrespective of whether the perturbing function f is both space and time dependent or depending only on time. Hence we can conclude that the system is integrable in the Painleve sense.The Lax method gave a different result. When the function f depends on both time and space, this equation is found to posses Lax pair. But when f is independent of space and depends only on time, it is found that the equation does not have a Lax pair. This means that the system may be integrable in the Lax sense only when f is both time and space dependent. But the perturbation depending only on time is of importance in the real cases. For example, in the case of propagation of optical solitons through fibers the refractive index of the fiber may be influenced by the change of temperature, pressure,.. etc which may depend only on time. It is found that the amplitude of the soliton herein, obtained by direct integration, decreases with increase in perturbation strength. The graphical studies reveal that beyond a critical value of the perturbation strength, the waveform undergoes déformation and splits into pulses. This illustrates the

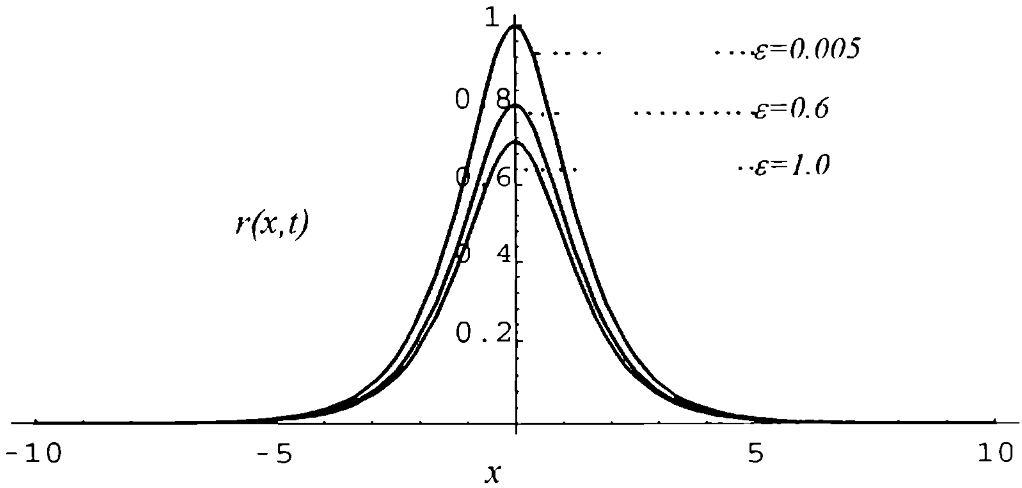

Fig.2(a). Variation of $r(x,t)$ w.r.to x and t for different values of ε

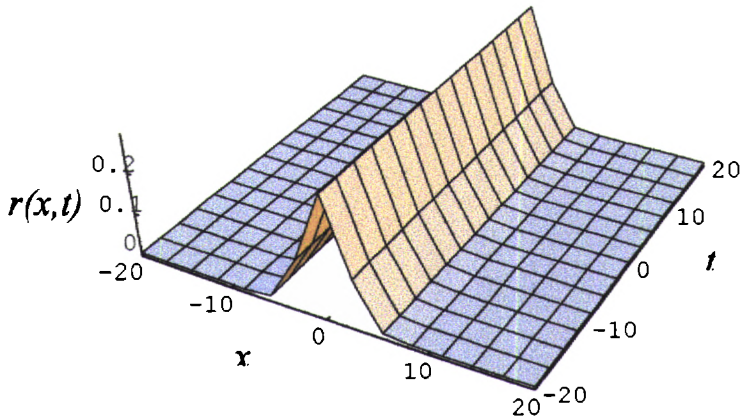

Fig. 2(b) The variation of *r(x,t)* with respect to *x* and *t* for *ε =0.005*

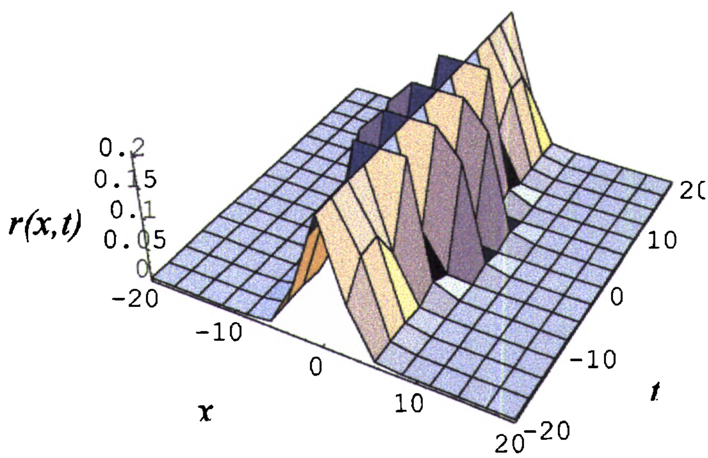

Fig.2(c) Variation of *r(x,t)* with respect to *x* and *t* for *ε=0.6*

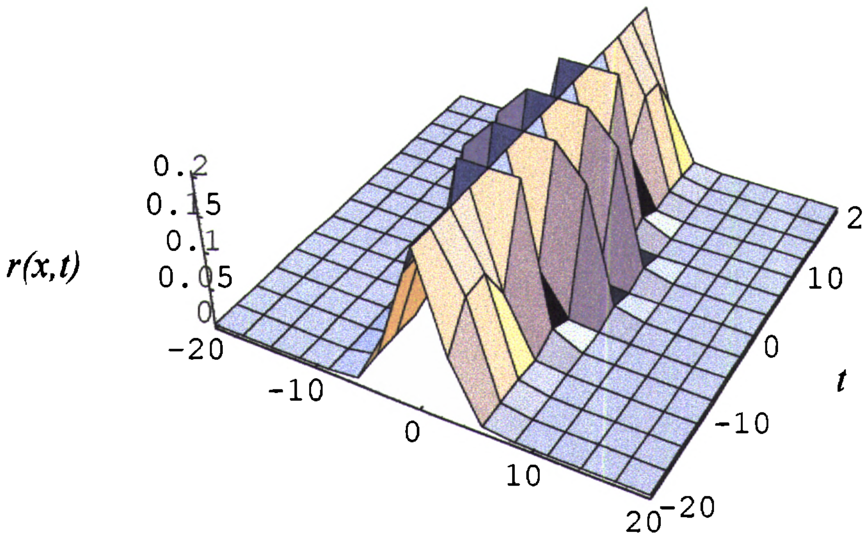

Fig.2(d) Variation of *r(x,t)* with respect to *x* and *t* for *ε=1.005*

influence of the inhomogeneous nonlinearity interacting with the soliton. For small value of the parameter ε this is not significant, but above a critical value it asserts itself. This proves the non integrability of the system with $f(t)$, as we obtained in the Lax method.

At least in the context of the perturbed NLSE we studied, it appears that the Lax method is a more refined indicator of integrability than the Painleve approach. We accordingly, conjecture that a fully integrable system will be simultaneously Painleve and Lax integrable under all circumstances, a criterion that is violated in the present case.

