STOCHASTIC MODELLING AND ANALYSIS

STOCHASTIC INVENTORY WITH/WITHOUT SERVICE TIME, RENEGING AND RETRIAL OF CUSTOMERS

THESIS SUBMITTED TO
THE COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
UNDER THE FACULTY OF SCIENCE

BY

JOSE K.P.

DEPARTMENT OF MATHEMATICS
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
KOCHI-682 022, INDIA

DECEMBER 2006

CERTIFICATE

This is to certify that the thesis entitled **Stochastic Inventory with/without Service Time, Reneging and Retrial of Customers** is a bonafide record of the research work carried out by Mr. Jose K.P. under my supervision in the Department of Mathematics, Cochin University of Science and Technology. The results embodied in the thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

Dr. A.Krishamoorthy

(Supervising Guide)

Professor, Department of Mathematics

Cochin University of Science and Technology

Kochi - 682 022.

Kochi-22,

28 December 2006.

Declaration

I hereby declare that the work presented in this thesis entitled **Stochastic Inventory with/without Service Time, Reneging and Retrial of Customers** is based on the original work done by me under the supervision of Prof. A.Krishnamoorthy, Department of Mathematics, Cochin University of Science and Technology and no part thereof has been presented for the award of any other degree or diploma.

Kochi-22,

28 December 2006.

Jose K.P.

Acknowledgement

This thesis gets materialized with the help of many people. I am deeply grateful to my supervisor Dr.A.Krishnamoorthy, Professor, Department of Mathematics, Cochin University of Science and Technology, for his personal guidance and valuable support throughout this work. I owe my gratitude to Dr.T.Thrivikraman, Academic Secretary, Kerala Mathematical Association (KMA), Formerly Professor and Head, Department of Mathematics, Cochin University of Science and Technology, for his sincere help and advice.

I thank Dr. M. Jathavedan, Head, Department of Mathematics, other faculty members, Librarian and administrative staff for providing excellent facilities and supportive conditions to the work. I also thank fellow researches in the department from whom I have benefited immeasurably.

I warmly thank Dr. Vishanath C. Narayanan, Lecturer, Govt. Engineering College, Srikrishnapuram, for his extensive discussions and suggestions throughout the work. I am very much indebted to Mr. Babu S., Department of Mathematics, University College, Thiruvananthapuram, and Mr. Indulal G., St.Aloysius College, Edathua for extending their sincere help during the course of the work. My sincere thanks go to my former colleagues at Model Engineering College, Kochi, for their constant encouragement to complete this work.

A special word of thanks goes to my brother-in-law Mr. Joseph Thanikunnel for extending his knowledge and help in writing **C** programmes for the purpose of numerical works.

I thank the Management, the Principal, the Head of the Department of Mathematics and my Colleagues in St.Peter's college, Kolenchery for their kind cooperation and keen interest in completing this work successfully.

Without the blessings of my parents and relatives, I could not have completed this work. I remember all of them with gratitude on this occasion of fulfillment of this thesis.

Contents

Chapter 1

Introduction

1.1 Description of Inventory Systems

In our daily life, we observe that a small retailer knows roughly the demand of his customers in a month or a week, and accordingly places orders with wholesaler to meet the demand of his customers. However, this is not the case with a manager of a big departmental store or a big retailer, because the stocking in such cases depends upon various factors like demand, time of ordering, lag between orders and actual receipts, etc. The study of such type of problems is known by the term inventory control. In inventory control, our aim is to determine the optimal level of inventory so that the total expected inventory cost is its minimum.

In broad sense, inventory may be defined as the stock of goods, commodities or other economic resources that are stored or reserved in order to ensure smooth and efficient running of business affairs. The design and implementation of an inventory system requires the knowledge of the stocks being maintained. In inventory, the availability of items is also needed in addition to the features required in queueing theory. This means that in queuing theory we do not look at the availability of resources used for service whereas this has to be taken into account in the inventory systems.

1.2 Description of Retrial Inventory Systems

Retrial queues (or queues with repeated calls, returning customers, repeated orders, etc.) are a type of network with re-servicing after blocking. The following feature characterizes these queues: a customer, arriving when all servers accessible to him are busy, leaves the service area but after some random time repeats his demand. This plays a special role in several computer and communication networks. In the simplest and the most important cases, this network contains two nodes: the main node where the blocking is possible and the delay node for repeated attempts. The methods of analysis and areas of applications allow us to divide the retrial queues into three large groups in a natural way: single-channel systems, multi-channel systems and structurally complex systems.

Inventory systems in which arriving customers, who find all items are out of stock, may retry for the items after a period are called retrial inventory. If the item is available, the same is supplied, may be with negligible or a positive service time. However, when at a demand epoch the item is out of stock, such items need not be backlogged or lost. At random epochs such customers retry until either the demand is met or finally the customer decides not to approach that establishment.

1.3 Motivation towards the Work

Queueing systems with reneging have received wide attention. However inventory or retrial inventory system with customer's reneging has not been studied. Artalejo, Krishnamoorthy and Lopez-Herrero [10] were the first attempt to study inventory policies with positive lead-time and retrial of customers who could not get the item during their earlier attempts to access the service station. In this thesis we introduce some inventory systems with customer's reneging and retrial. Reneging occurs when a waiting customer leaves the system due to impatience. In one of the inventory models discussed in this thesis, we introduce the reneging rate as $(i-1)\alpha$, when there are i customers in the system with the head of the queue in service and $i\alpha$ if there is no service going on. In some other models discussed it is assumed that arriving customers who find the inventory level zero (or server busy or buffer full) proceed to an orbit with some probability and are lost forever with the compliment probability. When a retrial customer in the orbit, finds the inventory level zero (or server busy or buffer full), returns to the orbit with some other probability and is lost forever with the compliment probability. We also introduce a *PH/PH/*1 inventory model with reneging and shortage. We study these inventory models using Matrix analytic method.

We construct a suitable cost function for each model. It is defined as the expected total cost (*ETC*) per unit time of the system. This can be found implicitly by considering the performance measures of the system and corresponding relevant inventory costs. Our aim is to determine the minimum expected total cost per unit time. Since the function is known implicitly one can determine the range for the costs that will yield nice analytical properties such as convexity of the cost function, which can then be exploited for arriving at an optimal solution.

1.4 Areas of Application

In addition to the obvious application to stocks of physical goods-light bulbs, toothpaste, raw materials to be used in some production process and the like- there exist many less obvious opportunities to use the models developed in inventory. For example, the number of engineers employed in a company or the number students enrolled in a college can be regarded as inventories. The amount of equity capital available for corporate growth can be regarded as inventory. As it is used up, it must be replenished through issuance of new stocks or bonds. Sometimes it is useful to think not of the physical items but of the space they occupy as the inventory. For example, the space available for new books in a library can be thought of as an inventory. As it is consumed, it must be replenished. The most classical application of a retrial queue arises from telephone traffic theory where a subscriber, whose call was congested, makes repeated attempts. A discussion of practical situations where retrial queues arise can be found in the monograph by Falin and Templeton [29].

1.5 Some Basic Concepts

Poisson Process

Consider a sequence $X_1, X_2, ...$ of positive, independent random variables with common probability distribution. Think of X_n as the time elapsed between the $(n-1)$th and nth occurrence of some specific event in a probabilistic situation. Let

$$S_0 = 0 \text{ and } S_n = \sum_{k=1}^{n} X_k \ , \ n = 1, 2, ...$$

Then S_n is the epoch at which the nth event occurs. For each $t \geq 0$ define the random variable $N(t)$ by

$$N(t) = \text{ the largest integer } n \geq 0 \text{ for which } S_n \leq t.$$

The random variable $N(t)$ represents the number of events up to time t.

Definition: The counting process $\{N(t), t \geq 0\}$ is called a Poisson process with rate λ, if the inter-occurrence times $X_1, X_2, ...$ have a common exponential distribution function

$$P\{X_n \leq x\} = 1 - e^{-\lambda x} \ , \ x \geq 0.$$

Stochastic Process

A stochastic Process is a family of random variables $\{X(t), t \in I\}$ taking values from a set E. The parameter t is generally interpreted as time. The sets I and E are called the index set and the state space of the process, respectively. There are four types of stochastic processes depending on whether I and E are discrete or not. A discrete parameter stochastic process is usually denoted by$\{X_n, n \in I\}$.

Markov Process

A stochastic Process $\{X(t), t \in I\}$ with index set I and state space E is said to be Markov process if it satisfies the following condition $\Pr\{X(t_n) \le x_n / X(t_0) = x_0, X(t_1) = x_1, \ldots\ldots, X(t_{n-1}) = x_{n-1}\} = \Pr\{X(t_n) \le x_n / X(t_{n-1}) = x_{n-1}\}$ for all $t_0 < t_1 < \ldots\ldots. < t_n$. This means that the distribution of any future state depends on the present state, but not on the past.

Exponential Distribution

The exponential distribution has the density

$$f(x) = \lambda e^{-\lambda x} \quad \text{, for } \; x \ge 0$$

and enjoy the status of a mainstay in applied probability for several reasons. Primarily, it has memoryless property

$$P[X > t + s / X > t] = P[X > s] \quad \text{for all } t, s \ge 0 .$$

This provides tremendous ease in conditioning arguments and results in a Markovian structure of models, which involve the exponential distribution.

Continuous-Time Phase type (PH) Distributions

Consider a finite Markov chain (MC) with m transient states and an absorbing state with infinitesimal generator Q portioned as

$$Q = \begin{bmatrix} T & \mathbf{T}^0 \\ \mathbf{0} & 0 \end{bmatrix},$$

where T is a matrix of order m and $\mathbf{T}^0$ is a column vector such that $T\mathbf{e} + \mathbf{T}^0 = \mathbf{0}$. The vector $\mathbf{e}$ is a column vector of 1's. For eventual absorption into the absorbing state, starting from the initial state, it is necessary and sufficient that T be nonsingular. Suppose that the initial state of the Markov chain is chosen according to the probability vector$(\boldsymbol{\alpha}, \alpha_{m+1})$. Here $\boldsymbol{\alpha} = (\alpha_1, \ldots\ldots, \alpha_m)$ is vector with i componentα_i. Let X denote the time until absorption. Then X is a continuous random variable taking nonnegative values with probability distribution function $F(x)$ given by

$$F(x) = 1 - \boldsymbol{\alpha}\, \mathrm{e}^{(Tx)}\, \mathbf{e} \ , \ x \geq 0 .$$

Such a probability function constructed from a finite MC with a single absorbing state is a continuous PH-distribution with representation$(\boldsymbol{\alpha}, T)$.

Note that

(i) the distribution $F(x)$ has a jump of magnitude α_{m+1} at $x = 0$ and the probability density function $f(x)$ on $(0, \infty)$ is given by

$$f(x) = \boldsymbol{\alpha}\, \mathrm{e}^{(Tx)}\, \mathbf{T}^{\mathbf{0}} .$$

(ii) The Laplace-Stieltjes transform $f^*(s)$ of X is given by

$$f^*(s) = \alpha_{m+1} + \boldsymbol{\alpha}(sI - T)^{-1}\mathbf{T}^{\mathbf{0}} \ , \text{for } \mathrm{Re}(s) \geq 0 .$$

(iii) The moments about origin are given by

$$E(X^k) = \mu'_k = (-1)^k k!(\boldsymbol{\alpha} T^{-k} \mathrm{e}) \ , \text{for } k \geq 0 .$$

(iv) When $m = 1, T = -\lambda$, the underlying PH-distribution becomes exponential.

(v) A generalized Erlang of order m is a PH-distribution with representation$(\boldsymbol{\alpha}, T)$, where

$$\boldsymbol{\alpha} = (1,0,0,\ldots,0), \qquad \text{and} \qquad T = \begin{pmatrix} -\lambda_1 & \lambda_1 & & & \\ & -\lambda_2 & \lambda_2 & & \\ & & \ddots & & \\ & & & & \lambda_{m-1} \\ & & & & -\lambda_m \end{pmatrix}.$$

(vi) A hyper-exponential of order m is a PH-distribution with representation $(\boldsymbol{\alpha}, T)$, where

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \quad \text{and} \qquad T = \begin{pmatrix} -\lambda_1 & & & & \\ & -\lambda_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -\lambda_m \end{pmatrix}.$$

For a complete discussion on PH-distribution and their usefulness in stochastic modelling, one can refer to Neuts [52], Latouche and Ramaswami [48], Chakravarthy [19].

PH-renewal Process

A renewal process whose inter-renewal times have a PH distribution is called a PH –renewal process. To construct a PH-renewal process we consider a continuous time Markov Chain with state space $\{1, 2, \ldots, m+1\}$ having infinitesimal generator

$$Q = \begin{bmatrix} T & \mathbf{T}^0 \\ \mathbf{0} & 0 \end{bmatrix}.$$

The $m \times m$ matrix T is taken to be nonsingular so that absorption to the state $m+1$ occurs with probability 1 from any initial state. Let $(\boldsymbol{\alpha}, 0)$ be the initial probability vector. When absorption occurs in the above chain we say that an event, may be in the form of an arrival, has occurred and the process immediately starts anew in one of the states in $\{1, 2, \ldots, m\}$ according to the probability vector $\boldsymbol{\alpha}$. Continuation of this process gives a non-terminating process and is called PH-renewal process. The class of PH-renewal process includes Poisson process, Compound Poisson Process, etc.

Level Independent Quasi-Birth-Death (LIQBD) Process

A Level Independent Quasi-Birth-Death Process is a Markov process on a state space $E = \{(0,j), 1 \le j \le n\} \cup \{(i,j), i \ge 1, 1 \le j \le m\}$ and with infinitesimal generator Q given by

$$Q = \begin{pmatrix} B_1 & B_0 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.1}$$

The above Q is obtained by partitioning the state space E into the levels $\{\underline{0}, \underline{1}, \underline{2}, \cdots\}$ where $\underline{0} = \{(0,j), 1 \le j \le n\}$ and $\underline{i} = \{(i,j), i \ge 1, 1 \le j \le m\}$. The matrix $(B_1)_{n \times n}$ denotes transition rates from states of level 0 to itself, $(B_0)_{n \times m}$ denotes transition rates from states at level 0 to those at level 1 and $(B_2)_{m \times n}$ denotes transition rates from states of level 1 to the level 0. A_2, A_1 and A_0 are square matrices of order m and denote transition rates from level i to the levels $i-1$, i and $i+1$ respectively.

Level Dependent Quasi-Birth-Death (LDQBD) Process

A Level Dependent Quasi-Birth-Death Process is a Markov process on a state space $E = \{(i,j), i \ge 0, 1 \le j \le n_i\}$ and with infinitesimal generator Q given by

$$Q = \begin{pmatrix} A_{1,0} & A_{0,0} & & & \\ A_{2,1} & A_{1,1} & A_{0,1} & & \\ & A_{2,2} & A_{1,2} & A_{0,2} & \\ & & A_{2,3} & A_{1,3} & A_{0,3} \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.2}$$

The generator Q is obtained in the above form by partitioning the state space E into levels $\underline{i} = \{(i, j), i \geq 0, 1 \leq j \leq n_i\}$. Here the transitions take place only to the adjacent levels. However, the transition rate may depend on the level i and therefore the spatial homogeneity of the associated process is lost.

Matrix Analytic Method

A matrix analytic approach to stochastic models was introduced by Neuts [52] to provide an algorithmic analysis for M/G/1 and GI/M/1 type of queueing models. Recent developments, however, have helped to show that the approach has far greater power and reach and provides information on fundamental properties of structural import for vast classes of models far beyond those originally envisaged.

The following theorem gives a brief description of Matrix Analytic Method applied for solving Quasi-Birth-Death Process (QBD).

Theorem: *A continuous time QBD with infinitesimal generator Q of the form* (1.2) *is positive recurrent if and only if, the minimal non negative solution R to the matrix quadratic equation*

$$R^2 A_2 + RA_1 + A_0 = 0 \tag{1.3}$$

has spectral radius less than 1 *and the finite system of equations*

$$\left.\begin{aligned} &x_0 B_0 + x_1 A_{2,1} = 0 \\ &x_{i-1} A_{0,N-1} + x_i A_{1,i} + x_{i+1} A_{2,i+1} = 0, (1 \leq i \leq N-2) \\ &x_{N-2} A_{0,N-2} + x_{N-1}(A_{1,N-1} + RA_2) = 0 \end{aligned}\right\} \tag{1.4}$$

has a unique solution for $x_0, \ldots\ldots, x_{N-1}$. *If the matrix* $A = A_0 + A_1 + A_2$ *is irreducible, then* $sp(R) < 1$ *if and only if*

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e} \ , \tag{1.5}$$

where $\boldsymbol{\pi}$ *is the stationary probability vector of the generator matrix A.*

If $\mathbf{x} = (x_0, x_1, ...)$ is the stationary probability vector of Q, then x_i's$(i \geq N)$ are given by

$$x_{N+r-1} = x_{N-1}R^r \quad for \quad r \geq 1 \tag{1.6}$$

To find the minimal solution of (1.3) one can use the iterative formula given by

$$R_{n+1} = -(R_n{}^2 A_2 + A_0)A_1^{-1}, \quad n = 1,2,3.... \text{ with } R_0 = 0.$$

Another method to find the matrix R is to use the relation

$$R = A_0(-A_1 - A_0 G)^{-1},$$

where the matrix G is the minimal nonnegative solution to the matrix quadratic equation

$$A_2 + A_1 G + A_0 G^2 = 0 \tag{1.7}$$

The matrix G will be stochastic if $sp(R) < 1$. The logarithmic Reduction Algorithm due to Ramaswamy (see Latouche and Ramaswami [48]) can be used to calculate G.

1.6 Review of Related Work

1.6.1 Works on (s, S) Inventory

The analysis of inventory problem was started by Harris [32] in 1915. He proposed the famous EOQ formula that was popularized by Wilson. Inventory systems of (s, S) policy had been extensively studied in the past. The first paper closely related to (s, S) policy is by Arrow *et al.* [3].They showed that cost function, incurred in an (s, S) policy, satisfied a renewal equation. A systematic approach of the (s, S) inventory system is proposed by Arrow *et al.* [4] based on renewal theory. Further details of work carried out in this field can be found in Hadley and Whitin [31], Veinott [66], Naddor [51], Gross and Harris [26], and Tijms [62]. Sivazlian [59] analyzed the continuous review (s, S) inventory system with arbitrarily distributed inter-arrival times and unit demands. He showed that the limiting distribution of the position inventory is uniform and independent of the interarrival time distribution when lead time is zero and no shortage is permitted. Srinivasan [60] extended Sivazlian's result to the case of random lead times. Sahin [57] discussed a continuous review inventory system with constant lead time, arbitrarily distributed inter-arrival time with continuous demand quantities. Manoharan et al. [50] extended Srinivasan's results to a non identically distributed interarrival times.

Kalpakam and Arivarignan [33] studied a single server item inventory model in which demands from a finite number of different types of sources form Markov chain. Chikan [20] and Sahin [58] discussed extensively a number of continuous review inventory systems in their books. Krishnamoorthy and Lakshmy [38] discussed problems with Markov dependent re-ordering levels and Markov dependent replenishment quantities. In 1991, they [39] considered an (s, S) inventory system in which the successive demand quantities form a Markov chain. An inventory system with varying reorder levels and random lead time is discussed by Krishnamoorthy and Manoharan [40].

Azoury and Brill [11] derived the steady state distribution of net inventory in which demand process is Poisson, ordering decisions are based on net inventory and lead times are random. In 1993, Kalpakam and Sapna [34] analyzed an (s, S) ordering policy in which items are procured on an emergency basis during stock out period.Krishnamoorthy and Varghese [46] considered a two commodity inventory problem with Markov shift in demand for the commodity. Krishnamoorthy and Merlymole [41] investigated a two commodity inventory problem with correlated demands. An (s, S) inventory system with lead time and N-policy has been introduced and by Krishanamoorthy and Raju [43]. Krishnamoorthy and Rekha [44] studied an (s, S) inventory system with lead time and T-policy in 1998.

1.6.2 Works on Inventory with Service

In all works reported in inventory prior to 1993, it was assumed that the time required to serve the items to the customer is negligible. Berman *et al.* [13] were the first attempt to introduce positive service time in inventory, where it was assumed that service time is a constant. Later Berman and Kim [14] extended this result to random service time. Berman and Sapna [15] studied inventory control at a service facility, where exactly one item from the inventory is used for each service provided. Under a specified cost structure, they derived the optimum ordering quantity that minimizes the long run expected cost rate. The system considered by them was having a finite state space and hence using Markov renewal theoretical approach they could determine the system state distributions uniquely. Arivarignan *et al.* [2] studied a perishable inventory system with service facility. Each customer requires single item, which is delivered through a service of random duration having exponential distribution. Deepak *et al.* [23] studied the queues with postponed work.

Recently, Krishnamoorthy *et al.* [37] discussed a method of effectively utilizing the idle time of the server in an inventory system with positive service time. In another paper, they [47] studied the control policy N, namely the optimal number N of customers to accumulate in the service station of an (s, S) inventory system from the epoch when a customer leaves the system with none left behind. They obtained expressions for the optimal values of the control variables s, S, N under the assumptions of Poisson arrival demands, exponentially distributed service time and zero lead time. Vishanath et al. [67] studied an (s, S) inventory policy with service time by considering the vacation to server and correlated lead time. In 2006, Maike *et al.* [49] discussed M/M/1 Queueing systems with inventory where service times and lead times are exponentially distributed. They derived stationary distribution of joint queue length and inventory processes in explicit product form. They proved that the limiting distribution of the queue length process is same as that in classical M/M/1/∞ - system.

1.6.3 Works on Retrial Queues and Retrial Inventory

A good account of the basic queueing theory is provided in the books by Cooper [22], Kleinrock ([35], [36]), Gross and Harris [27] and Takagi [61]. A detailed discussion on stochastic models is done in Cinlar [21] and Tijms [63]. Matrix Geometric Solutions to the Stochastic Models were first introduced in late1970's by Marcel F. Neuts [52]. Retrial queues or queues with repeated attempts have been extensively investigated. (see Yang and Templeton [68], Falin [28], and Falin and Templeton [29]). Some relevant works on retrial queues can be found in Artalejo [5], Artalejo and Gomez-Corral ([8], [9]), Gomez-Corral [30], Dudin and Klimenock ([24], [25]). In1990, Neuts and Rao [54] suggested an approximation with the help of the model where the retrial rate stays constant when the number of customers in orbit exceeds some level. The idea is to obtain a quasi-toeplitz matrix (repetitive pattern from say, level N). The level N is chosen so as to minimize the truncation error. For a systematic account of results published in retrial queues, one can refer to Artalejo ([6], [7]). Bright and Taylor [16] studied the equilibrium distribution in Level Dependent Quasi-Birth-and-Death Processes. They introduced a new method for finding truncation level N in the system.

Retrials of failed components for service were introduced into the reliability of k-out-of-n system by Krishnamoorthy and Ushakumari [45]. Artalejo, Krishnamoorthy and Lopez-Herrero [10] were the first attempt to study inventory policies with positive lead-time and retrial of customers who could not get service during their earlier attempts to access the service station. In 2004, Krishnamoorthy and Mohammad Ekramol Islam [42] analyzed an (s, S) inventory system with postponed demands. When the inventory level reaches zero due to demands, further demands are sent to a pool (of postponed demand) which has finite capacity. The service to the pooled customers would be considered only after replenishment against the order placed. Recently, Ushakumari [65] derived analytical solution to the problem investigated by Artalejo, Krishnamoorthy and Lopez-Herrero [10].

1.7 An Outline of the Present Work

This thesis is divided into six chapters including this introductory chapter. In second chapter, we consider an (s,S) inventory model with service, reneging of customers and finite shortage of items. We assume that customers arrive to the system according to a Poisson process with rate λ. The service times are exponentially distributed with rate $i\mu$, when there are i customers in the system. The lead-time is zero. That is, when the inventory level reaches s an order is placed and instantly the level is brought to the maximum S. Reneging rate is $(i-1)\alpha$, when there are i customers in the system with the head of the queue in service and $i\alpha$ if there is no service going on. A maximum of k $(k>0)$ shortages is allowed in the system. Using Matrix Analytic Method we perform the steady state analysis of the inventory model. Some measures of the system performance in the steady state are derived. A suitable cost function is defined and analyzed numerically.

In the third chapter, we consider an (s,S) inventory system with retrial of customers. Arrival of customers forms a Poisson process with rate λ. When the inventory level depletes to s due to demands, an order for replenishment is placed. Lead-time follows exponential distribution with rate β. An arriving customer who finds the inventory level zero proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. When a retrial customer in the orbit finds the inventory level zero he returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. The inter-retrial times have exponential distribution with rate $i\theta$, when there are i customers in the orbit (each customer in orbit tries to access the service station and so the net rate will be $i\theta$ when there are i customers in the orbit). A cost function is constructed based on the performance measures of the system and it is analyzed numerically.

In Chapter 4, we analyze and compare three (s,S) inventory systems with positive service time and retrial of customers. In all these systems, arrivals of customers form a Poisson process and service times are exponentially distributed. When the inventory level depletes to s due to services, an order for replenishment is placed. The lead-time follows an exponential distribution. In model I, an arriving customer, encountering the inventory dry or server busy, proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer in the orbit, finding the inventory dry or server busy, returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$.In addition to the description in model I, we provide a buffer of varying (finite) capacity equal to the current inventory level for model II and another having capacity equal to the maximum inventory level S for model III. In models II and III, an arriving customer, encountering the buffer full, proceeds to an orbit with probability γ and is lost forever with probability$(1-\gamma)$. A retrial customer in the orbit, finding the buffer full, returns to the orbit with probability δ and is lost forever with probability$(1-\delta)$. In all these models, the inter-retrial times are exponentially distributed with linear rate. Some measures of the system performance in the steady state are derived. A common suitable cost function is defined for all these models and it is analyzed numerically.

In chapter 5, we analyze and compare three production inventory systems with positive service time and retrial of customers. In all these systems, arrivals of customers form a Poisson process and service times are exponentially distributed. When the inventory level depletes to s due to services provided to the arriving customers, production starts, that is, the production is switched to ON mode. The production continues until the inventory level reaches the maximum value S. The time between two successive items are added to the inventory is exponential with parameter β. In model I, an arriving customer, encountering the inventory dry or server busy, proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer in the orbit, finding the inventory dry or server busy, returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. In addition to the description in the model I, we provide a buffer of varying (finite) capacity equal to the current inventory level for model II and another having capacity equal to the maximum inventory level S for model III. In models II and III, an arriving customer, encountering the buffer full, proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer in the orbit, finding the buffer full, returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. In all these models, the inter-retrial times are exponentially distributed with linear rate. We study these inventory models using Matrix Analytic Method. A common suitable cost function, which is defined based on the performance measures, is discussed numerically.

In chapter 6, we consider a $PH/PH/1$ inventory model with reneging of customers and finite shortage of items. We assume that arrivals occur according to a phase type renewal process. The inter-arrival distribution is of phase type with representation $(\boldsymbol{\alpha}, U)$. The service times have common phase type distribution with representation $(\boldsymbol{\beta}, V)$. The Lead-time is zero. Reneging rate is constant with value γ. A maximum of K $(K>0)$ shortages is allowed in the system. Some measures of the system performance in the steady state are derived. A suitable cost function is defined and analyzed numerically.

Chapter 2

Inventory with Service Time, Reneging of Customers and Shortage *

2.1 Introduction

In most of the models considered so far the inventory depletes at a rate equal to demand rate; but it becomes unrealistic for the service facilities where the stocked items are delivered to the customers after some service. Some related work includes Berman *et al.* [13], Berman and Kim [14] and Berman and Sapna [15]. The first published work on stochastic inventory with service time is due to Parthasarathy and Vijayalakshmi [55]. They studied the transient analysis of an $(S-1,S)$ inventory model with positive service time. Qui-Ming and Neuts [56] discussed the analysis of two *M/M/*1 Queues with Transfer of Customers. In that system, when the difference of the queue lengths reaches L (> 0), a batch of K $(0< K< L)$ customers is transferred from longer queue to the shorter queue. Zhao and Grassman [69] analyzed a shortest queue model with jockeying using matrix analytic method.

In this chapter, we consider an inventory model with service time, reneging of customers. The inter-arrival time between two successive demands is assumed to

* Accepted as a paper titled *Matrix Analytic Solution to an Inventory with Service, Reneging and Retrial of Customers* in OPSEARCH, 2007.

be exponentially distributed with parameter λ. Service times are exponentially distributed with parameter $i\mu$, when there are i customers in the system with positive inventory level. Reneging rate is $(i-1)\alpha$; when there are i customers in the system with the head of the queue in service and rate $i\alpha$, if there is no service. A maximum of $k(>0)$ shortages is allowed in the system. When the inventory level depletes due to services and reaches the re-order level s, it is brought back to S, without any delay. Customers require single item and it is delivered after completing service

This chapter is organized as follows. Section 2.2 deals mathematical modelling and analysis of the problem. Section 2.3 contains algorithmic analysis of the model. Section 2.4 discusses different measurers of the system performance. Section2.5 contains cost analysis and numerical examples.

2.2 Mathematical Modelling and Analysis

The following assumptions and notations are used in this model.

Assumptions

(i) Maximum inventory level is S.

(ii) Inter-arrival times of demands are exponentially distributed with parameter λ.

(iii) Lead-time is zero.

(iv) Service times are exponentially distributed with parameter $i\mu$ when there are i customers in the system.

(v) Reneging rate is $(i-1)\alpha$ when there are $i(\geq 1)$ customers in the system with the head of the queue in service and with rate $i\alpha$ if there is no service going on.

(vi) A maximum of $k(>0)$ shortages is allowed in the system.

Notations

$I(t)$: Inventory level at the time t.

$N(t)$: Number of customers in the system at time t.

e : $(1,1,\ldots\ldots,1)'$, an S component column vector of 1's.

Let $I(t)$ be the inventory level and $N(t)$ be the number of customers in the system at time t. Now $\{(N(t),I(t));t\geq 0\}$ is a Level Dependent Quasi-Birth Process on the state space $\{(0,j);0\leq j\leq S-1\}\cup\{(i,j);1\leq i\leq k-1;0\leq j\leq S\}\cup\{(i,j);i\geq k;1\leq j\leq S\}$. The coordinate i is the level and the second coordinate j, the phase of the state (i,j). The infinitesimal generator Q of the process is a block tri-diagonal matrix and it has the following form

$$Q=\begin{pmatrix} B_0 & A_{0,0} & 0 & 0 & 0 & \cdots \\ A_{2,1} & A_{1,1} & A_{0,1} & 0 & 0 & \cdots \\ 0 & A_{2,2} & A_{1,2} & A_{0,2} & 0 & \cdots \\ 0 & 0 & A_{2,3} & A_{1,3} & A_{0,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the blocks $B_0, A_{0,0}, A_{1,i} and A_{2,i} (i\geq 1)$ are given by

$$B_0 = \begin{pmatrix} -\lambda & 0 & 0 & \cdots & 0 \\ 0 & -\lambda & 0 & & 0 \\ 0 & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda \end{pmatrix}_{S\times S}, \qquad A_{0,0} = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 \end{pmatrix}_{S\times(S+1)},$$

$$A_{0,i} = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}_{(S+1)\times(S+1)} (1 \le i \le k-2), \quad A_{0,i} = \begin{pmatrix} 0 & 0 & 0 & \cdots & \lambda \\ \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}_{(S+1)\times S} (i = k-1),$$

$$A_{0,i} = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}_{S\times S} (i \ge k),$$

$$A_{1,i} = \begin{pmatrix} -(\lambda+i\alpha) & 0 & 0 & & 0 \\ 0 & -(\lambda+i\mu+(i-1)\alpha) & 0 & & 0 \\ 0 & 0 & -(\lambda+i\mu+(i-1)\alpha) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -(\lambda+i\mu+(i-1)\alpha) \end{pmatrix}_{(S+1)\times(S+1)} (1 \le i \le k-1),$$

$$A_{1,i} = \begin{pmatrix} -(\lambda+i\mu+(i-1)\alpha) & 0 & & 0 \\ 0 & -(\lambda+i\mu+(i-1)\alpha) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -(\lambda+i\mu+(i-1)\alpha) \end{pmatrix}_{S\times S} (i \ge k),$$

$$A_{2,1}=\begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & \mu & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}_{(S+1)\times S}, A_{2,i}=\begin{pmatrix} i\alpha & 0 & \cdots & 0 & 0 \\ i\mu & (i-1)\alpha & \cdots & 0 & 0 \\ 0 & i\mu & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & i\mu & (i-1)\alpha \end{pmatrix}_{(S+1)\times(S+1)} \quad (2\le i\le k-1),$$

$$A_{2,i}=\begin{pmatrix} i\mu & (i-1)\alpha & 0 & \cdots & 0 & 0 \\ 0 & i\mu & (i-1)\alpha & \cdots & 0 & 0 \\ 0 & 0 & i\mu & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & i\mu & (i-1)\alpha \end{pmatrix}_{S\times(S+1)} (i=k), \quad A_{2,i}=\begin{pmatrix} (i-1)\alpha & 0 & \cdots & 0 & i\mu \\ i\mu & (i-1)\alpha & \cdots & 0 & 0 \\ 0 & i\mu & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & i\mu & (i-1)\alpha \end{pmatrix}_{S\times S} (i\ge k+1).$$

As number of arriving customers increases, the rate of reneging also increases. When number of customers is sufficiently large and if is restricted to an appropriately chosen number $N(>k)$, the change in the equilibrium probability vector is minimal. This truncation (see Neuts and Rao [54]) modifies the infinitesimal generator Q to the following form, where $A_{0,i}=A_0, A_{1,i}=A_1$ and $A_{2,i}=A_2$ for $i\ge N$

$$Q=\begin{pmatrix} B_0 & A_{0,0} \\ A_{2,1} & A_{1,1} & A_{0,1} \\ & A_{2,2} & A_{1,2} & A_{0,2} \\ & & \ddots & \ddots & \ddots \\ & & & A_{2,N-2} & A_{1,N-2} & A_{0,N-2} \\ & & & & A_{2,N-1} & A_{1,N-1} & A_{0,N-1} \\ & & & & & A_2 & A_1 & A_0 \\ & & & & & & A_2 & A_1 & A_0 \\ & & & & & & & \ddots & \ddots & \ddots \end{pmatrix} \quad (2.1)$$

Define the generator A as $A = A_0 + A_1 + A_2$. Then

$$A = \begin{pmatrix} -N\mu & 0 & 0 & & N\mu \\ N\mu & -N\mu & 0 & & 0 \\ 0 & N\mu & -N\mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & N\mu & -N\mu \end{pmatrix}$$

Theorem 2.1: The steady state probability vector $\boldsymbol{\pi}$ satisfying $\boldsymbol{\pi}A = 0$ and $\boldsymbol{\pi}\mathbf{e} = 1$ is given by $\boldsymbol{\pi} = (1/S)\mathbf{e}'$.

Proof: Let $\boldsymbol{\pi} = \{\pi_1, \pi_2, ...\pi_S\}$ be the partition of $\boldsymbol{\pi}$. Then $\boldsymbol{\pi}A = 0$ and $\boldsymbol{\pi}\mathbf{e} = 1$ imply that $\pi_j = 1/S, \forall j = 1, 2, ..., S$. This completes the proof.

Theorem 2.2: The stability condition of the system under steady state is

$$\rho(N) < 1, \tag{2.2}$$

where $\rho(N) = \lambda/[N\mu + (N-1)\alpha]$.

Proof: From the well-known result (see Neuts [52]) on positive recurrence of Q, which states that $\boldsymbol{\pi}A_0\mathbf{e} < \boldsymbol{\pi}A_2\mathbf{e}$, the result $\lambda/[N\mu + (N-1)\alpha] < 1$ follows.