Chapter 3

Wave propagation through a 2D lattice

3.1 Introduction

The study of solitons on discrete lattices dates back to the early days of soliton theory[3] and is of great physical importance. The most important studies are on the effect of anisotropies and nonhomogeneities in the media on wave propagation[73]. Using lasers, it has been shown that the heat flow in solids is closely related to the flow of solitons. Davydov[74], by using some rules of solid state physics, had shown that the idea of soliton propagation is essential in the study of the chemical changes taking place in long protein molecules-which is the basis for the understanding of muscle contraction.

Generally, the relevant nonlinear equations which model these lattices cannot be solved analytically. Consequently, one looks for possible pulse soliton solutions in the continuum or longwavelength approximation. Only when this approach is not workable, one has to use numerical approaches or simulations. Nevertheless, there exist some lattice models for which the governing equations can be solved exactly[76]. The Fermi-Pasta-Ulam[77] problem together with the explanation of Zabusky and Kruskal can be considered to be the origin of lattice solitons. Zabusky[78,79] first showed that the continuum limit of FPU lattice was the KdV equation. This led to the discovery of lattice solitons. The most remarkable model for the study of lattice solitons is the Toda chain[80]. With nearest neighbour interaction, Toda chain happens to be the only integrable nonlinear model. Its applications in different fields like wave propagation in nerve systems, ladder circuit, chemical reaction in atoms and molecules and ecological systems make it very important and interesting from a physical point of view[81,82]. The general solution to the initial value problem of the Toda lattice has been found[83].

Recently, it has been found that, by considering the weak nonlinear case, it is possible to reduce a large number of one dimensional nonlinear systems to integrable ones. Some physically interesting cases in plasma physics, solid state physics etc.have been reduced to the well known simple model equations such as Burger's and KdV equations using weak nonlinear approximation(WNA)[84-85].

The weak nonlinear approximation rests mainly on two assumptions:

(1) the amplitude of the wave is small but finite, and

(2) the wave is a long wave or a modulation of a monochromatic wave.

As far as these two conditions are satisfied, this method is applicable to inhomogeneous systems including random systems. For such a system, it is desirable to have a consistent method to treat the weak nonlinear phenomena. It is found that the reductive perturbation method(RPM)[86,87] is very useful for carrying out weak nonlinear approximation. It takes into account a competition between nonlinearity and dispersion in a systematic manner. Various cases of nonlinear dynamics in fluids, nonlinear lattices and plasmas are reduced to soliton equations by RPM[88,89]. Then it becomes easy to study the waves analytically and explain the observation of soliton phenomena. If a time-dependent and homogeneous perturbation is added to the nonlinear system, we also obtain soliton systems[85]

Iizuka etal.[90] studied the propagation of nonlinear waves through an inhomogeneous lattice. They considered a one dimensional system and reduced the equation of motion to the known equations, Korteweg de-Vries(KdV), modified KdV and nonlinear Schrödinger(NLS) equations, for different perturbations using WNA. In this chapter, we extend our studies to a two dimensional lattice and investigate the propagation of nonlinear waves using the continuum approximation. Such models are associated with rather important problems in physics. The continuum approximation to lattice problems is used in many contexts because: (1) continuum approximation is easier for analytical as well as numerical study than its discrete counterpart, and (2) results can be conveniently related to the discrete version in many cases. This approach is regarded as an extension of the RPM and it is extremely useful in describing wave propagation in inhomogeneous media. Here we study the wave propagation through a 2D lattice for three special cases- quadratic nonlinearity, cubic nonlinearity and both of these together. In each case, the equation of motion reduces to different nonlinear equations. For quadratic nonlinearity, we get the well known Kadomtsev-Petviashvili(KP) equation and for cubic nonlinearity, modified KP equation. When both of them are applied together, we arrive at an integro-differential equation.

3.2 Reductive perturbation method(RPM)

In the study of the asymptotic behaviour of nonlinear dispersive waves, Gardner and Morikawa[87] introduced the scale transformations

$$\zeta = \epsilon^{\alpha}(x - \lambda t)$$

$$\tau = \epsilon^{\beta} t$$

This transformation is called the Gardner-Morikawa transformation, and may be derived from the linearized asymptotic behaviour of long waves. They combined this transformation with a perturbation expansion of the dependent variables so as to describe the asymptotic nonlinear behaviour. In that process they arrived at the KdV equation as a single tractable equation describing the asymptotic behaviour of a wave.

The perturbation method has been developed and formulated in a general way by Taniuti and his collaborators[88,89] and this method is now known as Reductive Perturbation Method(RPM) This method was first established for the reduction of a fairly general nonlinear system to a single tractable nonlinear equation.

3.3 Formulation of the problem

We consider a nonlinear lattice where the masses of the particles are not equal. The force due to the spring between two adjacent particles is assumed to be

$$F = K(\Delta + \alpha\Delta^2 + \beta\Delta^3 + ...) \tag{3.1}$$

where Δ is the elongation of the spring and K is the spring constant. Let m_i be the mass and a_i be the displacement of the i^{th} particle. Then the equation of motion for the i^{th} particle is

$$\begin{aligned} m_i\,\ddot{a}_i = K[a_{i+1} - a_i + \alpha(a_{i+1} - a_i)^2 + \beta(a_{i+1} - a_i)^3 + ...]- \\ K[a_i - a_{i-1} + \alpha(a_i - a_{i-1})^2 + \beta(a_i - a_{i-1})^3 + ...] \end{aligned} \tag{3.2}$$

We assume that the inhomogeneity is small and does not depend explicitly on time. Let us suppose that

$$m_i = \tilde{m}(1 + \rho) \tag{3.3}$$

$$\rho = \varepsilon\rho_1 + \varepsilon^2\rho_2 + \tag{3.4}$$

where $\tilde{m}$ is the average mass and ρ_1 , ρ_2.... are functions of the lattice site i. Let the lattice spacings be h in the x direction and k in the y direction. Hence $a_i = a_i(x, y, t)$

The following three wave motions will be considered separately:

(a) Slowly varying in x, y and t for quadratic nonlinearity, ie, $\alpha \neq 0, \beta = 0$

(b) Slowly varying in x, y and t for cubic nonlinearity, ie, $\alpha = 0,\ \beta > 0$

(c) Slowly varying in x, y and t for quadratic nonlinearity along with cubic nonlinearity, ie, $\alpha > 0,\ \beta > 0$

The continuum case is physically acceptable when the wavelength is very large compared to the spacing of particles in a lattice., ie the wave is so smooth that one can make the Taylor expansion on a_{i+1}. Since we are interested in wave propagation through a 2D lattice, we may expand a_{i+1} in a Taylor series for two variables:

$$a_{i+1} = a_i + ha_x + ka_y + \frac{1}{2}[h^2 a_{xx} + 2kha_x a_y + k^2 a_{yy}] + \qquad (3.5)$$

where a_x, a_y,...etc are corresponding derivatives of a_i

Case (a):Quadratic nonlinearity ($\alpha \neq 0, \beta = 0$)

For $\beta = 0$, eqn(3.2) becomes,

$$m_i \ddot{a}_i = K[a_{i+1} - a_i + \alpha(a_{i+1} - a_i)^2 + \ldots - (a_i - a_{i-1}) - \alpha(a_i - a_{i-1})^2 - \ldots.] \qquad (3.6)$$