2.2.1 Steady State Analysis

Let $\mathbf{x} = (x_0, x_1, ...x_{N-1}, x_N, ...)$ be the steady state probability vector of Q. Under the stability condition (2), x_is' $(i \geq N)$ are given by

$$x_{N+r-1} = x_{N-1}R^r \, (r \geq 1)$$

where R is the unique non- negative solution of the equation

$$R^2 A_2 + RA_1 + A_0 = 0$$

in which the spectral radius is less than one and the vectors $x_0, x_1, ...x_{N-1}$ are given by solving the following equations

$$\left.\begin{array}{l} x_0 B_0 + x_1 A_{2,1} = 0 \\ x_{i-1} A_{0,N-1} + x_i A_{1,i} + x_{i+1} A_{2,i+1} = 0, (1 \le i \le N-2) \\ x_{N-2} A_{0,N-2} + x_{N-1}(A_{1,N-1} + RA_2) = 0 \end{array}\right\} \quad (2.3)$$

subject to the normalizing condition

$$\left(\sum_{i=1}^{N-2} x_i + x_{N-1}(I-R)^{-1}\right)\mathbf{e} = 1 \quad (2.4)$$

2.3 Algorithmic Analysis

2.3.1 Evaluation of the Rate Matrix R

To high light the fact that R is a function of N, we shall write $R(N)$ instead of R, wherever necessary for clarity. Theoretically, $R(N)$ is given by $\lim_{n\to\infty} R_n(N)$, where the sequence $\{R_n(N)\}$ is defined by

$$\begin{aligned} R_0(N) &= 0 \\ R_{n+1}(N) &= -R_n^2(N)A_2(N)A_1^{-1}(N) - A_0(N)A_1^{-1}(N) \\ &= [R_n^2(N)A_2(N) + \lambda I_S]/[\lambda + N\mu + (N-1)\alpha] \end{aligned}$$

where I_S is the unit matrix of order S.

2.3.2 Choice of the Truncation Level N

In Neuts-Rao truncation, the equilibrium probabilities of the states with level $i \ge N$ depend largely on $\eta(N)$ the spectral radius of $R(N)$. As outlined in Neuts [52], Elsner's algorithm to evaluate the spectral radius is used to determine $\eta(N)$. To minimize the effect of the approximation on the probabilities, N must be chosen such that $\eta(N)$ is sufficiently close to $\eta(N+1)$. Starting with an initial value of N, one can progressively increase the value of N, until $|\eta(N) - \eta(N+1)| < \varepsilon$, where ε is an arbitrary small value.

2. 3. 3 Computation of the Boundary Probabilities

Let $\mathbf{x}^*$ be the partitioned vector $(x_0, x_1, \ldots\ldots, x_{N-1})$, corresponding to the boundary portion of Q as in (2.1) Then $\mathbf{x}^*$ is the stationary vector normalized by (2.4) of the infinitesimal generator T shown below

$$T = \begin{pmatrix} B_0 & A_{0,0} & & & & \\ A_{2,1} & A_{1,1} & A_{0,1} & & & \\ & A_{2,2} & A_{1,2} & A_{0,2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{2,N-2} & A_{1,N-2} & A_{0,N-2} \\ & & & & A_{2,N-1} & A_{1,N-1}+RA_2 \end{pmatrix}$$

Now the system (2.3) can be written as $\mathbf{x}^* T = 0$ .To solve this system, we use the block Gauss-Seidel iterative scheme. The vectors $x_0, x_1, \ldots\ldots, x_{N-1}$ in the $(n+1)$th iteration are given by

$$\begin{aligned} x_0(n+1) &= x_1(n) A_{2,1} B_0^{-1} \\ x_i(n+1) &= [x_{i+1}(n) A_{2,i+1} + x_{i-1}(n+1) A_{0,i-1}] A_{1,i}^{-1}, (1 \le i \le N-2) \\ x_{N-1}(n+1) &= -x_{N-2}(n+1) A_{0,N-2} (A_{1,N-1} + RA_2)^{-1} \end{aligned}$$

After each iteration, the elements of $\mathbf{x}^*$ may be scaled to satisfy (2.4).

2.4 System Performance Measures

The components of steady state probability vector $\mathbf{x} = (x_0, x_1, \ldots\ldots, x_{N-1}, x_N, \ldots\ldots)$ can be partitioned as

$$\begin{aligned} x_0 &= (y_{0,0}, y_{0,1}, y_{0,2}, \ldots\ldots, y_{0,S-1}) \\ x_i &= (y_{i,0}, y_{i,1}, y_{i,2}, \ldots\ldots, y_{i,S}), (1 \le i \le k-1) \\ x_i &= (y_{i,1}, y_{i,2}, y_{i,3}, \ldots\ldots, y_{i,S}), (i \ge k) \end{aligned}$$

Then we have

(i) Expected Inventory level, *EI*, in the system is given by

$$EI = \sum_{j=1}^{S} j \sum_{i=0}^{\infty} y_{i,j}$$

(ii) Expected number of departures after completing the service/unit time, *EDS*, is given by

$$EDS = \mu \sum_{i=1}^{\infty} \left(i \sum_{j=1}^{S} y_{i,j} \right)$$

(iii) Expected number of departures due to reneging of customers/unit time , *EDR*, is given by

$$EDR = \alpha \sum_{i=1}^{\infty} \left((i-1) \sum_{j=1}^{S} y_{i,j} \right)$$

(iv) Expected number of customers, *EC*, in the system is given by

$$EC = \left(\sum_{i=1}^{\infty} i x_i \right) \mathbf{e}$$

$$= \left((\sum_{i=1}^{N-1} i x_i) + x_N \left(N(I-R)^{-1} + R(I-R)^{-2} \right) \right) \mathbf{e}$$

(v) Expected reorder rate, *ERO*, is given by

$$ERO = [\lambda/(\lambda + (k-1)\alpha)] y_{k-1,0}$$

(vi) Expected shortages, *ES*, in the system is given by

$$ES = \left(\sum_{i=1}^{k-1} i x_i \right) \mathbf{e}$$

2.5 Cost Analysis and Numerical Examples

Cost Analysis

Define c_1 = setup cost

c_2 = holding cost of inventory /unit /unit time

c_3 = service cost/unit/unit time

c_4 = loss due to reneging of customers/ unit /unit time

c_5 = holding cost of customers / unit /unit time

c_6 = shortage cost/unit/unit time

c_7 = revenue (profit) due to service / unit /unit time

The expected total cost (ETC) of the system/unit time is given by

$$ETC = c_1 ERO + c_2 EI + (c_3 - c_7) EDS + c_4 EDR + c_5 (EC - 1) + c_6 ES$$

Numerical Examples

(i) Variation in Number of Shortages (k)

lambda=10; mu=3; alpha=1;S=20; N=50;
c_1=10; c_2=2.7; c_3=2; c_4=1; c_5 =1; c_6=1; c_7=1.

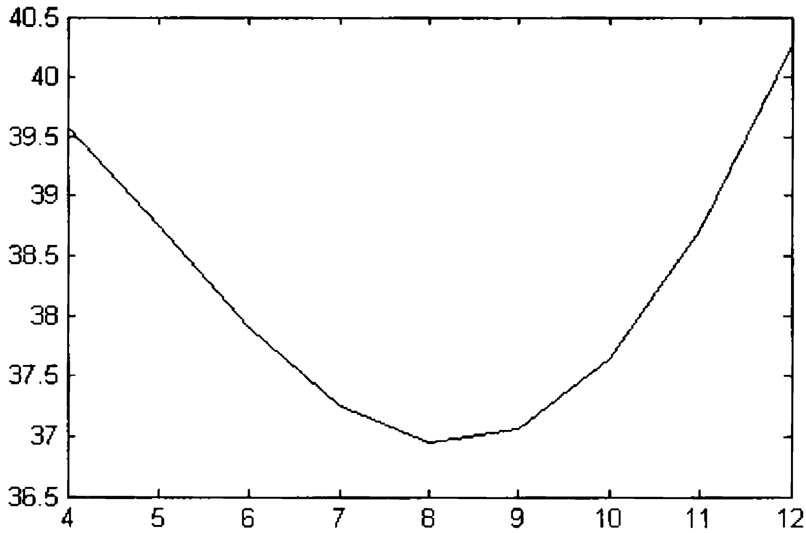

Fig.1 **k vs. ETC**

(ii) Variation in Maximum Inventory Level (S)

lambda=10; mu=2; alpha=1; k=4; N=60;
c_1=10; c_2 =1; c_3=43; c_4 =1; c_5=1; c_6=1; c_7=1;

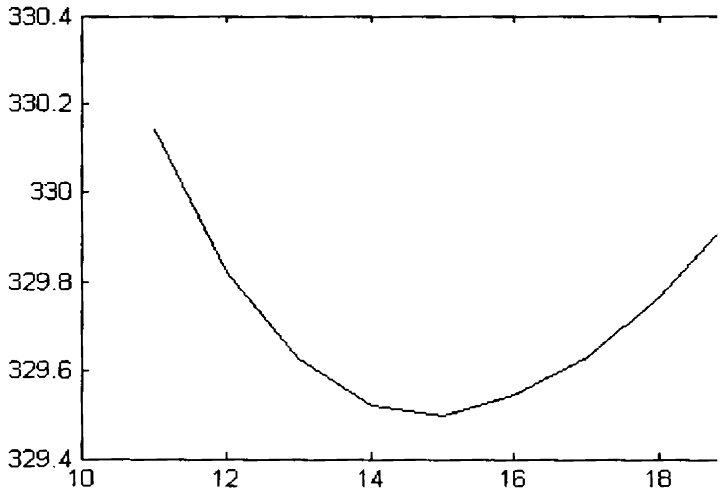

Fig.2 **S vs. ETC**

(iii) Variation in Service Rate (μ)

lambda =10; alpha =1; S=20; k=5; N=50;
c_1=10; c_2 =1; c_3 =2.46; c_4 =1; c_5=1; c_6=1; c_7=1

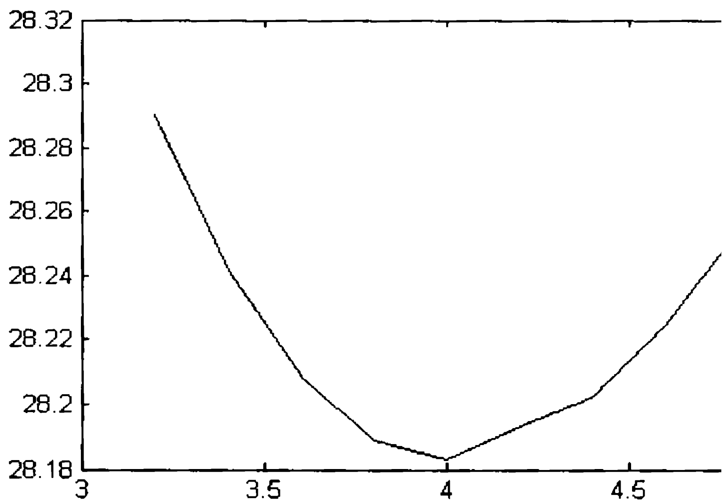

Fig.3 μ **vs. ETC**

(iv) Variation in Reneging Rate (α)

lambda=10; mu=2; S=20; k=4; N=60;
c_1=10; c_2=1; c_3 =1; c_4=5; c_5=1; c_6=1; c_7=1;

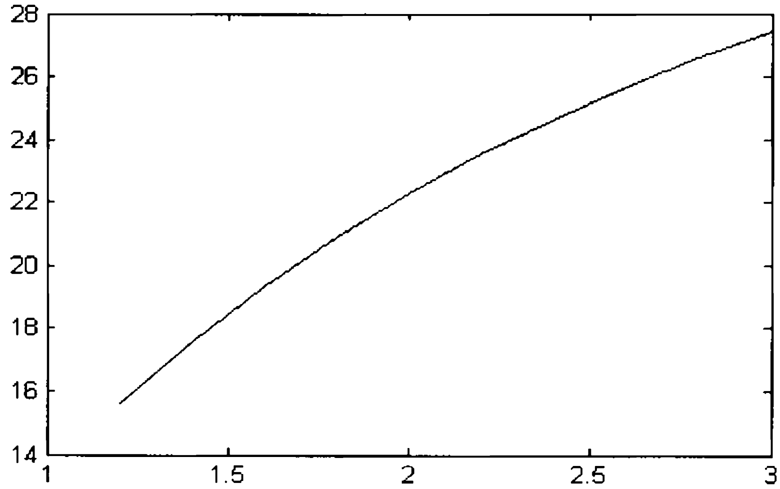

Fig.4 α **vs. ETC**

(v) Variation in Arrival Rate (λ)

mu=2; alpha=1; S=20; k=4; N=65;
c_1=10; c_2=1; c_3=2; c_4=1; c_5 =1; c_6=1; c_7=1.

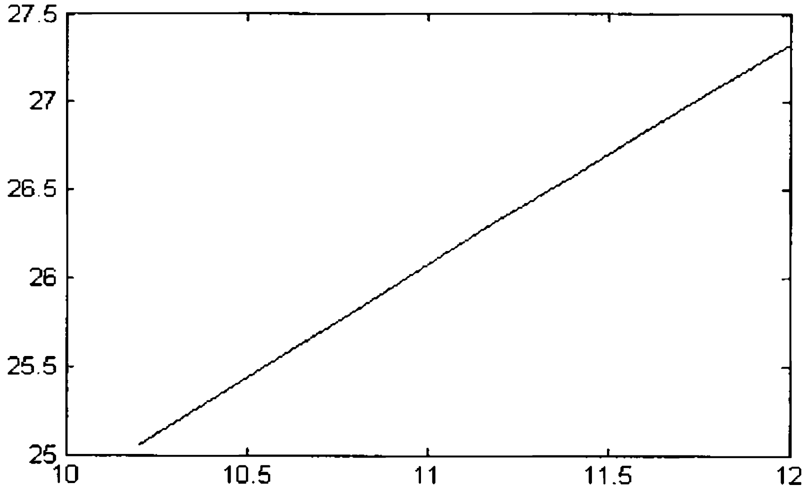

Fig.5 λ **vs. ETC**

Numerical Interpretations of the Graphs

One can determine the range for the costs that will yield nice analytical properties for the objective function, which can then be exploited for arriving at an optimal solution. In this model, the cost function has the following properties; (i) as the number of shortages k increases (keeping all other parameters fixed), the cost function is convex and attains its optimum (minimum) at $k = 8$. (see fig.1), (ii) as the maximum inventory level S increases the cost function is again convex and it has the minimum value at S = 15 (see fig.2), (iii) with the variation in the service rateμ, the cost function attains its minimum value at $\mu = 4$ (see Fig.3), (iv) as either the reneging rate α or the arrival rate λ increases, the cost function increases monotonically (see Fig.4 and Fig.5).

Chapter 3

Inventory with Positive Lead-time, Loss and Retrial of Customers *

3.1 Introduction

Queuing systems in which customers who find all servers and waiting position occupied may retry for service after a period of time are called retrial queues or queues with repeated attempts. For a detailed discussion of retrial queues one can refer to Yang and Templeton [68], Falin [28], and Falin and Templeton [29]. Artalejo, Krishnamoorthy and Lopez-Herrero [10] were the first attempt to study inventory policies with positive lead-time and retrial customers who could not get the item during their earlier attempts to access the service station. Except for [10] very little investigation is done in retrial inventory.

In this chapter, we consider an (s,S) inventory system with retrial of customers. Arrival of customers forms a Poisson process with rate λ. When the inventory level depletes to s due to demands, an order for replenishment is placed.

* Accepted as a paper titled '*An (s, S) Inventory Systems with Positive Lead-time, Loss and Retrial of Customers*' in Stochastic Modelling and Applications, 2005.

The lead-time follows exponential distribution with rate β. An arriving customer who finds the inventory level zero proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer in the orbit finds the inventory level zero returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. The inter-retrial time is an exponential distribution with linear rate $i\theta$, when there are i customers in the orbit (each customer in orbit tries to get the inventory and so the net rate will be $i\theta$ when there are i customers in the orbit).

This chapter is organized as follows. Section 3.2 deals with mathematical modeling and analysis of the problem. Section 3.3 contains algorithmic analysis of the model. In Section 3.4, we discuss different measurers of the system performance. Section 3.5 presents cost analysis and numerical examples.

3.2 Mathematical Modelling and Analysis of the Problem

The following assumptions and notations are considered in this chapter.

Assumptions

(i) Initially the inventory level is S.

(ii) Inter-arrival times of demands are exponentially distributed with parameter λ.

(iii) Lead-time follows exponential distribution with rate β.

(iv) Inter-retrial time is exponential with linear rate $i\theta$, when there are i customers in the orbit.

Notations

I (t) : Inventory level at the time t.

N(t) : Number of customers in the orbit at time t.

e $(1,1,....,1)'$ an S component column vector of 1's.

Here, consider an (s, S) inventory system with retrial of customers. Arrival of customers forms a Poisson process rate λ. When the inventory level depletes to s due to demands, an order for replenishment is placed. The lead-time is exponentially distributed with rate β. An arriving customer who finds the inventory level zero proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer in the orbit, who finds the inventory level zero, returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. The inter-retrial time follows an exponential distribution with linear rate $i\theta$, when there are i customers in the orbit.

Let $I(t)$ be the inventory level and $N(t)$ be the number of customers in the orbit at time t. Now $\{(N(t), I(t)); t \geq 0\}$ is a continuous time Markov chain on the state space $\{(i, j); i \geq 0, 0 \leq j \leq S\}$. The coordinate i is the level and the second coordinate j, the phase of the state (i, j). Then above model can be studied as a quasi-birth and death process. The infinitesimal generator Q, of the process is a block tri-diagonal matrix and it has the following form:

$$Q = \begin{pmatrix} A_{1,0} & A_0 & 0 & 0 & 0 & \cdots \\ A_{2,1} & A_{1,1} & A_0 & 0 & 0 & \cdots \\ 0 & A_{2,2} & A_{1,2} & A_0 & 0 & \cdots \\ 0 & 0 & A_{2,3} & A_{1,3} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.1}$$

where the blocks $A_0, A_{1,i}\,(i \geq 0) and A_{2,i}\,(i \geq 1)$ are square matrices of same order $(S+1)$ and they are given by

$$A_0 = \begin{pmatrix} \lambda\gamma & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$

$$A_{1,i} = \begin{pmatrix}
-(\lambda\gamma+\beta+i\theta(1-\delta)) & & & & & & & \beta & & & \\
\lambda & -(\lambda+\beta+i\theta) & & & & & & & \beta & & \\
 & \ddots & \ddots & & & & & & & \ddots & \\
 & & \ddots & \ddots & & & & & & \beta & \\
 & & & \lambda & -(\lambda+\beta+i\theta) & & & & & & \beta \\
 & & & & \lambda & -(\lambda+i\theta) & & & & & \\
 & & & & & \ddots & \ddots & & & & \\
 & & & & & & \lambda & -(\lambda+i\theta) & & & \\
 & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & \lambda & -(\lambda+i\theta)
\end{pmatrix}$$

$$A_{2,i} = \begin{pmatrix}
i\theta(1-\delta) & & & & & \\
i\theta & 0 & & & & \\
 & i\theta & 0 & & & \\
 & & \ddots & \ddots & & \\
 & & & i\theta & 0 & \\
 & & & & i\theta & 0
\end{pmatrix}$$

When the number of customers in the orbit is sufficiently large, majority of the customers fail to access the server and do not result in a change of the state of the system. Under this condition, if the number of customers in the orbit is restricted to an appropriately chosen number N, then the change on the equilibrium probability vector is minimal. This truncation (see Neuts and Rao [54]) modifies the infinitesimal generator Q to the following form, where, $A_{1,i} = A_1$ and $A_{2,i} = A_2$ for $i \geq N$:

$$Q = \begin{pmatrix} A_{1,0} & A_0 & & & & & & \\ A_{2,1} & A_{1,1} & A_0 & & & & & \\ & A_{2,2} & A_{1,2} & A_0 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & A_{2,N-1} & A_{1,N-1} & A_0 & & \\ & & & & A_2 & A_1 & A_0 & \\ & & & & & A_2 & A_1 & A_0 \\ & & & & & & \ddots & \ddots & \ddots \\ & & & & & & & \ddots & \ddots \end{pmatrix}$$

In the matrix Q, A_0 has entries representing rate of increase in orbit sizes essentially it increases by one since the arrivals are in single. Matrices on the main diagonal have entries providing rate of transitions among phases and those on the lower diagonal have entries representing rate of transitions to the immediate lower level (number of customers in the orbit).