From eqs (3.3), (3.4), (3.5)and(3.6), we have,

$$(1+\rho)\ddot{a}_i = \frac{K}{\tilde{m}}[h^2 a_{xx} + k^2 a_{yy} + 2hka_{xy} + 2\alpha h^3 a_x a_{xx}$$
$$+2\alpha hk^2 a_x a_{yy} + 2\alpha kh^2 a_y a_{xx} + 2\alpha k^3 a_y a_{yy} + \frac{h^4}{12} a_{xxxx} + ...] \qquad (3.7)$$

Now we introduce a change of independent variables x, y and t into η, ζ and τ:

$$\eta = \frac{\varepsilon}{h}(x - vt) \qquad (3.8)$$

$$\zeta = \frac{\varepsilon^2}{k} y \qquad (3.9)$$

$$\tau = \frac{\varepsilon^3}{24h} t \qquad (3.10)$$

Here v is the velocity of sound given by $v = h\sqrt{\frac{K}{\tilde{m}}}$. Again,

$$a(x,y,t) = \frac{-\varepsilon}{4\alpha}\phi(\eta,\zeta,\tau) \qquad (3.11)$$

Using eqs(3.8),(3.9),(3.10),and(3.11) along with eqn(3.4) in eqn(3.7), we arrive at

$$(1+\varepsilon\rho_1+\varepsilon^2\rho_2+....)(\frac{-\varepsilon^3 v^2}{4\alpha h^2}\phi_{\eta\eta} + \frac{\varepsilon^5 v}{48\alpha h^2}\phi_{\eta\tau} - \frac{\varepsilon^7}{2944\alpha h^2}\phi_{\tau\tau})$$
$$= \frac{K}{\tilde{m}}(\frac{-\varepsilon^3}{4\alpha}\phi_{\eta\eta} - \frac{\varepsilon^5}{4\alpha}\phi_{\zeta\zeta} - ....) \qquad (3.12)$$

Equating coefficients of equal powers of ε on either side of the equation;

ε^3 $$\frac{K}{\tilde{m}} = \frac{v^2}{h^2} \tag{3.13}$$

which gives the velocity v.

ε^4 $$\rho_1 \phi_{\eta\eta} = 2\phi_{\eta\zeta} \tag{3.14}$$

ε^5 $$-\rho_2\phi_{\eta\eta} + \frac{1}{12v}\phi_{\eta\tau} = -\phi_{\tau\zeta} + \frac{1}{2}\phi_\eta\phi_{\eta\eta} - \frac{1}{12}\phi_{\eta\eta\eta\eta} \tag{3.15}$$

Again applying the change of variables;

$$X = \eta + 12\int \rho_2(\tau)d\tau, \quad T = \tau, \quad Y = y \tag{3.16}$$

and

$$U(X, Y, T) = \phi_\eta(\eta, \zeta, \tau) \tag{3.17}$$

Then the equation(3.15) reduces to

$$\frac{\partial}{\partial X}(U_T - 6UU_X + U_{XXX}) = -12U_{YY} \tag{3.18}$$

or

$$U_{TX} - 6{U_X}^2 - 6UU_{XX} + U_{XXXX} + 12U_{YY} = 0 \tag{3.19}$$

Hence , for quadratic nonlinearity, we reduced the equation of motion into the 2 dimensional form of KdV equation (now known as KP equation).

Case (b): Cubic nonlinearity $(\alpha = 0, \beta > 0)$

In this case, the equation of motion becomes

$$m_i \ddot{a_i} = K[a_{i+1} - a_i + \beta(a_{i+1} - a_i)^3 + \ldots - (a_i - a_{i-1}) - \beta(a_i - a_{i-1})^3 - \ldots] \quad (3.20)$$

We define $a(x, y, t)$ as;

$$a(x,y,t) = \frac{1}{\sqrt{6\beta}}\phi(\eta, \zeta, \tau) \quad (3.21)$$

Using the the same transformations (3.8),(3.9) and (3.10) the equation of motion becomes,

$$(1+\rho)\ddot{a_i} = \frac{K}{\tilde{m}}[h^2 a_{xx} + k^2 a_{yy} + 2hk a_{xy} + 3\beta h^4 a_x^2 a_{xx} + 3\beta h^2 k^2 a_y^2 a_{xx} + 3\beta k^2 h^2 a_x^2 a_{yy}$$
$$+ 3\beta k^4 a_y^2 a_{yy} + \frac{h^4}{12} a_{xxxx} + 6\beta h^3 k a_x^2 a_{xy} + 6\beta k h^3 a_y^2 a_{xy} + \ldots \quad (3.22)$$

Substituting (3.4) along with (3.21), and equating powers of ε on either side , we get, for ε^4

$$\rho_2 \phi_{\eta\eta} - \frac{\phi_{\eta\tau}}{12v} = \phi_{\zeta\zeta} + \frac{1}{12}\phi_{\eta\eta\eta\eta} + \frac{1}{12}\phi_\eta^2 \phi_{\eta\eta} \quad (3.23)$$

Again introducing the change of variables as in the previous case, we arrive at,

$$\frac{\partial}{\partial X}(-U_T - U_{XXX} - 6U^2 U_X) = 12 U_{YY} \quad (3.24)$$

or

$$U_{TX} + U_{XXXX} + 6U^2U_{XX} + 12UU_X^2 + -12U_{YY} = 0 \qquad (3.25)$$

This equation is called modified KP equation.

Case (c): Quadratic nonlinearity along with cubic nonlinearity ($\alpha > 0,\ \beta > 0$)

In this case, the equation of motion becomes

$$m_i\ddot{a}_i = K[a_{i+1}-a_i+\alpha(a_{i+1}-a_i)^2+\beta(a_{i+1}-a_i)^3...-(a_i-a_{i-1})-\alpha(a_i-a_{i-1})^2-\beta(a_i \qquad (3.26)$$

We define $a(x,y,t)$ as;

$$a(x,y,t) = A\phi(\eta,\zeta,\tau) \qquad (3.27)$$

where A is a constant. Using the the same transformations (3.8),(3.9) and (3.10) the equation of motion becomes,

$$(1+\rho)\ddot{a}_i = \frac{K}{\tilde{m}}[h^2a_{xx} + k^2a_{yy} + 2hka_{xy} + 3\beta h^4a_x^2a_{xx}$$
$$+2\alpha hk^2a_xa_{yy} + 2\alpha kh^2a_ya_{xx} + 2\alpha k^3a_ya_{yy} + \frac{h^4}{12}a_{xxxx}$$
$$+3\beta h^2k^2a_y^2a_{xx} + 3\beta k^2h^2a_x^2a_{yy}$$
$$+3\beta k^4a_y^2a_{yy} + \frac{h^4}{12}a_{xxxx} + 6\beta h^3ka_x^2a_{xy} + 6\beta kh^3a_y^2a_{xy} + ...] \qquad (3.28)$$

Substituting (3.4) along with (3.27) in (3.28), and equating powers of ε on either side , we get, for ε^4

$$\rho_2\phi_{\eta\eta} - \frac{\phi_{\eta\tau}}{12v} = \phi_{\zeta\zeta} + \frac{1}{12}\phi_{\eta\eta\eta\eta} + \phi_\eta^2\phi_{\eta\eta} + 2\phi_\zeta\phi_{\eta\eta} \tag{3.29}$$

Again introducing the change of variables as in the previous case, we arrive at,

$$\frac{\partial}{\partial X}(U_X - 12U_{XXX} - 12U^2U_X) - 12U_{YY} = 24U_YU_X + 24\frac{\partial}{\partial Y}\int U dX \tag{3.30}$$

This equation represents an integro-differential equation. This can be identified as a modified form of KP equation with the terms on the right hand side representing perturbations. This means that the system becomes more perturbed as we apply the quadratic and cubic nonlinearities together.