Define the generator A as $A = A_0 + A_1 + A_2$. Then

$$A = \begin{pmatrix} -\beta & & & & & & & \beta & & & & \\ (\lambda+N\theta) & -(\lambda+\beta+N\theta) & & & & & & & \beta & & & \\ & \ddots & \ddots & & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \beta & \\ & & & (\lambda+N\theta) & -(\lambda+\beta+N\theta) & & & & & & & \beta \\ & & & & (\lambda+N\theta) & -(\lambda+N\theta) & & & & & & \\ & & & & & \ddots & \ddots & & & & & \\ & & & & & & \ddots & \ddots & & & & \\ & & & & & & & (\lambda+N\theta) & -(\lambda+N\theta) & & & \\ & & & & & & & & \ddots & \ddots & & \\ & & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & & (\lambda+N\theta) & -(\lambda+N\theta) \end{pmatrix}$$

Let $\boldsymbol{\pi}$ be the stationary probability vector of the Markov chain with infitesimal generator A. That is $\boldsymbol{\pi}A = 0$ and $\boldsymbol{\pi}\mathbf{e} = 1$. The vector $\boldsymbol{\pi}$ can be partitioned as $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots\ldots\ldots\ldots, \pi_S)$, where

$$\pi_k = \begin{cases} (\beta/(\lambda+N\theta))((\lambda+\beta+N\theta)/(\lambda+N\theta))^{k-1}\pi_0; (k=1,2,\ldots\ldots.s) \\ (\beta/(\lambda+N\theta))((\lambda+\beta+N\theta)/(\lambda+N\theta))^{s}\pi_0; (k=s+1,s+2,\ldots\ldots(S-s)) \\ (\beta/(\lambda+N\theta))\left[((\lambda+\beta+N\theta)/(\lambda+N\theta))^{s} - ((\lambda+\beta+N\theta)/(\lambda+N\theta))^{k-S+s-1}\right]\pi_0 \\ \qquad ;(k=S-s+1, S-s+2,\ldots\ldots S-1, S) \end{cases}$$

$$\pi_0 = \left[1 + (S-s)(\beta/(\lambda+N\theta))((\lambda+\beta+N\theta)/(\lambda+N\theta))^{s}\right]^{-1}$$

Now, we have the following theorem on system stability.

Theorem 3.1 The system is stable if

$$\rho(N) < 1, \tag{3.2}$$

where $\rho(N) = (\delta + (\lambda\gamma/N\theta))\left(1 + (S-s)(\beta/(\lambda+N\theta))((\lambda+\beta+N\theta)/(\lambda+N\theta))^{s}\right)^{-1}$.

Proof: From the well-known result (Neuts [52]) on positive recurrence of Q, which states that $\boldsymbol{\pi}A_0\mathbf{e} < \boldsymbol{\pi}A_2\mathbf{e}$, the result $(\delta + (\lambda\gamma/N\theta))\pi_0 < 1$ follows.

3.2.1 Steady State Probability Vector

Let $\mathbf{x} = (x_0, x_1, \ldots.. x_{N-1}, x_N, \ldots..)$ be the steady state probability vector of Q. Under the stability condition (3.2), $x_i s'$ $(i \geq N)$ are given by

$$x_{N+r-1} = x_{N-1}R^r \; (r \geq 1)$$

where R is the unique non- negative solution of the equation

$$R^2 A_2 + RA_1 + A_0 = 0$$

for which the spectral radius is less than one and the vectors $x_0, x_1 \ldots.. x_{N-1}$ are obtained by solving

$$\left.\begin{array}{l} x_0 A_{1,0} + x_1 A_{2,1} = 0 \\ x_{i-1} A_0 + x_i A_{1,i} + x_{i+1} A_{2,i+1} = 0 (1 \le i \le N-2) \\ x_{N-2} A_0 + x_{N-1} (A_{1,N-1} + RA_2) = 0 \end{array}\right\} \tag{3.3}$$

subject to the normalizing condition

$$\left[\sum_{i=1}^{N-2} x_i + x_{N-1}(I-R)^{-1}\right]\mathbf{e} = 1 \tag{3.4}$$

3.3 Algorithmic Analysis

3.3.1 Rate Matrix R and Truncation Level N

To evaluate the rate matrix R one can use the usual iterative method. Neuts-Rao truncation method is used to locate the truncation level N. For details, one can refer to the sub-section 2.3.1 and 2.3.2.

3.3.2 Computation of the Boundary Probabilities

Let $\mathbf{x}^*$ be the partitioned vector $(x_0, x_1, \ldots\ldots, x_{N-1})$ corresponding to the boundary portion of Q as in (1). Then $\mathbf{x}^*$ is the stationary vector of the infinitesimal generator T and is given by

$$T = \begin{pmatrix} A_{10} & A_0 & & & & \\ A_{2,1} & A_{1,1} & A_0 & & & \\ & A_{2,2} & A_{1,2} & A_0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{2,N-2} & A_{1,N-2} & A_0 \\ & & & & A_{2,N-1} & A_{1,N-1} + RA_2 \end{pmatrix}$$

Now the system (3.3) can be written as $\mathbf{x}^* T = 0$. In block Gauss-Seidel iterative scheme, the vectors $x_0, x_1, \ldots\ldots, x_{N-1}$ in the $(n+1)$th iteration are given by

$$x_0(n+1) = [x_1(n) A_{2,1}] A_{1,0}^{-1}$$
$$x_i(n+1) = \left[x_{i+1}(n) A_{2,i+1} + x_{i-1}(n+1) A_0\right] A_{1,i}^{-1}, (1 \le i \le N-2)$$
$$x_{N-1}(n+1) = -[x_{N-2}(n+1) A_0](A_{1,N-1} + R A_2)^{-1}$$

After each iteration, the elements of $\mathbf{x}^*$ may be scaled to satisfy (3.4).

3.4 System Performance Measures

We partition the components of the steady state probability vector $\mathbf{x} = (x_0, x_1, \ldots\ldots, x_{N-1}, x_N, \ldots\ldots)$ as,

$$x_i = (y_{i,0}, y_{i,1}, \ldots\ldots\ldots y_{i,S})(i \ge 0).$$

In terms of these we now express the following performance measures.

(i) Let, *ERO*, be the expected reorder rate. Then we have

$$ERO = \lambda \sum_{i=1}^{\infty} y_{i,s+1}$$

(ii) Expected Inventory level, EI, is given by

$$EI = \sum_{j=1}^{S} j \sum_{i=0}^{\infty} y_{i,j}$$

(iii) Expected number of customers in the orbit, EC, is given by

$$EC = \left(\sum_{i=1}^{\infty} i x_i\right) \mathbf{e}$$
$$= \left(\left(\sum_{i=1}^{N-1} i x_i\right) + x_N (N(I-R)^{-1} + R(I-R)^{-2})\right) \mathbf{e}$$

(iv) Expected number of external arrivals lost, EL_1, per unit time is given by

$$EL_1 = (1-\gamma) \lambda \sum_{i=1}^{\infty} y_{i,0}$$

(v) Expected number of customers lost from the orbit due to retrials per unit time is $$EL_2 = \theta(1-\delta)\sum_{i=1}^{\infty} iy_{i,0}$$

(vi) Overall rate of retrials (θ_1^*) is given by

$$\theta_1^* = \theta\left(\sum_{i=1}^{\infty} ix_i\right)\mathbf{e}$$

(vii) Successful rate of retrials (θ_2^*) is given by

$$\theta_2^* = \theta\sum_{i=1}^{\infty} i\left(\sum_{j=1}^{S} y_{ij}\right)$$

3.5 Cost Analysis and Numerical Examples

In order construct the cost function we define the following costs

C = fixed cost

c_1 = procurement cost/unit/unit time

c_2 = holding cost of inventory /unit /unit time

c_3 = holding cost of customers / unit /unit time

c_4 = cost due to the loss of customers before entering the orbit / unit /unit time

c_5 = cost due to the loss of customers from the orbit after the retrials / unit / unit time

The expected total cost (ETC) of the system/unit time is given by

$$ETC = (C + (S-s)c_1)ERO + c_2EI + c_3EC + c_4EL_1 + c_5EL_2$$

The following tables represent the overall and successful rate of retrials.

$\beta = 3$; $\theta = 0.75$; $\delta = 0.7$; $\gamma = 0.8$;
$s = 5$; $S = 15$; $N = 45$

λ	θ_1^*	θ_2^*
1.5	0.985962	0.867395
2.0	1.316079	1.132484
2.5	1.649399	1.388083
3.0	1.986962	1.635355
3.5	2.329638	1.875292
4.0	2.678155	2.108742
4.5	3.033131	2.336430
5.0	3.395102	2.558982
5.5	3.764530	2.776938
6.0	4.141819	2.990758

Table 1
(Variations in arrival rate λ)

$\lambda = 5$; $\theta = 0.75$; $\delta = 0.7$; $\gamma = 0.8$;
$s = 5$; $S = 15$; $N = 45$.

β	θ_1^*	θ_2^*
10.2	3.126780	2.881824
10.4	3.118772	2.878356
10.6	3.110679	2.874632
10.8	3.102511	2.870666
11.0	3.094272	2.866473
11.2	3.085969	2.862069
11.4	3.077603	2.857463
11.6	3.069184	2.852671
11.8	3.060714	2.847703
12.0	3.052197	2.842570

Table 2
(Variations in replenishment rate β)

$\lambda = 5$; $\beta = 3$; $\delta = 0.7$; $\gamma = 0.8$;
$s = 5$; $S = 15$; $N = 45$.

θ	θ_1^*	θ_2^*
0.1	1.896967	1.595484
0.2	2.720087	2.221597
0.3	3.103348	2.483966
0.4	3.277260	2.582240
0.5	3.349626	2.603762
0.6	3.377594	2.592706
0.7	3.390126	2.570754
0.8	3.398726	2.546344
0.9	3.406638	2.521812
1.0	3.414523	2.497621

Table 3
(Variations in retrial rate (θ)

$\lambda = 5$; $\beta = 3$; $\theta = 0.75$; $\delta = 0.7$;
$s = 5$; $S = 15$; $N = 45$.

γ	θ_1^*	θ_2^*
0.71	2.987861	2.277120
0.72	3.032734	2.308477
0.73	3.077688	2.339814
0.74	3.122723	2.371133
0.75	3.167838	2.402432
0.76	3.213033	2.433712
0.77	3.258307	2.464972
0.78	3.303661	2.496212
0.79	3.349098	2.527434
0.80	3.394610	2.558634

Table 4
(Variations in probability γ of primary arrivals joining the orbit)

$\lambda = 5$; $\beta = 3$; $\theta = 0.75$; $\gamma = 0.8$;
$s = 5$; $S = 15$; $N = 45$.

δ	θ_1^*	θ_2^*
0.71	3.438521	2.519706
0.72	3.463222	2.542282
0.73	3.488680	2.565388
0.74	3.514954	2.589061
0.75	3.542108	2.613348
0.76	3.570207	2.638291
0.77	3.599323	2.663938
0.78	3.629527	2.690339
0.79	3.660892	2.717542
0.80	3.693476	2.745587

Table 5
(Variations in return probability δ of retrial customers)

Interpretations of the Numerical Results in the Tables

As the arrival rate λ increases, the number of customers in the orbit becomes larger so that the overall and successful rate of retrials from the orbit will increase (see table 1). As the replenishment rate β increases the arriving customers will get the inventory more rapidly so that the number of customers in the orbit decreases. In this case, the overall and successful rate of retrials will decrease (see table 2). Table 3 indicates that as the retrial rate θ of customers in the orbit increases, the overall and successful rate of retrials from the orbit will increase. With the increase in probability γ of primary arrival joining the orbit or increase in return probability δ of retrial customers, the orbit size increases. In that case, overall and successful rate of retrials will increase (see tables 4 and 5).

Next we provide graphical illustrations of the Performance measures of the model.

β= 3; θ=0.75; δ= 0.7; γ=0.8;
s = 5; S=15; N=45; K=10;
c_1=1; c_2=5.6; c_3=1; c_4=1; c_5=2.

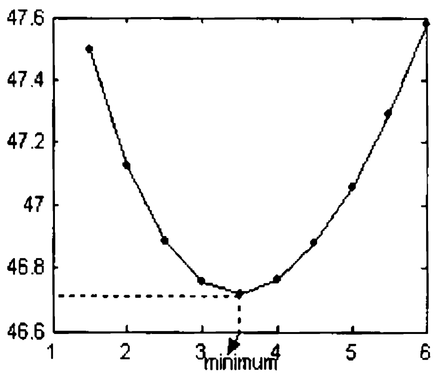

Fig.1 **lambda vs. ETC**

λ= 5; θ=0.75; δ= 0.7; γ=0.8;
s = 5; S=15; N=45; K=10;
c_1=1; c_2=3.2; c_3=3.99; c_4=1; c_5=1.

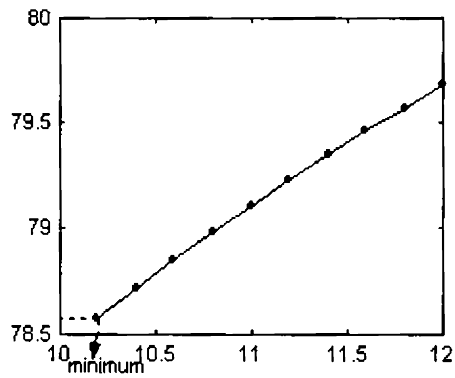

Fig.2 **beta vs. ETC**

λ= 5; β= 3; δ= 0.7; γ=0.8;
s = 5; S=15; N=45; K=10;
c_1=1; c_2=1; c_3=1; c_4=1; c_5=86.

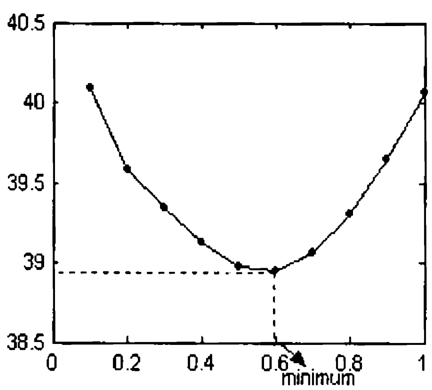

Fig.3 **theta vs. ETC**

λ= 5; β= 3; θ=0.75; γ=0.8;
s = 5; S=15; N=45; K=10;
c_1=1; c_2=1; c_3=4; c_4=1; c_5=14.2.

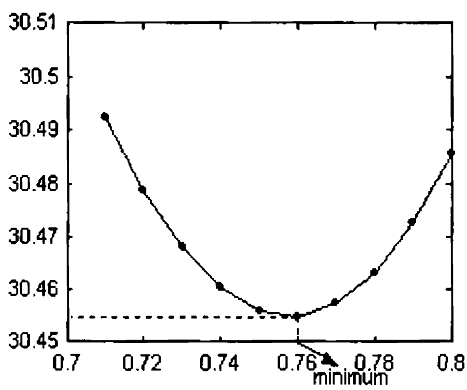

Fig.4 **delta vs. ETC**

$\lambda = 5$; $\beta = 3$; $\theta = 0.75$; $\delta = 0.7$;
$s = 5$; $S = 15$; $N = 45$; $K = 10$;
$c_1 = 1$; $c_2 = 1$; $c_3 = 4$; $c_4 = 1$; $c_5 = 1$.

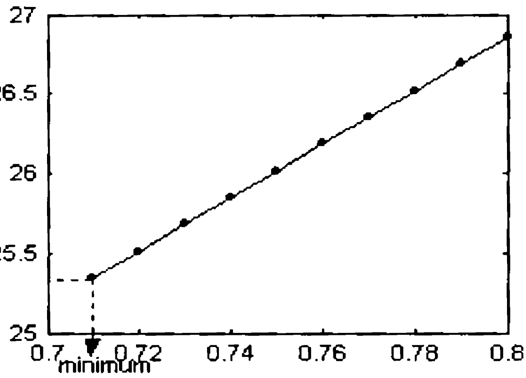

Fig.5 **gamma vs. ETC**

Interpretations of the Graphs

The objective is to minimize the total expected cost per unit time by varying the parameters one at a time and keeping others fixed. Since the objective function is known only implicitly the analytical properties such as convexity of the analytic function cannot be studied in general. One can determine the range for the costs that will yield nice analytical properties for the objective function, which can then be exploited for arriving at an optimal solution. By fixing all the parameters except the arrival rate λ, it is clear from the fig.1 that the cost function is convex in λ; for given parameter values this function attains its minimum value 46.719 at $\lambda = 3.5$. As replenishment rate β increases (keeping other parameters fixed), the cost function is linear and hence convex. For given parameter values this function attains its minimum value 78.569 at $\beta = 10.2$ in the interval [10.2, 12] of β. (see fig.2). One can observe the convexity of the objective function by changing other parameters θ, δ and γ (see fig.3, fig.4, and fig.5). Therefore, in all examples considered here, the cost function has the convexity property.