3.4 Conclusion

In this chapter, we have performed the problem of nonlinear wave propagation through a two dimensional lattice with nonuniform mass distribution. We have considered weak nonlinear approximation for (a) quadratic nonlinearity, (b) cubic nonlinearity and (c)quadratic nonlinearity along with cubic nonlinearity. Using RPM we reduced the equations of motion into three nonlinear equations for the three different cases. We derived Kadomtsev Petviashvili(KP) equation, modified KP and an integro-differential equation respectively for these three cases. The question of integrability of these equations is examined in the next chapter.

Chapter 4

Integrability studies on KP equations

4.1 Introduction

The mathematical structure of the theory of solitons is well established in one dimensional media. To a large extent, this has been done in the case of two dimensional media also[91-93]. The theory of solitons in three and higher dimensional media is still far from being understood; its construction may require basically new ideas on nonlinear processes. All soliton equations, in whatever guise they appear, have a common property that makes them all the same creature, namely, the integrable dynamical system.As explained earlier, the properties and behaviours of integrable dynamical systems are atypical[94]. The most important integrable equations in (1+1)dimension which model equations related to real world are the KdV and NLS equations. They are universal models describing one dimensional propagation of weakly nonlinear waves in weak and strong dispersion regimes repectively just like the harmonic oscillator in the universal description of motion around a stable equilibrium position. Their importance leads scientists to study these equations in two dimensions.

Ablowitz, Fokas and others[95] have carried out intense researches in understanding nonlinear dispersive waves in higher dimensions. The Kadomtsev-Petviashvili(KP)[96] and Davey- Stewartson(DS)[97] are some of the well studied (2+1) dimensional equations which are natural generalizations of the (1+1) dimensional KdV and NLS equations respectively[1-3]. Naturally, these (2+1) dimensional equations are richer in structure, where boundary conditions play a crucial role.

In the case of NLSE type equations, Calogero[98] explained how this type of equations can be extended to yield universal equations of N-wave interaction type, which of course, turn out to be widely applicable and generally integrable. According to him, the integrable equations may be obtained from very large classes of nonlinear evolution equations, by a procedure that is asymptotically exact in the limit of weak nonlinearity. In the absence of nonlinear effects, the amplitude of the solution would be a constant. By introducing appropriate scale variables to account for the space and time variation of the amplitude, it is found that in these variables, the amplitude generally evolves according to an equation belonging to a group of universal evolution PDEs of NLS type. The important point is that the derivation of these evolution equations from the original nonlinear evolution equation is exact(in an asymptotic sense, as the parameter ε that controls the weakness of nonlinearity vanishes). For this to happen, it is sufficient that the very large class of evolution equations from which they are obtainable contains just one integrable equation.

4.2 KP equation and its importance

The most important step in the study of higher dimensional equations is the discovery of the Lax pair of a (2+1) dimensional partial differential equation of physical interest- the KP equation. That is, when we consider a single spatial variable x, then the Lax equation is

$$L_t = [L, M] \tag{4.1}$$

But when L contains derivatives with respect to several variables then the Lax representation is radically different. Hence a generalisation of this is applied with

$$L = \alpha\frac{\partial}{\partial y} - M \tag{4.2}$$

where M is an ordinary differential operator in x.Then equation (4.1) can be written as

$$[\ \alpha\tfrac{\partial}{\partial y} - M, \tfrac{\partial}{\partial t} - A] = 0 \quad or \quad [U, V] = 0 \tag{4.3}$$

where $U = \alpha\frac{\partial}{\partial y} - M$ and $V = \frac{\partial}{\partial t} - A$

This gives the general form of the consistency relation for Lax pairs in (2+1) dimensions. As an example, let us take

$$M = -\frac{\partial^2}{\partial x^2} - u(x, y, t) \tag{4.4}$$

$$A = -4\frac{\partial^3}{\partial x^3} - 6u\frac{\partial}{\partial x} - 3u_x + 3\alpha w \tag{4.5}$$

Then (4.3) reduces to the system

$$w_x = u_y \tag{4.6}$$

$$u_t - 6uu_x - u_{xxx} - 3\alpha^2 u_{yy} = 0 \tag{4.7}$$

which is equivalent to the equation

$$\frac{\partial}{\partial x}(U_{TX} - 6UU_X + U_{XXX}) \pm U_{YY} = 0 \tag{4.8}$$

This is the generalized KdV equation for the two dimensional case and is known a as the KP equation because Kadomtsev and Petviashvili[96] showed that the equations govern slowly varying waves in dispersive media. The equation with +sign ($\alpha^2 = -1$) arises in the study of plasmas and so in the modulation of long weakly nonlinear water waves which propagate nearly in one dimension(i.e, nearly in a vertical plane). The equation with -sign ($\alpha^2 = +1$) arises in acoustics and lattice dynamics. Researches into physical, earth and life sciences have led to the discovery of hundreds more nonlinear evolution equations. Of these, only a few are known to have soliton solutions.

4.3 Painleve analysis (WTC Method)

In order to check the integrability of the obtained equations, we use the Painleve method(P-test) for partial differential equations[8]. As explained earlier, this method consists of determining the presence or absence of movable, noncharacteristic, critical singular manifolds. When the system is free from movable critical manifolds, the P-property holds, suggesting P-integrability. Main steps involved in the P-analysis of pdes are: (i) Determination of leading order behaviours, (ii) identification of powers at which arbitrary functions can enter into the Laürent series called resonances, (iii) verifying that at the resonance values, sufficient number of arbitrary functions exist without the introduction of movable critical manifolds. In the following, this method is applied to different systems;

4.3.1 (a) The KP equation

The equation is

$$(U_T + U_{XXX} - 6UU_X)_X + 12U_{YY} = 0 \tag{4.9}$$

In the leading order analysis, we take the first term in the Laurent series

$$U = \phi^\alpha \sum_{j=0}^{\infty} U_j \phi^j$$

as the solution of the equation (4.9). ie, Let

$$U = U_0 \phi^\alpha \tag{4.10}$$

Substituting this in eqn(4.9) and equating the leading order terms , we get, $\alpha = -2$ and $U_0 = 2\phi_X^2$

For obtaining the resonance values, we take

$$U = \sum_{j=0}^{\infty} U_j \phi^{j-2} \tag{4.11}$$

Again, substituting in equation(4.9) and equating leading order terms, we get,

$$(j+1)(j-4)(j-5)(j-6) = 0 \tag{4.12}$$

ie, the resonances are at j = -1, 4, 5 and 6.