Chapter 4

Comparison of Three Inventory Systems having Service Time, Positive Lead-time, Loss and Retrial of Customers

4.1 Introduction

The first published work on stochastic inventory with service time is due to Parthasarathy and Vijayalakshmi [55]. They studied the time dependent solution of the system state probabilities. Arivarignan et al. [2] considered a perishable inventory system with service facility with arrival of customers forming a Poison process. Each customer requires single item, which is delivered through a service of random duration having exponential distribution. Krishnamoorthy and Mohammad Ekramol Islam [42] analyzed an (s,S) inventory system with postponed demands. When the inventory level reaches zero due to demands, further demands are sent to a pool (of postponed demand) which has finite capacity. The service to the pooled customers would be considered only after replenishment against the order placed.

In this chapter, we compare three (s,S) inventory systems with service and retrial of customers. Arrival of customers forms a Poisson process with rate λ and service times are exponentially distributed with parameter μ. When the inventory level depletes to s due to service, an order for replenishment is placed. The lead-time follows an exponential distribution with rate β. In model I, we assume that if an arriving customer finds the inventory level zero or server busy, proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer in the orbit, who finds the inventory level zero or server busy, returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. In models II and III, we assume that an arriving customer finds buffer full, proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer from the orbit, who finds buffer full, returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. In all these cases, the inter-retrial times follow an exponential distribution with linear rate $i\theta$ when there are i customers in the orbit.

In this chapter the following assumptions and notations are used.

Assumptions

(i) Inter-arrival times of demands are exponentially distributed with parameter λ.

(ii) Service time follows exponential distribution with parameter μ.

(iii) Lead-time follows exponential distribution with parameter β.

(iv) Inter-retrial times are exponential with linear rate $i\theta$, when there are i customers in the orbit.

Notations

$I(t)$: Inventory level at time t.

$N(t)$: Number of customers in the orbit at time t.

$M(t)$: Number of customers in the buffer at time t.

$$C(t) : \begin{cases} 0, \text{ if the server is idle} \\ 1, \text{ if the server is busy} \end{cases}$$

e : $(1,1,.....1)'$ a column vector of 1's of appropriate order.

For convenience in the representation of the infinitesimal generator of the process we use the following notations:

$$\sigma = -(\lambda + i\theta)$$
$$\omega = -(\lambda + \beta + i\theta)$$
$$\eta = -(\lambda + \mu + i\theta)$$
$$\Theta = -(\lambda + \beta + \mu + i\theta)$$
$$\Delta = -(\lambda\gamma + \beta + i\theta(1-\delta))$$
$$\nabla = -(\lambda\gamma + \mu + i\theta(1-\delta))$$
$$\Omega = -(\lambda\gamma + \beta + \mu + i\theta(1-\delta))$$

This chapter is organized as follows. Sections 4.2, 4.3 and 4. 4 provide the analysis of the model I, II and III respectively. Section 4.5 presents the cost analysis and numerical results.

4.2 Mathematical Modelling and Analysis of Model I

Let $I(t)$ be the inventory level and $N(t)$ be the number of customers in the orbit at time t. Let $C(t)$ be the sever status which is equal to 0 if the server is idle and 1 if the sever is busy. Now $\{X(t), t \geq 0\}$, where $X(t) = (N(t), C(t), I(t))$ is a Level Dependent Quasi Birth-Death (LDQBD) process on the state space $\{(i,0,j); i \geq 0, 0 \leq j \leq S\} \cup \{(i,1,j); i \geq 0, 1 \leq j \leq S\}$. The infinitesimal generator Q of the process is a block tri-diagonal matrix and it has the following form:

$$Q = \begin{pmatrix} A_{1,0} & A_0 & 0 & 0 & 0 & \cdots \\ A_{2,1} & A_{1,1} & A_0 & 0 & 0 & \cdots \\ 0 & A_{2,2} & A_{1,2} & A_0 & 0 & \cdots \\ 0 & 0 & A_{2,3} & A_{1,3} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{4.1}$$

where the blocks $A_0, A_{1,i}\,(i \geq 0) and A_{2,i}\,(i \geq 1)$ are square matrices, each of order $(2S+1)$; they are given by

$$A_0 = \begin{pmatrix} B_0 & 0 \\ 0 & \lambda\gamma I_S \end{pmatrix}, \quad A_{1,i} = \begin{pmatrix} E_0 & E_1 \\ E_2 & E_3 \end{pmatrix}, \quad A_{2,i} = \begin{pmatrix} C_0 & C_1 \\ 0 & i\theta(1-\delta) I_S \end{pmatrix},$$

$$B_0 = \begin{pmatrix} \lambda\gamma & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{(S+1)x(S+1)},$$

$$E_0 = \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ (s+1) \\ \vdots \\ (S-s) \\ (S-s+1) \\ \vdots \\ S \end{array} \begin{pmatrix} \Delta & & & & & & \beta & & \\ & \omega & & & & & & \beta & \\ & & \ddots & & & & & & \\ & & & \omega & & & & & \beta \\ & & & & \sigma & & & & \\ & & & & & \ddots & & & \\ & & & & & & \sigma & & \\ & & & & & & & \sigma & \\ & & & & & & & & \\ & & & & & & & & \sigma \end{pmatrix}_{(S+1)\times(S+1)},$$

$$E_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix}_{(S+1)\times S}, \qquad E_2 = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & \mu & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu & 0 \end{pmatrix}_{S\times(S+1)},$$

$$
E_3 = \begin{array}{c} 1 \\ 2 \\ \vdots \\ s \\ (s+1) \\ \vdots \\ (S-s) \\ (S-s+1) \\ \vdots \\ S \end{array}
\begin{pmatrix}
\Omega & & & & & \beta & & \\
& \Omega & & & & & \beta & \\
& & & & & & & \\
& & \Omega & & & & & \beta \\
& & & \nabla & & & & \\
& & & & & & & \\
& & & & & \nabla & & \\
& & & & & & \nabla & \\
& & & & & & & \\
& & & & & & & \nabla
\end{pmatrix}_{S \times S},
$$

$$
C_0 = \begin{pmatrix} i\theta(1-\delta) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{(S+1)\times(S+1)} \quad \text{and} \quad C_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ i\theta & 0 & \cdots & 0 & 0 \\ 0 & i\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & i\theta & 0 \\ 0 & 0 & & 0 & i\theta \end{pmatrix}_{(S+1)\times S}
$$

4.2.1 System Stability

For the model under consideration we define the following Lyapunov test function (see Falin and templeton [29]):

$$\phi(s) = i \text{, if s is a state in the level } i$$

The mean drift y_s for any s belonging to the level $i \geq 1$ is given by

$$
\begin{aligned}
y_s &= \sum_{p \neq s} q_{sp}(\phi(p) - \phi(s)) \\
&= \sum_{u} q_{su}(\phi(u) - \phi(s)) + \sum_{v} q_{sv}(\phi(v) - \phi(s)) + \sum_{w} q_{sw}(\phi(w) - \phi(s))
\end{aligned}
$$

where u, v, w vary over the states belonging to the levels $(i-1), i,$ and $(i+1)$, respectively. Then by the definition of ϕ, $\phi(u) = i-1$, $\phi(v) = i$ and $\phi(w) = i+1$

so that
$$y_s = -\sum_u q_{su} + \sum_w q_{sw}$$
$$= \begin{cases} -i\theta & \text{, if the server is idle} \\ -i\theta(1-\delta) + \lambda\gamma & \text{, otherwise} \end{cases}$$

Since $(1-\delta) > 0$, for any $\varepsilon > 0$, we can find N' large enough that $y_s < -\varepsilon$ for any s belonging the level $i \geq N'$. Hence by Tweedi's [64] result, the system under consideration is stable.

4.2.2 System Performance Measures

We partition the $(i+1)$th component of the steady state probability vector $\mathbf{x} = (x_0, x_1, \ldots\ldots, x_{N-1}, x_N, \ldots\ldots)$ as

$$x_i = (y_{i,0,0}, y_{i,0,1}, \ldots\ldots y_{i,0,S}, y_{i,1,1}, y_{i,1,2}, \ldots\ldots y_{i,1,S}).$$

Then,

i) Expected Inventory level, EI, is given by

$$EI = \sum_{i=0}^{\infty} \sum_{j=0}^{S} j y_{i,0,j} + \sum_{i=0}^{\infty} \sum_{j=1}^{S} j y_{i,1,j}$$

ii) Expected number of customers, EC, in the orbit is given by

$$EC = \left(\sum_{i=1}^{\infty} i x_i\right)\mathbf{e}$$
$$= \left(\left(\sum_{i=1}^{N-1} i x_i\right) + x_N\left(N(I-R)^{-1} + R(I-R)^{-2}\right)\right)\mathbf{e}$$

iii) Expected reorder rate, ERO, is given by

$$ERO = \mu \sum_{i=0}^{\infty} y_{i,1,s+1},$$

iv) Expected number of departures, EDS, after completing service is given by

$$EDS = \mu \sum_{i=0}^{\infty} \sum_{j=1}^{S} y_{i,1,j}$$

v) Expected number of customers lost, EL_1, before entering the orbit per unit time is given by

$$EL_1 = (1-\gamma)\lambda\sum_{i=0}^{\infty}\left(y_{i,0,0} + \sum_{j=1}^{S} y_{i,1,j}\right)$$

vi) Expected number of customers lost, EL_2, due to retrials per unit time is given by

$$EL_2 = \theta(1-\delta)\sum_{i=1}^{\infty} i\left(y_{i,0,0} + \sum_{j=1}^{S} y_{i,1,j}\right)$$

vii) Overall retrial rate, ORR, is given by

$$ORR = \theta\left(\sum_{i=1}^{\infty} ix_i\right)\mathbf{e}$$

viii) Successful retrial rate, SRR, is given by

$$SRR = \theta\sum_{i=0}^{\infty} i\left(\sum_{j=1}^{S} y_{i,0,j}\right)$$

4.3 Mathematical Modelling and Analysis of Model II

Here, in addition to the description in model I, we assume that there is a buffer of varying (finite) capacity, equal to the current inventory level. Customers, finding the buffer full, are directed an orbit. Let $M(t)$ be the number of customers in the buffer at time t. Now $\{X(t), t \geq 0\}$, where $X(t) = (N(t), I(t), M(t))$ is an LDQBD on the state space $\{(i,j,k); i \geq 0, 0 \leq j \leq S, 0 \leq k \leq j\}$. Then the generator has the form (4.1), where the blocks $A_0, A_{1,i}\,(i \geq 0)\,and A_{2,i}\,(i \geq 1)$ are square matrices of the same order $\frac{1}{2}(S+1)(S+2)$ and they are given by

$$A_0 = \begin{array}{c} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{S} \end{array} \begin{pmatrix} B_0 & & & \\ & B_1 & & \\ & & B_2 & \\ & & & \ddots \\ & & & & B_S \end{pmatrix} \qquad A_{2,i} = \begin{array}{c} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{S} \end{array} \begin{pmatrix} C_0 & & & \\ & C_1 & & \\ & & C_2 & \\ & & & \ddots \\ & & & & C_S \end{pmatrix}$$

$$A_{1,i} = \begin{array}{c} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{s} \\ \underline{s+1} \\ \\ \underline{S-s} \\ \underline{S-s+1} \\ \underline{S-s+2} \\ \vdots \\ \underline{S} \end{array} \begin{pmatrix} H_0 & & & & & & & U_0 & & & \\ P_1 & H_1 & & & & & & & U_1 & & \\ & P_2 & H_2 & & & & & & & U_2 & \\ & & \ddots & \ddots & & & & & & & \\ & & & P_s & H_s & & & & & & U_s \\ & & & & P_{s+1} & G_{s+1} & & & & & \\ & & & & & \ddots & \ddots & & & & \\ & & & & & & P_{S-s} & G_{S-s} & & & \\ & & & & & & & P_{S-s+1} & G_{S-s+1} & & \\ & & & & & & & & P_{S-s+2} & G_{S-s+2} & \\ & & & & & & & & & \ddots & \ddots \\ & & & & & & & & & P_S & G_S \end{pmatrix}$$

where

$$B_0 = (\lambda\gamma)_{1\times 1}, \qquad B_n = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \lambda\gamma \end{pmatrix}_{(n+1)\times(n+1)} \qquad (n = 1,2,.....S),$$

$$H_0 = (\Delta)_{1\times 1}, \qquad C_0 = (i\theta(1-\delta))_{1\times 1},$$

$$C_n = \begin{pmatrix} 0 & i\theta & & & & & \\ & 0 & i\theta & & & & \\ & & \ddots & \ddots & & & \\ & & & 0 & i\theta & & \\ & & & & 0 & i\theta & \\ & & & & & i\theta(1-\delta) \end{pmatrix}_{(n+1)\times(n+1)} \qquad (n=1,2,\ldots..S),$$

$$H_n = \begin{pmatrix} \omega & \lambda & & & & \\ & \Theta & \lambda & & & \\ & & \Theta & \lambda & & \\ & & & \ddots & \ddots & \\ & & & & \Theta & \lambda \\ & & & & & \Omega \end{pmatrix}_{(n+1)\times(n+1)} \qquad (n=1,2,\ldots..S),$$

$$G_n = \begin{pmatrix} \sigma & \lambda & & & & \\ & \eta & \lambda & & & \\ & & \eta & \lambda & & \\ & & & \ddots & \ddots & \\ & & & & \eta & \lambda \\ & & & & & \nabla \end{pmatrix}_{(n+1)\times(n+1)} \qquad (n=s+1,s+2,\ldots..S),$$

$$U_n = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ s \end{array} \begin{pmatrix} \beta & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \beta & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta & 0 & \cdots & 0 \end{pmatrix}_{(n+1)\times((S-s)+n+1)} \qquad (n=0,1,2,\ldots..s),$$

$$P_n = \begin{pmatrix} 0 & 0 & & 0 \\ \mu & 0 & & 0 \\ 0 & \mu & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \mu \end{pmatrix}_{(n+1)\times n} \qquad (n = 1,2,.....S).$$

4.3.1 System Stability

Here mean drift y_s is given by

$$y_s = \begin{cases} -i\theta(1-\delta)+\lambda\gamma & \text{, if the buffer is full} \\ -i\theta & \text{, otherwise} \end{cases}$$

Since $(1-\delta) > 0$, for any $\varepsilon > 0$, we can find N' large enough that $y_s < -\varepsilon$ for any s belonging the level $i \geq N'$. Hence by Tweedi's [64] result, the system under consideration is stable.

4.3.2 System Performance Measures

For computing various measures of performance we judiciously obtain a truncation level N. To find N we adopt the procedure in the sub-section 2.3.2. Here again we partition the steady state probability vector $\mathbf{x} = (x_0, x_1, \ldots.., x_{N-1}, x_N, \ldots..)$ such that its $(i+1)$th component is

$x_i = (y_{i,0,0}, y_{i,1,0}, y_{i,1,1}, y_{i,2,0}, y_{i,2,1}, y_{i,2,2}, \ldots.., y_{i,S,0}, y_{i,S,1}, y_{i,S,2}, \ldots.., y_{i,S,S})$

Then,

i) Expected Inventory level, EI, in the system is given by

$$EI = \sum_{i=0}^{\infty} \sum_{j=0}^{S} \sum_{k=0}^{j} j y_{i,j,k}$$

ii) Expected number of customers, EC, in the orbit is given by

$$EC = \left(\sum_{i=1}^{\infty} i x_i\right)\mathbf{e}$$

$$= \left((\sum_{i=1}^{N-1} i x_i) + x_N \left(N(I-R)^{-1} + R(I-R)^{-2}\right)\right)\mathbf{e}$$

iii) Expected number of customers, EB, in the buffer is given by

$$EB = \sum_{i=0}^{\infty}\sum_{j=0}^{S}\sum_{k=0}^{j} k y_{i,j,k}$$

iv) Expected reorder rate, ERO, is given by

$$ERO = \mu\sum_{i=0}^{\infty}\sum_{k=1}^{s+1} y_{i,s+1,k}$$

v) Expected number of departures, EDS, after completing service is

$$EDS = \mu\sum_{i=0}^{\infty}\sum_{j=1}^{S}\sum_{k=1}^{j} y_{i,j,k}$$

vi) Expected number of external customers lost, EL_1, before entering the orbit per unit time is given by

$$EL_2 = (1-\gamma)\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{S} y_{i,j,j}$$

vii) Expected number of customers lost, EL_2, due to retrials per unit time is

$$EL_2 = \theta(1-\delta)\sum_{i=1}^{\infty} i\left(\sum_{j=1}^{S} y_{i,j,j}\right)$$

viii) Overall rate of retrials, ORR, is given by

$$ORR = \theta\left(\sum_{i=1}^{\infty} i x_i\right)\mathbf{e}$$

ix) Successful rate of retrials, SRR, is given by

$$SRR = \theta\sum_{i=0}^{\infty} i\left(\sum_{j=1}^{S}\sum_{k=0}^{j-1} y_{i,j,k}\right)$$

4.4 Mathematical Modelling and Analysis of Model III

The distinguishing feature of this model from model II is that the capacity of the buffer is equal to S, the maximum inventory level, irrespective of the inventory held at any given instant of time. Here $\{X(t), t \geq 0\}$, where $X(t) = (N(t), I(t), M(t))$, is an LDQBD on the state space $\{(i, j, k); i \geq 0, 0 \leq j \leq S, 0 \leq k \leq S\}$. Then the generator has the form (4.1), where the blocks $A_0, A_{1,i}\ (i \geq 0) and A_{2,i}\ (i \geq 1)$ are square matrices of the same order $(S+1)^2$ and they are given by

$$A_0 = \begin{array}{c} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{S} \end{array} \begin{pmatrix} B_0 & & & & \\ & B_1 & & & \\ & & B_2 & & \\ & & & & \\ & & & & B_S \end{pmatrix}, \quad A_{2,i} = \begin{array}{c} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{S} \end{array} \begin{pmatrix} C_0 & & & & \\ & C_1 & & & \\ & & C_2 & & \\ & & & \ddots & \\ & & & & C_S \end{pmatrix},$$

$$A_{1,i} = \begin{array}{c} \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{s} \\ \underline{s+1} \\ \\ \underline{S-s} \\ \underline{S-s+1} \\ \underline{S-s+2} \\ \vdots \\ \underline{S} \end{array} \begin{pmatrix} H_0 & & & & & & & U_0 & & & & \\ P_1 & H_1 & & & & & & & U_1 & & & \\ & P_2 & H_2 & & & & & & & U_2 & & \\ & & \ddots & \ddots & & & & & & & \ddots & \\ & & & P_s & H_s & & & & & & & U_s \\ & & & & P_{s+1} & G_{s+1} & & & & & & \\ & & & & & \ddots & \ddots & & & & & \\ & & & & & & P_{S-s} & G_{S-s} & & & & \\ & & & & & & & P_{S-s+1} & G_{S-s+1} & & & \\ & & & & & & & & P_{S-s+2} & G_{S-s+2} & & \\ & & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & & P_S & G_S \end{pmatrix},$$

where

$$C_n = \begin{pmatrix} 0 & i\theta & & & & & \\ & 0 & i\theta & & & & \\ & & \ddots & \ddots & & & \\ & & & 0 & i\theta & & \\ & & & & 0 & i\theta & \\ & & & & & i\theta(1-\delta) \end{pmatrix}_{(S+1)\times(S+1)} , \quad B_n = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \lambda\gamma \end{pmatrix}_{(S+1)\times(S+1)} ,$$

$$(n = 0,1,2,.....S) \qquad (n = 0,1,2,.....S)$$

$$H_0 = \begin{pmatrix} \omega & \lambda & & & \\ & \omega & \lambda & & \\ & & \ddots & \ddots & \\ & & & \omega & \lambda \\ & & & & \Delta \end{pmatrix}_{(S+1)\times(S+1)} , \quad H_n = \begin{pmatrix} \omega & \lambda & & & & \\ & \Theta & \lambda & & & \\ & & \Theta & \lambda & & \\ & & & \ddots & \ddots & \\ & & & & \Theta & \lambda \\ & & & & & \Omega \end{pmatrix}_{(S+1)\times(S+1)} \quad (n=1,2,....S),$$

$$G_n = \begin{pmatrix} \sigma & \lambda & & & & \\ & \eta & \lambda & & & \\ & & \eta & \lambda & & \\ & & & \ddots & \ddots & \\ & & & & \eta & \lambda \\ & & & & & \nabla \end{pmatrix}_{(S+1)\times(S+1)} , \quad U_n = \begin{pmatrix} \beta & & & & \\ & \beta & & & \\ & & \beta & & \\ & & & \ddots & \\ & & & & \beta \end{pmatrix}_{(S+1)\times(S+1)} ,$$

$$(n = s+1, s+2,.....S) \qquad (n = 0,1,2,.....s)$$

$$P_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mu & 0 & & 0 & 0 \\ 0 & \mu & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & \mu & 0 \end{pmatrix}_{(S+1)\times(S+1)} \qquad (n = 1,2,.....S) .$$

4.4.1 System Stability

Here mean drift y_s is given by

$$y_s = \begin{cases} -i\theta(1-\delta)+\lambda\gamma & \text{, if the buffer is full} \\ -i\theta & \text{, otherwise} \end{cases}$$

Since $(1-\delta) > 0$, for any $\varepsilon > 0$, we can find N' large enough that $y_s < -\varepsilon$ for any s belonging the level $i \geq N'$. Hence by Tweedi's [64] result, the system under consideration is stable.

4.4.2 System Performance Measures

To find the truncation level N we adopt the procedure as in sub-section 2.3.2. The $(i+1)$th component of $\mathbf{x} = (x_0, x_1, ..., x_{N-1}, x_N, ...)$, the steady state probability vector, can be partitioned as

$$x_i = (y_{i,0,0}, y_{i,0,1}, y_{i,0,2}, ..., y_{i,0,S}, y_{i,1,0}, y_{i,1,1}, y_{i,1,2}, ..., y_{i,1,S}, ..., y_{i,S,0}, y_{i,S,1}, y_{i,S,2}, ..., y_{i,S,S})$$

Then,

(i) Expected Inventory level, EI, is given by

$$EI = \sum_{i=0}^{\infty} \sum_{j=0}^{S} \sum_{k=0}^{S} j y_{i,j,k}$$

(ii) Expected number of customers, EC, in the orbit is given by

$$EC = \left(\sum_{i=1}^{\infty} ix_i\right)\mathbf{e}$$

$$= \left((\sum_{i=1}^{N-1} ix_i) + x_N\left(N(I-R)^{-1} + R(I-R)^{-2}\right)\right)\mathbf{e}$$

(iii) Expected number of customers, EB, in the buffer is given by

$$EB = \sum_{i=0}^{\infty}\sum_{j=0}^{S}\sum_{k=0}^{S} ky_{i,j,k}$$

(iv) Expected reorder rate, ERO, is given by

$$ERO = \mu\sum_{i=0}^{\infty}\sum_{k=1}^{S} y_{i,s+1,k}$$

(v) Expected number of departures, EDS, after completing service is given by

$$EDS = \mu\sum_{i=0}^{\infty}\sum_{j=1}^{S}\sum_{k=1}^{S} y_{i,j,k}$$

(vi) Expected number of external customers lost, EL_1, before entering the orbit per unit time is given by

$$EL_1 = (1-\gamma)\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{S} y_{i,j,S}$$

(vii) Expected number of customers lost, EL_2, due to retrials per unit time is given by

$$EL_2 = \theta(1-\delta)\sum_{i=1}^{\infty} i\left(\sum_{j=1}^{S} y_{i,j,S}\right)$$

(viii) Overall rate of retrials, ORR, is given by

$$ORR = \theta\left(\sum_{i=1}^{\infty} ix_i\right)\mathbf{e}$$

(ix) Successful rate of retrials, SRR, is given by

$$SRR = \theta\sum_{i=0}^{\infty} i\left(\sum_{j=0}^{S}\sum_{k=0}^{S-1} y_{i,j,k}\right)$$

4.5 Cost Analysis and Numerical Examples

To construct the cost function we define the following costs as

C = fixed cost

c_1 = procurement cost/unit

c_2 = holding cost of inventory /unit /unit time

c_3 = holding cost of customers / unit /unit time

c_4 = cost due to loss of customers / unit /unit time

c_5 = cost due to service / unit /unit time

c_6 = revenue from service/unit/unit time.

In terms of these costs we define the expected total cost function as

$$ETC = (C + (S - s)c_1)ERO + c_2 EI + c_3(EC + EB) + c_4(EL_1 + EL_2) + (c_5 - c_6)EDS .$$

In the following tables we provide a comparison among the overall and successful rate of retrials for models I to III.

μ=1.5; β=1.0;θ=0.75;γ=0.2;δ=0.4;N=62;
s=1;S=4;c_1=1;c_2=1;c_3=0.6;c_4=1;c_5=2;c_6=1.

λ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
2.6	0.288076	0.081170	1.250731	0.660523	0.561902	0.145595
2.7	0.307286	0.087625	1.274185	0.662086	0.605659	0.149230
2.8	0.326964	0.094214	1.297996	0.663758	0.649895	0.153096
2.9	0.347096	0.100919	1.322157	0.665546	0.694298	0.157210
3.0	0.367663	0.107724	1.346658	0.667455	0.738639	0.161582
3.1	0.388647	0.114610	1.371489	0.669494	0.782765	0.166204
3.2	0.410028	0.121562	1.396640	0.671670	0.826577	0.171040
3.3	0.431785	0.128564	1.422100	0.673990	0.870013	0.176015
3.4	0.453897	0.135600	1.447860	0.676462	0.913040	0.180994
3.5	0.476343	0.142656	1.473909	0.679093	0.955641	0.185780

Table 1(**Variations in arrival rate** λ)

λ=2.0; μ=1.5; θ=0.75; γ=0.2; δ=0.4; N=48;
s=1; S=4; c_1=1; c_2=5; c_3=0.5; c_4=1; c_5=2; c_6=1

β	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
1.1	0.169276	0.041360	1.106037	0.695448	0.311534	0.199893
1.2	0.156979	0.037028	1.095219	0.692633	0.293366	0.201550
1.3	0.146188	0.033169	1.085242	0.690077	0.277385	0.201175
1.4	0.136660	0.029745	1.076010	0.687745	0.263261	0.199345
1.5	0.128201	0.026713	1.067440	0.685611	0.250709	0.196505
1.6	0.120650	0.024032	1.059460	0.683649	0.239489	0.192984
1.7	0.113876	0.021662	1.052012	0.681840	0.229406	0.189028
1.8	0.107770	0.019564	1.045041	0.680167	0.220297	0.184812
1.9	0.102244	0.017706	1.038504	0.678616	0.212029	0.180464
2.0	0.097222	0.016058	1.032359	0.677173	0.204490	0.176075

Table 2 (**Variations in replenishment rate** β)

λ=2.0; β=1.0; θ=0.75; .γ=0.2; δ=0.4; N=37;
s=1; S=4;c_1=1;c_2=1; c_3=1; c_4=1; c_5=2; c_6=1.

μ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
2.1	0.174671	0.045655	1.920751	1.497005	0.366320	0.290650
2.2	0.173506	0.045573	1.866814	1.442761	0.362866	0.286314
2.3	0.172406	0.045494	1.812846	1.388586	0.359369	0.281763
2.4	0.171364	0.045416	1.758860	1.334492	0.355847	0.276986
2.5	0.170376	0.045341	1.704867	1.280491	0.352321	0.271972
2.6	0.169438	0.045268	1.650881	1.226597	0.348820	0.266708
2.7	0.168546	0.045196	1.596919	1.172826	0.345380	0.261181
2.8	0.167697	0.045127	1.543002	1.119197	0.342050	0.255375
2.9	0.166888	0.045059	1.489156	1.065732	0.338890	0.249274
3.0	0.166116	0.044994	1.435410	1.012452	0.335977	0.242856

Table 3(**Variations in service rate** μ)

λ=2.0; β=1.0; μ=1.5; θ=0.75; δ=0.4; N=56;
s=1; S=4;c_1=1;c_2=5; c_3=1; c_4=1; c_5=2; c_6=1.

γ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
0.24	0.235962	0.058532	1.166675	0.716265	0.373307	0.197246
0.28	0.294115	0.071712	1.217041	0.734260	0.422046	0.201751
0.32	0.357502	0.085542	1.268831	0.752496	0.477661	0.209205
0.36	0.425575	0.099809	1.321993	0.770936	0.538306	0.219249
0.40	0.497650	0.114299	1.376494	0.789548	0.601978	0.231259
0.44	0.573019	0.128825	1.432317	0.808309	0.667075	0.244629
0.48	0.651051	0.143245	1.489458	0.827201	0.732477	0.258886
0.52	0.731269	0.157471	1.547918	0.846208	0.797423	0.273698
0.56	0.813361	0.171460	1.607709	0.865318	0.861393	0.288838
0.60	0.897137	0.185199	1.668843	0.884520	0.924018	0.304154

Table 4 (**Variations in probability γ of primary arrivals joining the orbit)**

λ=2.0; β=1.0; μ=1.5; θ=0.75;γ =0.2; N=51;
s=1; S=4; c_1=1; c_2=1.6; c_3=0.32; c_4=1; c_5=2; c_6=1.

δ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
0.35	0.179867	0.044661	1.077717	0.684408	0.325449	0.190152
0.40	0.183381	0.045382	1.117815	0.698563	0.332249	0.195492
0.45	0.187421	0.046189	1.162668	0.714154	0.338966	0.201308
0.50	0.192090	0.047095	1.213261	0.731433	0.345442	0.207632
0.55	0.197515	0.048111	1.270888	0.750721	0.351484	0.214490
0.60	0.203860	0.049248	1.337303	0.772436	0.356886	0.221897
0.65	0.211348	0.050516	1.414941	0.797138	0.361457	0.229859
0.70	0.220371	0.051921	1.507329	0.825602	0.365094	0.238379
0.75	0.232606	0.053478	1.619805	0.858943	0.367846	0.247469
0.80	0.266273	0.055392	1.760960	0.898851	0.369974	0.257184

Table 5(**Variations in return probability δ of retrial customers)**

λ=2.0; β=1.0; μ=1.5; γ =0.2; δ=0.4; N=28;
s=1; S=4; c_1=1; c_2=1; c_3=1; c_4=60; c_5=2; c_6=1.

θ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
1.1	0.200807	0.035823	1.137674	0.284382	0.309113	0.049809
1.2	0.204819	0.036609	1.142996	0.297763	0.310083	0.053750
1.3	0.208480	0.037415	1.148167	0.312566	0.311154	0.058208
1.4	0.211829	0.038242	1.153192	0.329041	0.312339	0.063285
1.5	0.214901	0.039085	1.158074	0.347505	0.313649	0.069106
1.6	0.217728	0.039944	1.162817	0.368356	0.315098	0.075831
1.7	0.220334	0.040814	1.167424	0.392112	0.316703	0.083670
1.8	0.222745	0.041689	1.171901	0.419448	0.318480	0.092899
1.9	0.224979	0.042563	1.176251	0.451269	0.320448	0.103889
2.0	0.227055	0.043427	1.180478	0.488808	0.322629	0.117151

Table 6(**Variations in retrial rate** θ)

Interpretations of the Numerical Results in the Tables

As the arrival rate λ increases the number of customers in the orbit becomes larger so that the overall and successful rates of retrials from the orbit will increase (see table 1). As either the replenishment rate β or the service rate μ increases the arriving customers will get the inventory served more rapidly thereby the number of customers in the orbit gets decreased. In that case, the overall and successful rates of retrials will decrease (see tables 2 and 3). With the increase in probability γ of primary arrivals joining the orbit or increase in return probability δ of retrial customers, the orbit size increases. Here again, overall and successful rates of retrials increase (see tables 4 and 5). Table 6 indicates that as the retrial rate θ of customers in the orbit increases, the overall and successful rates of retrials from the orbit will increase.

Next we provide graphical illustrations of the Performance measures of the above described models.

$\mu=1.5$; $\beta=1.0$; $\theta=0.75$;$\gamma=0.2$;$\delta=0.4$;N=62;
s=1;S=4;$c_1=1$;$c_2=1$;$c_3=0.6$;$c_4=1$;$c_5=2$;$c_6=11$

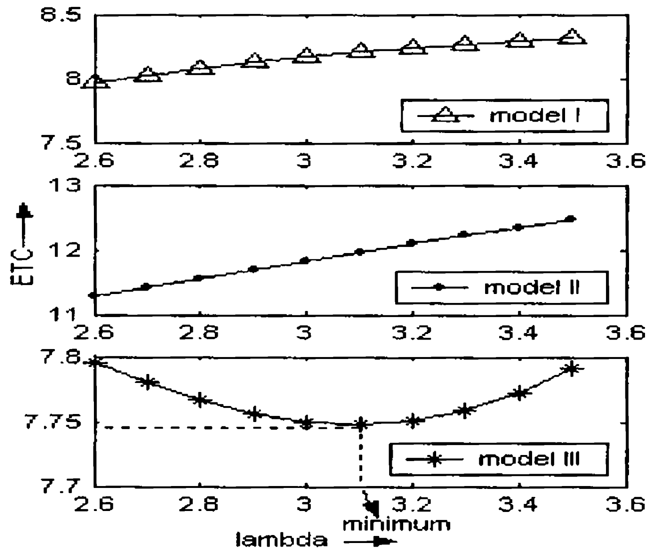

fig.1 **(arrival rate vs. ETC)**

$\lambda=2.0$; $\mu=1.5$; $\theta=0.75$; $\gamma=0.2$; $\delta=0.4$; N=48;
s=1; S=4;$c_1=1$;$c_2=5$; $c_3=0.5$; $c_4=1$; $c_5=2$; $c_6=1$

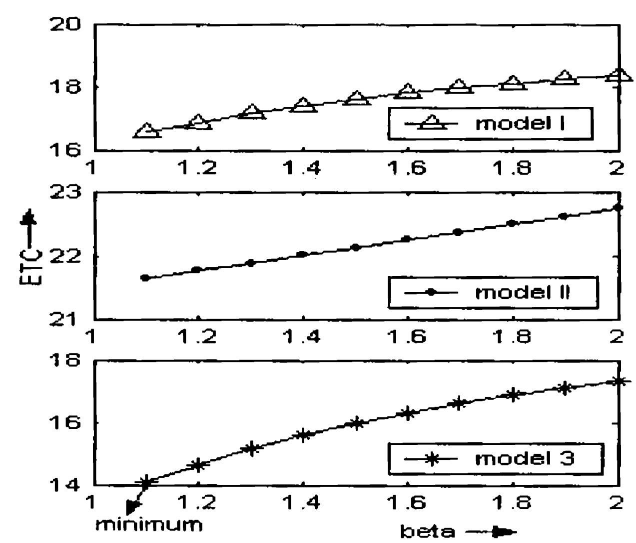

fig.2 **(replenishment rate vs. ETC)**

$\lambda=2.0; \beta=1.0; \theta=0.75; \gamma=0.2; \delta=0.4; N=37;$
$s=1; S=4; c_1=1; c_2=1; c_3=1; c_4=1; c_5=2; c_6=1.$

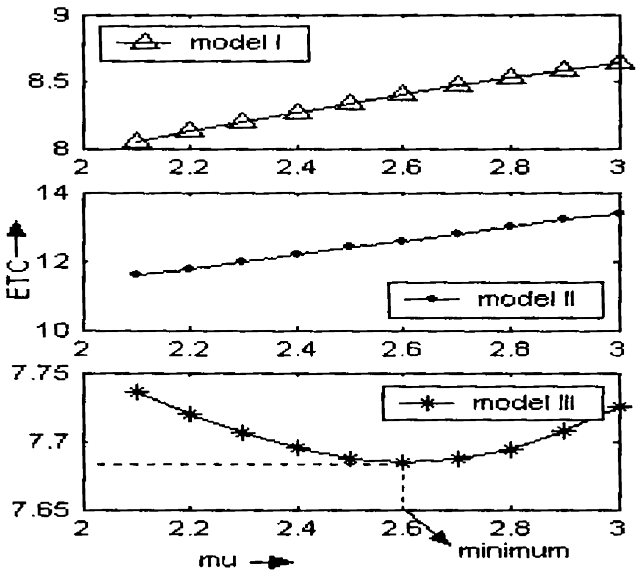

fig.3 **(service rate vs. ETC)**

$\lambda=2.0; \beta=1.0; \mu=1.5; \theta=0.75; \delta=0.4; N=56;$
$s=1; S=4; c_1=1; c_2=5; c_3=1; c_4=1; c_5=2; c_6=1.$