These values of j correspond to points where arbitrary functions of (X,Y,T) are introduced into the expansion Here $j = -1$ corresponds to the arbitrary singularity manifold ($\phi(X,Y,T) = 0$). To prove the arbitrariness of other values, we subtitute the Laurent series $U = \sum_{j=0}^{\infty} U_j \phi^{j-2}$ in the equation(4.9). From the recursion relations we find that,

$$j = 0, \qquad U_0 = 2\phi_X^2 \tag{4.13}$$

$$j = 1, \qquad U_1 = 0 \tag{4.14}$$

$$j = 2, \qquad U_2\phi_X^2 - \phi_T\phi_X - \phi_Y^2 = 0 \tag{4.15}$$

$$j = 3, \qquad U_3 = \phi_{YY} + \phi_{XT} \tag{4.16}$$

$$j = 4, \qquad \frac{\partial^2}{\partial X^2}(\phi_X\phi_T + \phi_Y^2 + U_2\phi_X^2) = 0 \tag{4.17}$$

By eqn(4.15), U_4 is arbitrary.

$j = 5$, again U_5 is arbitrary if the compatibility condition

$$\frac{\partial^2}{\partial X^2}(\phi_{XT} + \phi_{YY} - U_2\phi_X^2) = 0 \tag{4.18}$$

By eqn(4.15), this is so.

Similarly, for $j = 6$, we obtain U_6 as arbitrary.

By the above considerations, the KP equation is found to pass the P-test and hence it is integrable in the P-sense.

4.3.2 (b) The mKP equation

Here the equation is

$$(U_T + U_{XXX} - 6U^2U_X)_X - 12U_{YY} = 0 \tag{4.19}$$

As in the previous case, for the leading order analysis, we take the first term in the Laürent series

$$\phi^\alpha \sum_{j=0}^{\infty} U_j \phi^j$$

as the solution of the equation (4.19). i.e.,

$$U = U_0 \phi^\alpha \tag{4.20}$$

Substituting this in eqn(4.19) and equating the leading order terms , we get, $\alpha = -1$ and $U_0 = \phi_X$

For obtaining the resonance values, we take

$$U = \sum_{j=0}^{\infty} U_j \phi^{j-1} \tag{4.21}$$

Again , substituting in equation(4.19) and equating leading order terms, we get,

$$(j+1)(j-3)(j-4)(j-4) = 0 \tag{4.22}$$

ie, the resonances are at j = -1, 3 and 4,

These values of j correspond to points where arbitrary functions of (X,Y,T) are introduced into the expansion Since these values do not give sufficient number of arbitrary functions, we can say that the equation is not integrable in the Painleve sense.

4.3.3 The perturbed mKP equation

The equation is

$$\frac{\partial}{\partial X}(U_X - 12U_{XXX} - 12U^2U_X) - 12U_{YY} = 24U_YU_X + 24\frac{\partial}{\partial Y}\int UdX \quad (4.23)$$

Due to the presence of the integral term, it is not possible to apply the P-analysis on this equation and hence it does not belong to the integrable class. It has been proved that a perturbed nonlinear equation is integrable when the perturbation is homogeneous. From the equation itself, it is clear that the perturbation is inhomogeneous, which means that it is nonintegrable.

4.4 Lax method

In the section 4.2 we have introduced the Lax pair for the KP equation from which we derived the integrable KP equation[5]. It is possible to find the Lax pair for the KP hierarchy. This can be done by generalizing the evolution operator V keeping the spectral operator U same. The general form of V is accordingly taken to be

$$V = \frac{\partial}{\partial t} + \beta \frac{\partial^{2m+1}}{\partial x^{2m+1}} + \Sigma_{n=1}^{m} [u \frac{\partial^n}{\partial x^n} - \frac{\partial^n}{\partial x^n} u] - B \qquad (4.24)$$

Now substituting this equation along with eqn (4.4) in eqn (4.3), we get the KP hierarchy. For $m = 1$, the system reduces to the first member in the KP hierarchy which is the same equation given by (4.8). For $m = 2$, we get the second member in the KP hierachy, namely,

$$\frac{\partial}{\partial x}[u_t - 3u_{xxx} + u_{xxxxx} + 3uu_x + 3uu_{xx} - \alpha u_{yxxx}] = 3\alpha^2 u_{yy} \qquad (4.25)$$

Thus, by assigning different values for m, it is possible to find different members of the KP hierarchy.

In the case of mKP equation(4.19) obtained in the case of quadratic nonlinearity, and the integro-differential equation(4.23) obtained when both cubic and quadratic nonlinearities are considered, we could not find any Lax pair. The non existence of Lax pairs clearly indicates the nonintegrability of the corresponding system(in the Lax sense).

4.5 Solitary wave solutions

Solutions of the KP equation and mKP equation can be found using the travelling wave method. Accordingly, we assume the solution of the equation(4.9) as

$$U(X, Y, T) = f(\xi) \tag{4.26}$$

where

$$\xi = X + Y - cT \tag{4.27}$$

Now we substitute this in equation (4.9), we get,

$$\frac{\partial}{\partial \xi}(-cf_{\xi} - 6ff_{\xi} + f_{\xi\xi\xi}) + 12f_{\xi\xi} = 0 \tag{4.28}$$

On integration, we obtain,

$$(-cf_{\xi} - 6ff_{\xi} + f_{\xi\xi\xi}) + 12f_{\xi} + A = 0 \tag{4.29}$$

where A is the constant of integration. Integrating again with respect to ξ, we get,

$$-cf - 3f^2 + f_{\xi\xi} + 12f + Af + B = 0 \tag{4.30}$$

Multiplying with f_{ξ} and integrating again, we finally arrive at,

$$(f_{\xi})^2 = 2(\frac{f^3}{3} + (\frac{c}{2} - 6 - \frac{A}{2})f^2 - Bf - C) \tag{4.31}$$

or

$$\xi - \xi_0 = \int \frac{df}{\sqrt{2(\frac{f^3}{3} + (\frac{c}{2} - 6 - \frac{A}{2})f^2 - Bf - C)}} \tag{4.32}$$

If eqn(4.29) is defined on the infinite domain, and applying the boundary condition f, f_ξ, $f_{\xi\xi}...\rightarrow 0$ as $\xi \rightarrow {}_+ \infty$, then it is easy to deduce from eqn(4.30), (4.31) and (4.32) that all constants of integration are zero. In this case, the quadrature (4.32) reduces to

$$\xi - \xi_0 = \int \frac{df}{f\sqrt{2f + (c-12)}} \tag{4.33}$$

which leads to the solution

$$f(\xi) = \frac{-1}{2}(c-12)sech^2(\frac{1}{2}\sqrt{(c-12)}(\xi - \xi_0)) \tag{4.34}$$

or

$$U(X,Y,T) = \frac{-1}{2}(c-12)sech^2(\frac{1}{2}\sqrt{(c-12)}(\xi - \xi_0)) \tag{4.35}$$

where $\xi = X + Y - cT$ This gives the soliton solution of the KP equation. The time evolution of this equation is as indicated in figure (4.1).