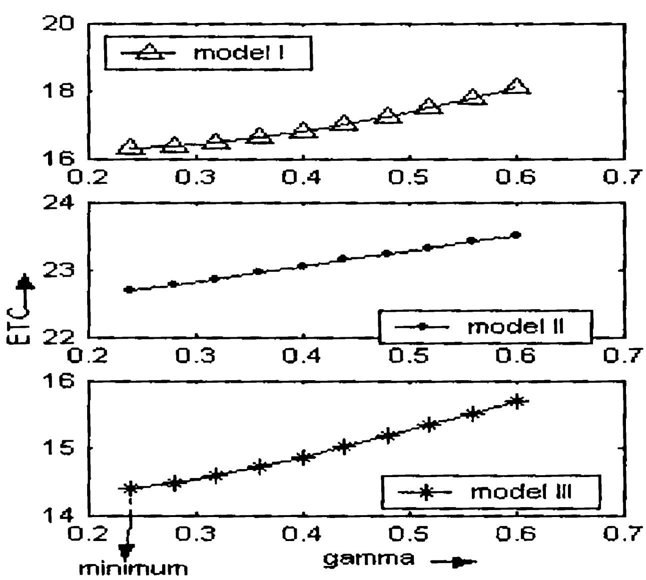

fig.4 **(gamma vs. ETC)**

$\lambda=2.0$; $\beta=1.0$; $\mu=1.5$; $\theta=0.75$; $\gamma=0.2$; $N=51$; $s=1$;
$S=4$; $c_1=1$; $c_2=1.6$; $c_3=0.32$; $c_4=1$; $c_5=2$; $c_6=1$

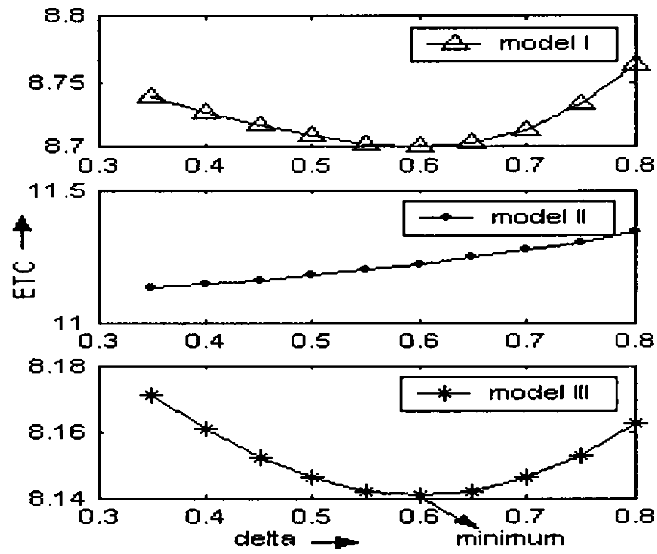

fig.5 **(delta vs. ETC)**

$\lambda=2.0$; $\beta=1.0$; $\mu=1.5$; $\gamma=0.2$; $\delta=0.4$; $N=28$; $s=1$;
$S=4$; $c_1=1$; $c_2=1$; $c_3=1$; $c_4=60$; $c_5=2$; $c_6=1$.

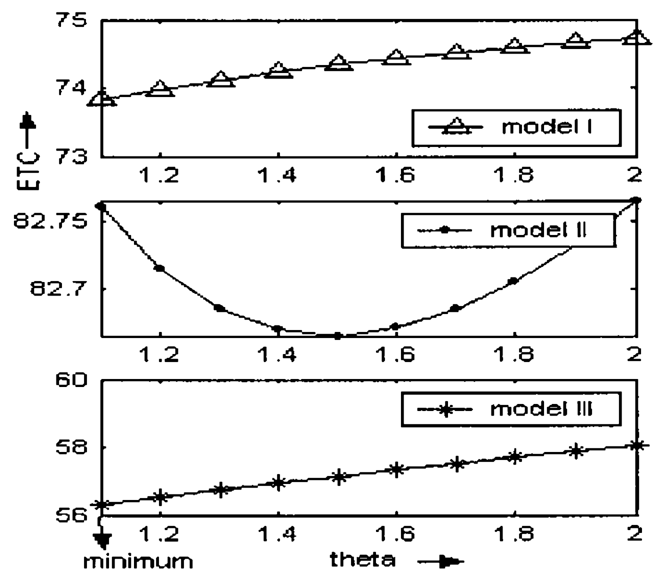

fig.6 **(theta vs. ETC)**

Interpretations of the Graphs

The objective is to compare the three models and identify the one which is more profitable. For this, we compute the value of the expected total cost per unit time by varying the parameters one at a time keeping others fixed and then look for the minimum of these values. By fixing all the parameters except the arrival rate λ, it is clear from the fig.1 that the cost function is convex in λ for model III; for given parameter values this function attains the following minimum values (a) 7.966 at $\lambda = 2.6$ for model I, (b) 11.287 at $\lambda = 2.6$ for model II and (c) 7.747 at $\lambda = 3.1$ for model III. It may be noted that the model III performs better in the range for λ from 2.6 to 3.8. Therefore model III is the best for minimum cost per unit time. As replenishment rate β increases (keeping other parameters fixed), one can observe that cost function attains the minimum values $16.543, 21.647$, 14.057 at $\beta = 1.1$ for the models I, II and III respectively in the range [1.1, 2] of β. Here also the model III is best for minimum cost per unit time. One can observe the minimum value of the objective function by changing other parameters μ, γ, δ and θ (see fig.3, fig.4, fig.5 and fig.6). In all examples considered here, the cost function has the minimum value for the model III. Therefore model III (model with buffer size equal to the maximum inventory level S) can be considered as the best model for practical applications in the ranges specified for parameter values.

Chapter 5

Comparison of Three Production Inventory Systems having Service Time, Loss and Retrial of Customers

5.1 Introduction

Inventory systems in which arriving customers who find items out of stock may retry for the items after a period are called retrial inventory. If the item is available, then the same is supplied with negligible or a positive service time. However, when at a demand epoch the item is out of stock, such items need not be backlogged or lost; instead they are directed to an orbit. At random epochs such customers retry until either the demand is met or finally the customer decides not to approach that establishment. Artalejo et al. [10] were the first to study inventory policies with positive lead-time and retrial of customers who could not get the item during their earlier attempts to access the service station.

In this chapter, we compare three production inventory systems with service and retrial of customers. Arrival of customers forms a Poisson process rate λ and service times are exponentially distributed with parameter μ. When the inventory level depletes to s due to service provided to the arriving customers, production starts, that is, the production is switched to ON mode. The time between additions of two successive items (by production) to the inventory is exponential with rate β. In model 1, an arriving customer who finds the inventory level zero, proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer in the orbit who finds the inventory level zero, returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. In models II and III, an arriving customer who finds the buffer full proceeds to an orbit with probability γ and is lost forever with probability $(1-\gamma)$. A retrial customer from the orbit who finds the buffer full returns to the orbit with probability δ and is lost forever with probability $(1-\delta)$. In all these systems, inter-retrial times follow an exponential distribution with linear rate $i\theta$ when there are i customers in the orbit.

The following assumptions and notations are used in this chapter.

Assumptions

(i) Inter-arrival times of demands are exponentially distributed with parameter λ.

(ii) Service time follows exponential distribution with rate μ.

(iii) Time between two successive items added to the inventory follows exponential distribution with rate β.

(iv) Inter-retrial times are exponential with linear rate $i\theta$, when there are i customers in the orbit.

Notations

$I(t)$: Inventory level at time t.

$N(t)$: Number of customers in orbit at time t.

$M(t)$: Number of customers in the buffer at time t.

$$C(t) : \begin{cases} 0, & \text{if the server is idle} \\ 1, & \text{if the server is busy} \end{cases}$$

$$J(t) \quad : \begin{cases} 0, \text{ if the production is in OFF mode} \\ 1, \text{ if the production is in ON mode} \end{cases}$$

e : $(1,1,.....1)'$ a column vector of 1's of appropriate order.

For convenient in the representation of the infinitesimal generator of the process we introduce the following notations:

$$\sigma = -(\lambda + i\theta); \omega = -(\lambda + \beta + i\theta); \eta = -(\lambda + \mu + i\theta); \Theta = -(\lambda + \beta + \mu + i\theta);$$
$$\Delta = -(\lambda\gamma + \beta + i\theta(1-\delta)); \nabla = -(\lambda\gamma + \mu + i\theta(1-\delta)); \Omega = -(\lambda\gamma + \beta + \mu + i\theta(1-\delta))$$

This chapter is organized as follows. Sections 5.2, 5.3 and 5.4 provide the analysis of the model I, II and III. Section 5.5 presents the cost analysis and numerical results.

5.2 Mathematical Modelling and Analysis of Model I

Let $I(t)$ be the inventory level and $N(t)$ be the number of customers in the orbit at time t. Let $C(t)$ be the sever status at time t, which is equal to 1 if the server is busy and 0 if the sever is idle. Let $J(t)$ be the production which is equal to 1 if the production is in ON mode and 0 if the production is in OFF mode. Now $\{X(t), t \geq 0\}$, where $X(t) = (N(t), C(t), J(t), I(t))$ is a level dependent quasi birth-death process on the state space $\{(i,k,0,j); i \geq 0; k = 0,1; s+1 \leq j \leq S\} \cup \{(i,0,1,j); i \geq 0; 0 \leq j \leq S-1\} \cup \{(i,1,1,j); i \geq 0; 1 \leq j \leq S-1\}$. The infinitesimal generator Q, of the process is a block tri-diagonal matrix and it has the following form:

$$Q = \begin{pmatrix} A_{1,0} & A_0 & 0 & 0 & 0 & \cdots \\ A_{2,1} & A_{1,1} & A_0 & 0 & 0 & \cdots \\ 0 & A_{2,2} & A_{1,2} & A_0 & 0 & \cdots \\ 0 & 0 & A_{2,3} & A_{1,3} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.1}$$

where the blocks $A_0, A_{1,i}\,(i \ge 0)\, and A_{2,i}\,(i \ge 1)$ are square matrices, each of order $(4S-2s-1)$; they are given by

$$A_0 = \begin{array}{c} \underline{0,0} \\ \underline{0,1} \\ \underline{1,0} \\ \underline{1,1} \end{array} \begin{pmatrix} 0 & & & \\ & \lambda\gamma B_1 & & \\ & & \lambda\gamma I_{S-s} & \\ & & & \lambda\gamma I_{S-1} \end{pmatrix},$$

$$A_{1,i} = \begin{array}{c} \underline{0,0} \\ \underline{0,1} \\ \underline{1,0} \\ \underline{1,1} \end{array} \begin{pmatrix} -(\lambda+i\theta)I_{S-s} & 0 & \lambda I_{S-s} & 0 \\ B_3 & B_4 & 0 & \lambda B_2 \\ B_5 & B_6 & -(\lambda+i\theta(1-\delta)+\mu)I_{S-s} & 0 \\ 0 & B_7 & B_8 & B_9 \end{pmatrix} (i \ge 0),$$

$$A_{2,i} = \begin{array}{c} \underline{0,0} \\ \underline{0,1} \\ \underline{1,0} \\ \underline{1,1} \end{array} \begin{pmatrix} 0 & 0 & i\theta I_{S-s} & 0 \\ 0 & i\theta(1-\delta)B_1 & 0 & i\theta B_2 \\ 0 & 0 & i\theta(1-\delta)I_{S-s} & 0 \\ 0 & 0 & 0 & i\theta(1-\delta)I_{S-1} \end{pmatrix} (i \ge 0),$$

where

$$B_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{SxS}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & & 1 \end{pmatrix}_{S\times(S-1)},$$

$$B_3 = \begin{pmatrix} 0 & 0 & & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}_{S\times(S-s)}, \qquad B_4 = \begin{pmatrix} \Delta & \beta & & \\ & \omega & \ddots & \\ & & \ddots & \beta \\ & & & \omega \end{pmatrix}_{S\times S},$$

$$B_5 = \begin{pmatrix} 0 & & & \\ \mu & 0 & & \\ & \mu & & \\ & & \ddots & \\ & & \mu & 0 \end{pmatrix}_{(S-s)\times(S-s)}, \qquad B_6 = \begin{array}{c} s+1 \\ s+2 \\ \vdots \\ S \end{array}\begin{pmatrix} 0 & \cdots & 0 & \mu & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(S-s)\times S},$$

$$B_7 = \begin{pmatrix} \mu & 0 & & & \\ & \mu & 0 & & \\ & & \ddots & \ddots & \\ & & & \mu & 0 \end{pmatrix}_{(S-1)\times S}, \qquad B_8 = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & & 0 & 0 \\ 0 & \cdots & 0 & \beta \end{pmatrix}_{(S-1)\times(S-1)},$$

$$B_9 = \begin{pmatrix} \Omega & \beta & & \\ & \Omega & \ddots & \\ & & \ddots & \beta \\ & & & \Omega \end{pmatrix}_{(S-1)\times(S-1)},$$

5.2.1 System Stability

For the model under consideration we define the following Lyapunov test function (see Falin and Templeton [29]):

$\phi(s) = i$, if s is a state in the level i

The mean drift y_s for any s belonging to the level $i \geq 1$ is given by

, if the server is idle and inventory level is positive

, otherwise

$$y_s = \begin{cases} -i\theta \\ -i\theta(1-\delta)+\lambda\gamma \end{cases}$$

Since $(1-\delta)>0$, for any $\varepsilon>0$, we can find N' large enough that $y_s<-\varepsilon$ for any s belonging to the level $i \geq N'$. Hence by Tweedi's [64] result, the system under consideration is stable.

5.2.2 System Performance Measures

We partition the steady state probability vector $\mathbf{x}=(x_0, x_1, \ldots\ldots, x_{N-1}, x_N, \ldots.)$ such that its $(i+1)$th component is given by

$x_i = (y_{i,0,0,s+1}, y_{i,0,0,s+2}, \ldots y_{i,0,0,S}, y_{i,0,1,0}, y_{i,0,1,1}, \ldots y_{i,0,1,S-1}, y_{i,1,0,s+1}, y_{i,1,0,s+1}, \ldots y_{i,1,0,S}, y_{i,1,1,1}, y_{i,1,1,2}, \ldots y_{i,1,1,S-1})$

Then,

(i) Expected Inventory level, EI, in the system is given by

$$EI = \sum_{i=0}^{\infty}\sum_{k=0}^{1}\sum_{j=0}^{S} jy_{i,k,0,j} + \sum_{i=0}^{\infty}\sum_{k=0}^{1}\sum_{j=1}^{S-1} jy_{i,k,1,j}$$

(ii) Expected number of customers, EC, in the orbit is given by

$$EC = \left(\sum_{i=1}^{\infty} ix_i\right)\mathbf{e}$$

$$= \left((\sum_{i=1}^{N-1} ix_i) + x_N\left(N(I-R)^{-1} + R(I-R)^{-2}\right)\right)\mathbf{e}$$

(iii) Expected switching rate, ESR, is given by

$$ESR = \mu\sum_{i=0}^{\infty} y_{i,1,0,s+1,}$$

(iv) Expected number of departures, EDS, after completing service is

$$EDS = \mu\sum_{i=0}^{\infty}\left(\sum_{j=s+1}^{S} y_{i,1,0,j} + \sum_{j=1}^{S-1} y_{i,1,1,j}\right)$$

(v) Expected number of external customers lost, EL_1, before entering the orbit per unit time is

$$EL_1 = (1-\gamma)\lambda\sum_{i=0}^{\infty}\left(y_{i,0,1,0} + \sum_{j=s+1}^{S} y_{i,1,0,j} + \sum_{j=1}^{S-1} y_{i,1,1,j}\right)$$

(vi) Expected number of customers lost, EL_2, due to retrials per unit time is

$$EL_2 = \theta(1-\delta)\sum_{i=1}^{\infty} i\left(y_{i,0,1,0} + \sum_{j=s+1}^{S} y_{i,1,0,j} + \sum_{j=1}^{S-1} y_{i,1,1,j}\right)$$

(vii) Overall rate of retrials, ORR, is given by

$$ORR = \theta\left(\sum_{i=1}^{\infty} ix_i\right)\mathbf{e}$$

(viii) Successful rate of retrials, SRR, is given by

$$SRR = \theta\sum_{i=0}^{\infty} i\left(\sum_{j=s+1}^{S} y_{i,0,0,j} + \sum_{j=1}^{S-1} y_{i,0,1,j}\right)$$