We use the same method to find the solution of the mKP equation. For this, we substitute eqn(4.26) in eqn(4.19). Then we get,

$$\frac{\partial}{\partial \xi}(-cf_\xi - 6^2 ff_\xi + f_{\xi\xi\xi}) + 12f_{\xi\xi} = 0 \tag{4.36}$$

On integration, we obtain,

$$(-cf_\xi - 6f^2 f_\xi + f_{\xi\xi\xi}) + 12 f_\xi + A = 0 \tag{4.37}$$

where A is the constant of integration. Integrating again with respect to ξ, we get,

$$-cf - 2f^3 + f_{\xi\xi} + 12f + Af + B = 0 \tag{4.38}$$

Multiplying with f_ξ and integrating again, we finally arrive at,

$$(f_\xi)^2 = (f^4 + (c - 12 - A)f^2 - 2Bf - 2C) = 0 \tag{4.39}$$

Again, by using the boundary conditions as in the previous case, the constants of integration can be neglected. Hence, we finally arrive at the equation,

$$\xi - \xi_0 = \int \frac{df}{f \sqrt{(f^2 + (c-12))}} \tag{4.40}$$

which gives the solution

$$f(\xi) = -(c - 12) sech(\sqrt{(c - 12)}(\xi - \xi_0)) \tag{4.41}$$

Thus we arrived at the solitary wave solution of the mKP equation The evolution of this solution for different times is sketched in figure (4.2).

Now, in the case of pertubed mKP equation, we could not find any steady state solution using the travelling wave method. This may be because of the presence of the additional term which represents an inhomogeneous perturbation. Hence we can conclude that the system represented by this equation shows chaotic behaviour.

4.6 Conclusion

In this chapter we considered the KP equation which is a two dimensional version of KdV equation. We have carried out integrability studies on the KP equation along with the modified KP and integro-differential equation using Painleve method and Lax method. In chapter 3, we obtained these equations by considering weak nonlinear approximation for (a) quadratic nonlinearity,(b) cubic nonlinearity and (c)quadratic nonlinearity along with cubic nonlinearity. The KP equation is found to pass P-test and hence it is integrable in the Painleve sense. i.e., the equation passes the P-test. The Lax pair for this equation is also known(given). Since this equation passes both the integrability criteria, it can be concluded that the system represented by this equation is completely integrable. For the mKP equation, we did not get sufficient number of resonances and hence this equation is not completely integrable in the P-sense. The non existence of Lax pairs proves its nonintegrability. We derived the steady state solutions of

The Evolution of solitary wave solution of KP equation at different time

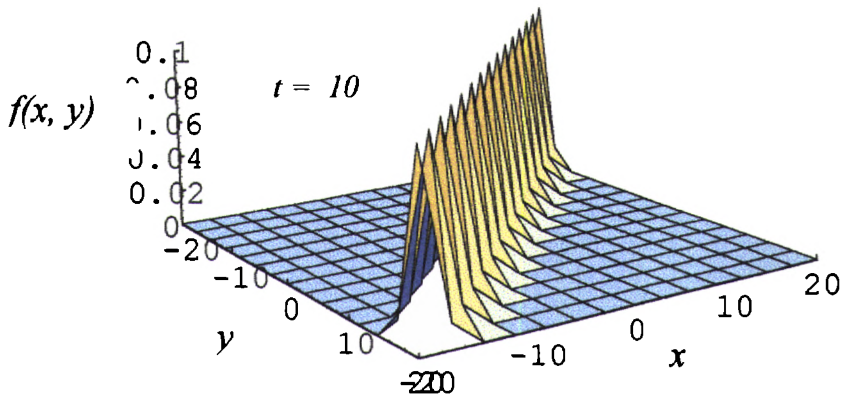

Fig. 4.1(a)

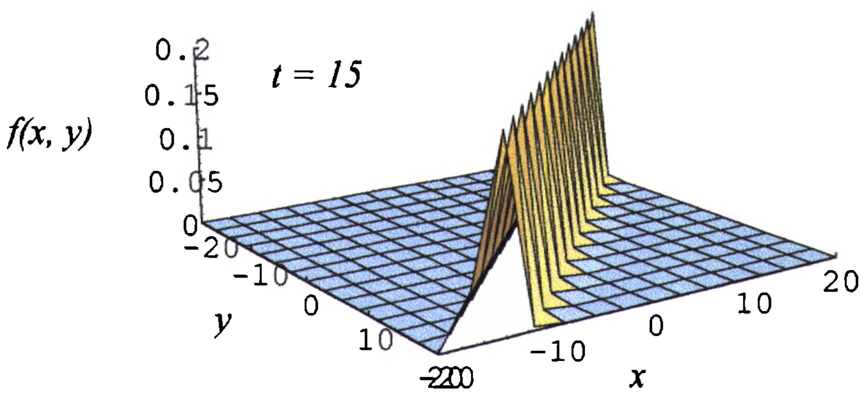

Fig. 4.1(b)

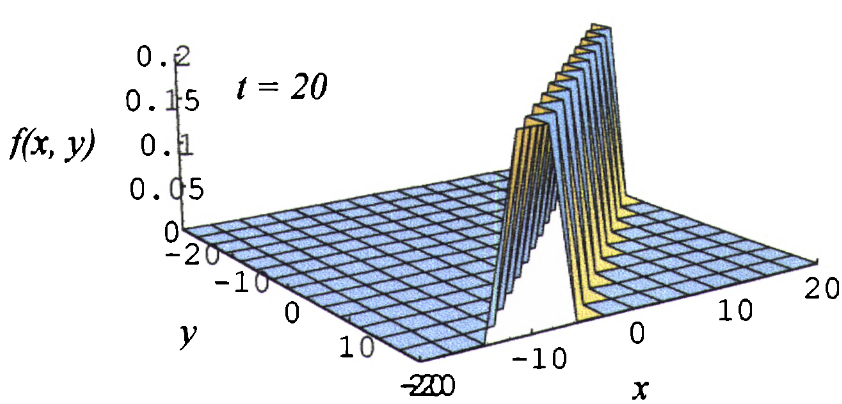

Fig. 4.1(c)

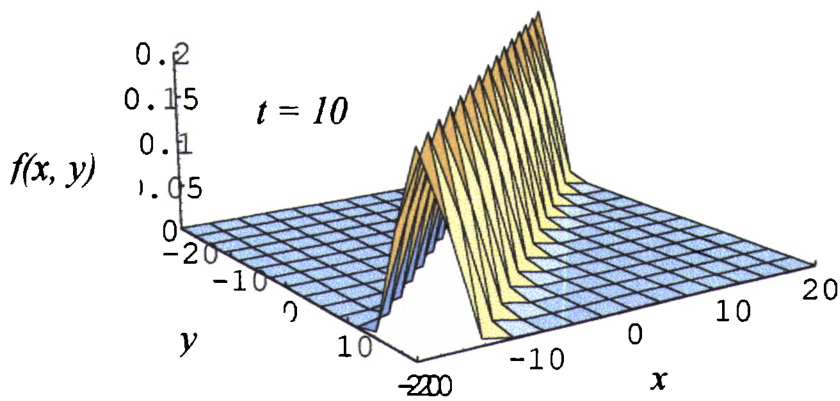

Fig. 4.2(a)

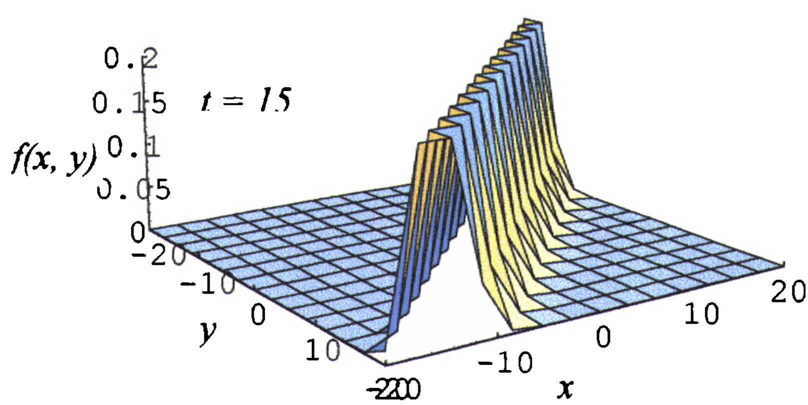

Fig. 4.2(b)

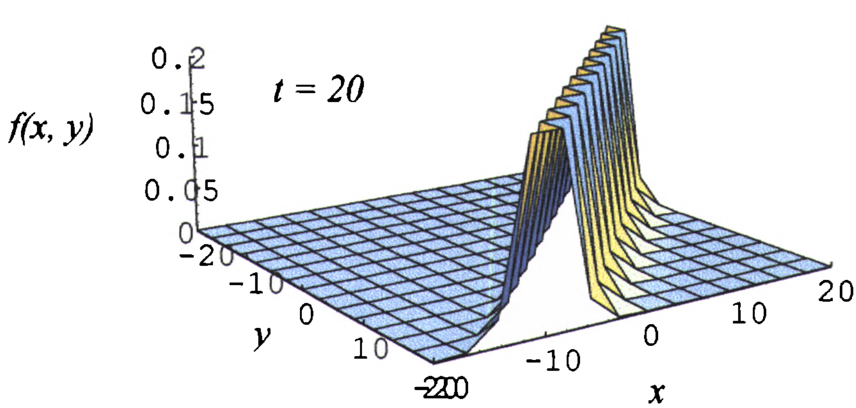

Fig. 4.2(c)

Evolution of solitary wave solution of mKP equation at different time

the KP and mKP equations. But in the case of the third equation, which is a form of modified KP equation with perturbation, we could not find any Lax pair and the P-analysis cannot be applied to it. Also, there is no steady state solution for this equation. Hence we can conclude that the system with quadratic nonlinearity is completely integrable while the system with cubic nonlinearity is partially integrable(in the P-sense). When the quadratic and cubic nonliearities act together, the system is not integrable and may become chaotic. As it is known, nonlinearity in evolution equations may cause quite different behaviours, from chaos to regular motion.

Chapter 5

Nonlinear quintic Schrödinger equation

5.1 Introduction

Even if the possibility of effective optical communication through fibres in the form of solitons was theoretically predicted by Hasegawa[99], the experimental observation was done only after ten years. During the last twenty years optical solitons have been discovered and investigated in a large number of different systems and more results and discoveries are still to come.

As explained in chapter 2, the Nonlinear Schrödinger equation[NLSE] plays an important role in both experimental and theoretical studies in optical fiber communication. Soliton solutions of NLSE arises in nature because the nonlinearity exactly balances the pulse broadening due to dispersion and they are usually called envelope solitons. Several model equations, obtained by slowly varying envelope approximation, are close to integrable equations[100]. As a consequence, many nonlinear optical effects in both passive and active media are presently controlled by the spectral theory of solitons.

In terms of complex amplitude $U(x,t)$, the nonlinear cubic Schrodinger equation(NLCSE) may be expressed as

$$iU_t + \gamma U_{xx} + \delta \mid U \mid^2 U = 0 \tag{5.1}$$

where γ and δ are constants. Here the second term originates from the group velocity dispersion(GVD). GVD occurs because the group velocity is different for different frequencies in the signal. The third term in equation(5.1) originates from the nonlinear effect, due to the fact that the wavelength depends upon the inensity of the wave. In NLCSE, it is the Kerr effect that provides the nonlinearity property in an optical fiber. It produces a change in refractive index of the fiber material. This is due to the deformation of the electron orbits in glass molecules by electric field of the incident radiation. The Kerr coefficient changes the refractive index from n_0 to $n_0 + n_2 \mid E \mid^2$. The change $n_2 \mid E \mid^2$ is very small for a relatively large electric field but such a small nonlinear effect is sufficient to compensate the small group velocity dispersion. Thus the negative GVD along with the nonlinear change in refractive index provides necessary and sufficient condition for optical solitons to be propagated in an optical fiber.

5.2 The Nonlinear quintic Schrödinger equation

The nonlinear Quintic Schrödinger equation (NLQSE) is obtained by modifying the NLCSE by concidering the different orders of nonlinear response of the material of the fiber to electromagnetic radiation. Here we consider the fifth order nonlinearity term in the electrical susceptibility of the medium in developing modified NLSE. The first step towards this was taken by Pushkarov etal[101]., and later by Cowan etal[102]. They considered the effect of quintic term appearing along with the conventional NLCSE. They have obtained the solutions of this nonlinear cubic-quintic Schrödinger equation(NLCQSE). This equation is derived by taking the fourth order in the expansion of the refractive index of an isotropic medium given by

$$n = n_0 + n_2 \mid E \mid^2 + n_4 \mid E \mid^4$$

In general, the coefficients n_2 and n_4 can be positive or negative depending on the medium and frequency selected. The NLCQSE is of the form;

$$iU_t + \gamma U_{xx} + \delta \mid U \mid^2 U + \delta \mid U \mid^4 U = 0 \qquad (5.2)$$

The solitary wave solution to this equation has been obtained as

$$U(x,t) = \frac{C_1 exp(ikt)}{\sqrt{C_2 cosh\sqrt{C_1}x + 1}} \tag{5.3}$$

where $C_1 = \sqrt{2k}$, $C_2 = \sqrt{1 + 48k}$

Cowan etal.,[102] have tested the stability of this solution using numerical methods.

Mohanachandran etal.,[103] developed a nonlinear Schrödinger equation in which there is only the quintic term as nonlinearity and hence the equation is called nonlinear Quintic Schrödinger Equation(NLQSE). While deriving NLQSE, it is assumed that the refractve index n_2 is absent and only n_4 is active. This may be achieved by doping a fiber with appropriate materials. The solutions to this NLQSE are found to possess soliton behavior They also studied the stability of this solution using variation method and it is found to be advantageous over the solutions to the conventional NLCSE. The form of the solutions of the NLCSE and NLQSE are given in the figure(5.1). Since the pulse width of the soliton solution of NLQSE is less than that of NLCSE, the solution of NLQSE is more stable. We have studied the integrability of NLQSE along the lines reported in the next section.

5.3 Integrability

5.3.1 Painleve analysis

The equation under study is

$$iU_t + \gamma U_{xx} + \delta \mid U \mid^4 U = 0 \tag{5.4}$$

Before taking the solution in the form of Laurent series, let us split the above equation into two separate equations. For this, let $U = q$ and $U^* = p$. Then the equation becomes

$$iq_t + \gamma q_{xx} + \delta q^3 p^2 = 0 \tag{5.5}$$

$$-ip_t + \gamma p_{xx} + \delta p^3 q^2 = 0 \tag{5.6}$$

Now, taking the solutions of these equations in the form

$$q = \phi^{\alpha} \Sigma_{j=0}^{n} U_j \phi^j \tag{5.7}$$

$$p = \phi^{\beta} \Sigma_{j=0}^{n} U_j \phi^j \tag{5.8}$$

Inorder to find the leading orders, we consider only the first terms in the solutions and substituting that in the equations(5.5) and (5.6) we get $\alpha = \beta = -\frac{1}{2}$ and $p_0^2 q_0^2 = -\frac{\gamma}{\delta}\frac{3}{4}$

To get the resonant values, we substitute equations(5.7) and (5.8) with $\alpha = -\frac{1}{2}$ in (5.5) and (5.6). Equating the leading order terms it is found that the resonances are at $j = -1, 0, 2$ and 3 irrespective of the values of γ and δ. As explained earlier, the resonances at $j = -1$ and $j = 0$ imply the fact that $\psi(t)$ is arbitrary and that there is only one equation defining U_0 and V_0 (ie., either U_0 or V_0 is arbitrary).

Arbitrary analysis at $j = 1$ gives the values of U_1 and V_1 as

$$U_1 = -iU_0\phi_t$$

$$V_1 = iV_0\phi_t$$

Similarly, $j = 2$ and $j = 3$ leads to the arbitrariness of U_2 or V_2 and U_3 and V_3. Hence the NLQSE is found to be integrable in the Painleve sense.

It is also possible to find the bilinear form of the NLQSE. For this, we take $U = \frac{g}{f}$ where f is a real function and g is a complex function. Substituting in the equation (5.4) we get

$$i(D_t + D_x^2)g.f + \frac{1}{2}D_x^2 g.f + \frac{g^3 g^{*2}}{f^5} = 0 \qquad (5.9)$$

This gives the bilinear form of NLQSE.

5.3.2 Lax method

The Lax method requires that if a given nonlinear equation be written in terms of two linear operators U and V, then the compatibility condition for them is the equation;

$$U_t - V_x + [U, V] = 0 \tag{5.10}$$

which leads to the original nonlinear equation.

Here also we shall confine ourselves to the case where U and V are 2×2 matrices and U is a linear function of λ, and assume $U(x,t)$ to be of the form,

$$U = i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \tag{5.11}$$

where q and r are complex valued functions of x and t. We shall so choose the matrix V(λ) that equation (2.10) is reduced to certain pde in q and r. V is chosen as follows ,

$$V = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & q_x \\ -r_x & 0 \end{pmatrix} - i\frac{\delta}{2} \begin{pmatrix} rq^2 & 0 \\ 0 & -rq^2 \end{pmatrix} \tag{5.12}$$

Substituting equations (5.11) and (5.12) in equation (5.10) and solving, we find that equation (5.10) is equivalent to the system of equations;

$$ir_t + \gamma r_{xx} + \delta r^3 q^2 = 0 \tag{5.13}$$

$$iq_t + \gamma q_{xx} - \delta q^3 r^2 = 0 \tag{5.14}$$

Setting q $= r^*$ we get,

$$ir_t + \gamma r_{xx} + \delta \mid r \mid^4 r = 0 \tag{5.15}$$

and for r = q^*,

$$iq_t + \gamma q_{xx} - \delta | q |^4 q = 0 \tag{5.16}$$

provided

$$\frac{\partial[| q |^4 - | q |^2]}{\partial x} = 0 \tag{5.17}$$

The existence of Lax pairs indicates that the system given by equations (5.15) and (5.16) may be integrable. When we consider the cubic term also, the U operator remains the same but the V operator changes and is given as

$$V = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & q_x \\ -r_x & 0 \end{pmatrix} - i\frac{\delta}{2} \begin{pmatrix} rq & 0 \\ 0 & -rq \end{pmatrix} - i\frac{\delta}{2} \left(\begin{matrix} rq^2 \\ 0 \end{matrix} \right. \tag{5.18}$$

Substituting the values of U and V in the compatibility condition we arrive at the NLCQSE

$$iU_t + \gamma U_{xx} + \delta \mid U \mid^2 U + \delta \mid U \mid^4 U = 0 \tag{5.19}$$

5.4 Conclusion

In this chapter we considered the nonlinear quintic Schrödinger equation in which the cubic nonlinearity is absent and only the effect of quintic nonlinearity is dominant. The single soliton solution, already given in [103], shows that it is more stable than that of simple NLCSE. The integrability studies of this equation using P-analysis show that this equation is

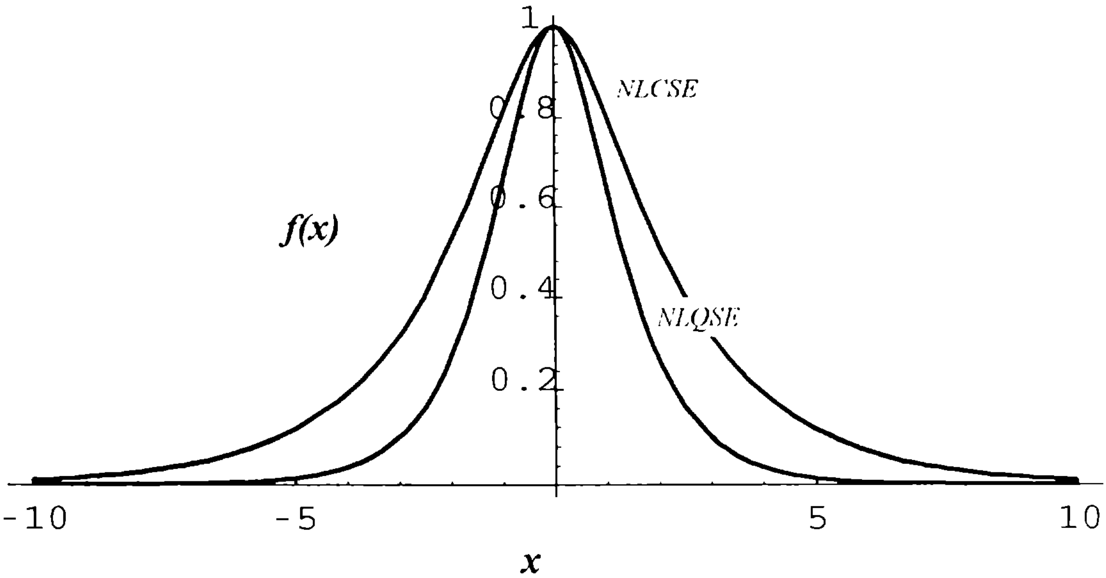

Fig 5.1 Solitary wave forms of NLCSE and NLQSE

integrable in the P-sense. Even if the α value is a fraction, we could prove the arbitrariness of the resonant values using the Kruskal ansatz[30]. But the Lax method imposes some conditions for the existence of Lax pairs. The integrability studies reveal the integrability properties depend on the order of nonlinearity and may be destroyed when the effect of higher order terms increases.

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