5.3 Mathematical Modelling and Analysis of Model II

Here, in addition to the description in model I, we assume that there is a buffer of varying (finite) capacity, equal to the current inventory level. Customers finding the buffer full are directed to an orbit. Let M (t) be the number of customers in the buffer at time t. Assume that the capacity of the buffer is equal to the number of inventoried items at that instant of time. Now $\{X(t), t \geq 0\}$, where $X(t) = (N(t), J(t), I(t), M(t))$ is an LDQBD on the state space $\{(i,0,j,k); i \geq 0, s+1 \leq j \leq S, 0 \leq k \leq j\} \cup \{(i,1,j,k); i \geq 0, 0 \leq j \leq S-1, 0 \leq k \leq j\}$. Then the generator has the form (5.1), where the blocks $A_0, A_{1,i}\ (i \geq 0)$ and $A_{2,i}\ (i \geq 1)$ are square matrices of the same order $\frac{1}{2}[(S-s)(S+s+3) + S(S+1)]$ and they are given by

$$A_0 = \begin{pmatrix} C_{s+1} & & & & & & \\ & C_{s+2} & & & & & \\ & & \ddots & & & & \\ & & & C_S & & & \\ & & & & C_1 & & \\ & & & & & C_2 & \\ & & & & & & \ddots \\ & & & & & & & C_{S-1} \end{pmatrix},$$

$$A_{2,i}=\begin{pmatrix}
E_{s+1} & & & & & & \\
 & E_{s+2} & & & & & \\
 & & E_S & & & & \\
 & & & E_0 & & & \\
 & & & & E_1 & & \\
 & & & & & & \\
 & & & & & & E_{S-1}
\end{pmatrix},$$

$$A_{1,i}=\begin{array}{c}
\underline{0,s+1}\\ \underline{0,s+2}\\ 0,s+3\\ \vdots\\ \underline{0,S}\\ \underline{1,0}\\ \underline{1,1}\\ \underline{1,2}\\ \vdots\\ \underline{1,s}\\ \vdots\\ \underline{1,S-2}\\ \underline{1,S-1}
\end{array}
\begin{pmatrix}
G_{s+1} & & & & & & & & & P_{s+1} & & & \\
P_{s+2} & G_{s+2} & & & & & & & & & & & \\
 & P_{s+3} & G_{s+3} & & & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & & & \\
 & & & P_S & G_S & & & & & & & & \\
 & & & & \ddots & H_0 & U_0 & & & & & & \\
 & & & & & P_1 & H_1 & U_1 & & & & & \\
 & & & & & & P_2 & H_2 & U_2 & & & & \\
 & & & & & & & \ddots & \ddots & \ddots & & & \\
 & & & & & & & & P_s & H_s & U_s & & \\
 & & & & & & & & & \ddots & \ddots & \ddots & \\
 & & & & & & & & & & P_{S-2} & H_{S-2} & U_{S-2} \\
 & & & & U_{S-1} & & & & & & & P_{S-1} & H_{S-1}
\end{pmatrix},$$

where

$$C_0=\left(\lambda\gamma\right)_{1\times1}\ ,\quad C_n=\begin{pmatrix}0 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & 0\\ 0 & \cdots & 0 & \lambda\gamma\end{pmatrix}_{(n+1)\times(n+1)}\quad (n=1,2,....S)\ ,\ E_0=\left(i\theta(1-\delta)\right)_{1\times1}\ ,$$

$$C_n = \begin{pmatrix} 0 & i\theta & & & & & \\ & 0 & i\theta & & & & \\ & & \ddots & \ddots & & & \\ & & & 0 & i\theta & & \\ & & & & 0 & i\theta & \\ & & & & & i\theta(1-\delta) \end{pmatrix}_{(n+1)\times(n+1)} \quad (n=1,2,....S)\ ,\ H_0 = \left(-(\lambda\gamma + \beta + i\theta(1-\delta))\right)_{1\times 1}\ ,$$

$$H_n = \begin{pmatrix} \omega & \lambda & & & & \\ & \eta & \lambda & & & \\ & & \eta & \lambda & & \\ & & & \ddots & \ddots & \\ & & & & \eta & \lambda \\ & & & & & \Omega \end{pmatrix}_{(n+1)\times(n+1)} \quad (n = 1,2,.....S-1)\ ,$$

$$G_n = \begin{pmatrix} \sigma & \lambda & & & & \\ & \eta & \lambda & & & \\ & & \eta & \lambda & & \\ & & & \ddots & \ddots & \\ & & & & \eta & \lambda \\ & & & & & \nabla \end{pmatrix}_{(n+1)\times(n+1)} \quad (n = s+1, s+2,.....S)\ ,$$

$$U_n = \begin{pmatrix} \beta & 0 & & 0 & 0 & 0 \\ 0 & \beta & & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & & \beta & 0 & 0 \\ 0 & 0 & & 0 & \beta & 0 \end{pmatrix}_{(n+1)\times(n+2)} \quad (n = 0,1,.....S-1)\ ,$$

$$P_n = \begin{pmatrix} 0 & 0 & & 0 \\ \mu & 0 & \cdots & 0 \\ 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \mu \end{pmatrix}_{(n+1)\times n} \qquad (n = 1, 2, \ldots..S)$$

5.3.1 System stability

Here mean drift y_s is given by

$$y_s = \begin{cases} -i\theta(1-\delta) + \lambda\gamma & \text{, if the buffer is full} \\ -i\theta & \text{, otherwise} \end{cases}$$

Since $(1-\delta) > 0$, for any $\varepsilon > 0$, we can find N' large enough that $y_s < -\varepsilon$ for any s belonging to the level $i \geq N'$. Hence by Tweedi's [64] result, the system under consideration is stable.

5.3.2 System Performance Measures

We partition the steady state probability vector $\mathbf{x} = (x_0, x_1, \ldots..x_{N-1}, x_N, \ldots..)$ such that its the $(i+1)$ th component is

$x_i = (y_{i,0,s+1,0}, y_{i,0,s+1,1}, \ldots, y_{i,0,s+1,s+1}, y_{i,0,s+2,0}, y_{i,0,s+2,1}, \ldots, y_{i,0,s+2,s+2}, \ldots, y_{i,0,S,0}, y_{i,0,S,1}, \ldots, y_{i,0,S,S},$ $y_{i,1,0,0}, y_{i,1,1,0}, y_{i,1,1,1}, y_{i,1,2,0}, y_{i,1,2,1}, y_{i,1,2,2}, \ldots\ldots, y_{i,1,S-1,0}, y_{i,1,S-1,1}, \ldots\ldots, y_{i,1,S-1,S-1})$.

Then,

(i) Expected Inventory level, EI, in the system is given by

$$EI = \sum_{i=0}^{\infty}\sum_{j=s+1}^{S}\sum_{k=0}^{j} jy_{i,0,j,k} + \sum_{i=0}^{\infty}\sum_{j=0}^{S-1}\sum_{k=0}^{j} jy_{i,1,j,k}$$

(ii) Expected number of customers, EC, in the orbit is given by

$$EC = \left(\sum_{i=1}^{\infty} i x_i\right)\mathbf{e}$$

$$= \left(\left(\sum_{i=1}^{N-1} i x_i\right) + x_N \left(N(I-R)^{-1} + R(I-R)^{-2} \right) \right)\mathbf{e}$$

(iii) Expected number of customers, EB, in the buffer is given by

$$EB = \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} \sum_{k=0}^{j} k y_{i,0,j,k} + \sum_{i=0}^{\infty} \sum_{j=0}^{S-1} \sum_{k=0}^{j} k y_{i,1,j,k}$$

(iv) Expected switching rate, ESR, is given by

$$ESR = \mu \sum_{i=0}^{\infty} \sum_{k=1}^{s+1} y_{i,0,s+1,k}$$

(v) Expected number of departures, EDS, after completing service is

$$EDS = \mu \left(\sum_{i=0}^{\infty} \sum_{j=s+1}^{S} \sum_{k=1}^{j} y_{i,0,j,k} + \sum_{i=0}^{\infty} \sum_{j=1}^{S-1} \sum_{k=1}^{j} y_{i,1,j,k} \right)$$

(vi) Expected number of external customers lost, EL_1, before entering the orbit per unit time is given by

$$EL_1 = (1-\gamma)\lambda \left(\sum_{i=0}^{\infty} \sum_{j=s+1}^{S} y_{i,0,j,j} + \sum_{i=0}^{\infty} \sum_{j=0}^{S-1} y_{i,0,j,j} \right)$$

(vii) Expected number of customers lost, EL_2, due to retrials per unit time is

$$EL_2 = \theta(1-\delta) \left[\sum_{i=1}^{\infty} i \left(\sum_{j=s+1}^{S} y_{i,0,j,j} + \sum_{j=0}^{S-1} y_{i,1,j,j} \right) \right]$$

(viii) Overall rate of retrials, ORR, is given by

$$ORR = \theta \left(\sum_{i=1}^{\infty} i x_i \right)\mathbf{e}$$

(ix) Successful rate of retrials, SRR, is given by

$$SRR = \theta \left[\sum_{i=1}^{\infty} i \left(\sum_{j=s+1}^{S} \sum_{k=0}^{j-1} y_{i,0,j,k} \right) + \sum_{i=1}^{\infty} i \left(\sum_{j=1}^{S-1} \sum_{k=0}^{j-1} y_{i,1,j,k} \right) \right]$$

5.4 Mathematical Modelling and Analysis of Model III

The distinguishing factor of this model from model II is that the capacity of the buffer is equal to S, the maximum inventory level, irrespective of the inventory held at any given instant of time. Now $\{X(t), t \geq 0\}$, where $X(t) = (N(t), J(t), I(t), M(t))$ is a LDQBD on the state space $\{(i,0,j,k); i \geq 0, s+1 \leq j \leq S, 0 \leq k \leq S\} \cup \{(i,1,j,k); i \geq 0, 0 \leq j \leq S-1, 0 \leq k \leq S\}$.

Then the generator has the form (5.1), where the blocks $A_0, A_{1,i}\,(i \geq 0)$ and $A_{2,i}\,(i \geq 1)$ are square matrices of same order $(S+1)(2S-s)$ and they are given by

$$A_0 = \begin{array}{c} \underline{s+1} \\ \underline{s+2} \\ \\ \underline{S} \\ \underline{0} \\ \underline{1} \\ \\ \underline{S-1} \end{array} \begin{pmatrix} F_{s+1} & & & & & & \\ & F_{s+2} & & & & & \\ & & \ddots & & & & \\ & & & F_S & & & \\ & & & & F_0 & & \\ & & & & & F_1 & \\ & & & & & & \ddots \\ & & & & & & & F_{S-1} \end{pmatrix},$$

$$A_{2,i} = \begin{pmatrix} M_{s+1} & & & & & & \\ & M_{s+2} & & & & & \\ & & \ddots & & & & \\ & & & M_S & & & \\ & & & & M_0 & & \\ & & & & & M_1 & \\ & & & & & & \ddots \\ & & & & & & & M_{S-1} \end{pmatrix},$$

$$
A_{1,i} = \begin{array}{c}
\underline{0,s+1} \\
\underline{0,s+2} \\
0,s+3 \\
\vdots \\
\underline{0,S} \\
\underline{1,0} \\
\underline{1,1} \\
\underline{1,2} \\
\vdots \\
\underline{1,s} \\
\vdots \\
\underline{1,S-2} \\
\underline{1,S-1}
\end{array}
\begin{pmatrix}
K_{s+1} & & & & & & & & & J_{s+1} & & & \\
J_{s+2} & K_{s+2} & & & & & & & & & & & \\
 & J_{s+3} & K_{s+3} & & & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & & & \\
 & & & J_S & K_S & & & & & & & & \\
 & & & & \ddots & L_0 & V_0 & & & & & & \\
 & & & & & J_1 & L_1 & V_1 & & & & & \\
 & & & & & & J_2 & L_2 & V_2 & & & & \\
 & & & & & & & \ddots & \ddots & \ddots & & & \\
 & & & & & & & & J_s & L_s & V_s & & \\
 & & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & & J_{S-2} & L_{S-2} & V_{S-2} \\
 & & & & V_{S-1} & & & & & & & J_{S-1} & L_{S-1}
\end{pmatrix},
$$

$$
F_n = \begin{pmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
0 & & 0 & \lambda\gamma
\end{pmatrix}_{(S+1)\times(S+1)} \qquad (n=1,2,....S-1) \;,
$$

$$
M_n = \begin{pmatrix}
0 & i\theta & & & & \\
 & 0 & i\theta & & & \\
 & & \ddots & \ddots & & \\
 & & & 0 & i\theta & \\
 & & & & 0 & i\theta \\
 & & & & & i\theta(1-\delta)
\end{pmatrix}_{(S+1)\times(S+1)} \qquad (n=0,1,.....S),
$$

$$
V_n = \begin{pmatrix}
\beta & & & \\
 & \beta & & \\
 & & \beta & \\
 & & & \\
 & & & \beta
\end{pmatrix}_{(S+1)\times(S+1)} \qquad (n=0,1,.....,S-1),
$$

$$L_0 = \begin{pmatrix} \omega & \lambda & & & & \\ & \omega & \lambda & & & \\ & & \ddots & & & \\ & & & & \omega & \lambda \\ & & & & & \Delta \end{pmatrix}_{(S+1)\times(S+1)} ,$$

$$L_n = \begin{pmatrix} \omega & \lambda & & & & \\ & \Theta & \lambda & & & \\ & & \Theta & \lambda & & \\ & & & \ddots & \ddots & \\ & & & & \Theta & \lambda \\ & & & & & \Omega \end{pmatrix}_{(S+1)\times(S+1)} \qquad (n = 1,2,.....S-1),$$

$$K_n = \begin{pmatrix} \sigma & \lambda & & & & \\ & \eta & \lambda & & & \\ & & \eta & \lambda & & \\ & & & \ddots & & \\ & & & & \eta & \lambda \\ & & & & & \nabla \end{pmatrix}_{(S+1)\times(S+1)} \qquad (n = s+1, s+2,.....S) ,$$

$$J_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mu & 0 & \cdots & 0 & 0 \\ 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu & 0 \end{pmatrix}_{(S+1)\times(S+1)} \qquad (n = 1, 2,.....S) ,$$

5.4.1 System Stability

Here mean drift y_s is given by

$$y_s = \begin{cases} -i\theta(1-\delta)+\lambda\gamma & \text{, if the buffer is full} \\ -i\theta & \text{, otherwise} \end{cases}$$

Since $(1-\delta) > 0$, for any $\varepsilon > 0$, we can find N' large enough that $y_s < -\varepsilon$ for any s belonging to the level $i \geq N'$. Hence by Tweedi's [64] result, the system under consideration is stable.

5.4.2 System Performance Measures

The $(i+1)$th component of $\mathbf{x} = (x_0, x_1, \ldots\ldots, x_{N-1}, x_N, \ldots\ldots)$ can be partitioned as

$$x_i = (y_{i,0,s+1,0}, y_{i,0,s+1,1}, \ldots, y_{i,0,s+1,S}, y_{i,0,s+2,0}, y_{i,0,s+2,1}, \ldots, y_{i,0,s+2,S}, \ldots, y_{i,0,S,0}, y_{i,0,S,1}, \ldots, y_{i,0,S,S},$$
$$y_{i,1,0,0}, y_{i,1,0,1}, \ldots\ldots, y_{i,1,0,S} y_{i,1,1,0}, y_{i,1,1,1}, \ldots\ldots, y_{i,1,1,S}, \ldots\ldots, y_{i,1,S-1,0}, y_{i,1,S-1,1}, \ldots\ldots, y_{i,1,S-1,S}).$$

Then,

(i) Expected Inventory level, EI, in the system is given by

$$EI = \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} \sum_{k=0}^{S} j y_{i,0,j,k} + \sum_{i=0}^{\infty} \sum_{j=0}^{S-1} \sum_{k=0}^{S} j y_{i,1,j,k}$$

(ii) Expected number, EC, in the orbit is given by

$$EC = \left(\sum_{i=1}^{\infty} i x_i\right)\mathbf{e}$$

$$= \left((\sum_{i=1}^{N-1} i x_i) + x_N \left(N(I-R)^{-1} + R(I-R)^{-2}\right)\right)\mathbf{e}$$

(iii) Expected number of customers, EB, in the buffer is given by

$$EB = \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} \sum_{k=0}^{S} k y_{i,0,j,k} + \sum_{i=0}^{\infty} \sum_{j=0}^{S-1} \sum_{k=0}^{S} k y_{i,1,j,k}$$

(iv) Expected switching rate, ESR, is given by

$$ESR = \mu \sum_{i=0}^{\infty} \sum_{k=1}^{S} y_{i,0,s+1,k}$$

(v) Expected number of departures, EDS, after completing service is

$$EDS = \mu\left(\sum_{i=0}^{\infty}\sum_{j=s+1}^{S}\sum_{k=1}^{S} y_{i,0,j,k} + \sum_{i=0}^{\infty}\sum_{j=1}^{S-1}\sum_{k=1}^{S} y_{i,1,j,k}\right)$$

(vi) Expected number of customers lost, EL_1, before entering the orbit per unit time is given by

$$EL_1 = (1-\gamma)\lambda\left(\sum_{i=0}^{\infty}\sum_{j=s+1}^{S} y_{i,0,j,S} + \sum_{i=0}^{\infty}\sum_{j=0}^{S-1} y_{i,0,j,S}\right)$$

(vii) Expected number of customers lost, EL_2, after retrials per unit time is

$$EL_2 = \theta(1-\delta)\left[\sum_{i=1}^{\infty} i\left(\sum_{j=s+1}^{S} y_{i,0,j,S} + \sum_{j=0}^{S-1} y_{i,1,j,S}\right)\right]$$

(viii) Overall rate of retrials, ORR, is given by

$$ORR = \theta\left(\sum_{i=1}^{\infty} i x_i\right)\mathbf{e}$$

(ix) Successful rate of retrials, SRR , is given by

$$SRR = \theta\left[\sum_{i=1}^{\infty} i\left(\sum_{j=s+1}^{S}\sum_{k=0}^{S-1} y_{i,0,j,k} + \sum_{j=0}^{S-1}\sum_{k=0}^{S-1} y_{i,1,j,k}\right)\right]$$

5.5 Cost analysis and Numerical Examples

Here we introduce the following costs

C = fixed cost

c_1 = procurement cost/unit

c_2 = holding cost of inventory /unit /unit time

c_3 = holding cost of customers / unit /unit time

c_4 = cost due to loss of customers / unit /unit time

c_5 = cost due to service / unit /unit time

c_6 = revenue from service/unit/unit time.

In terms of these costs we define the expected total cost function as

$$ETC = (C+(S-s)c_1)ESR + c_2 EI + c_3(EC+EB) + c_4(EL_1+EL_2) + (c_5-c_6)EDS$$

In the following tables we provide a comparison among the overall and successful rate of retrials for models I to III.

$\mu=2.0;\beta=1.0;\theta=0.8;\gamma=0.2;\delta=0.3;N=34;$
$s=1;S=3;c_1=1;c_2=10.11;c_3=1;c_4=1;c_5=2;c_6=8.2.$

λ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
2.1	0.323675	0.098596	0.352202	0.131951	0.156996	0.075470
2.2	0.344573	0.102470	0.378501	0.139123	0.165689	0.078928
2.3	0.365644	0.106222	0.405016	0.146133	0.174396	0.082355
2.4	0.386883	0.109858	0.431720	0.152987	0.183115	0.085752
2.5	0.408287	0.113386	0.458589	0.159689	0.191844	0.089122
2.6	0.429854	0.116812	0.485604	0.166245	0.200586	0.092466
2.7	0.451582	0.120142	0.512749	0.172661	0.209342	0.095787
2.8	0.473470	0.123381	0.540008	0.178942	0.218114	0.099086
2.9	0.495516	0.126535	0.567370	0.185096	0.226905	0.102367
3.0	0.517720	0.129609	0.594824	0.191128	0.235720	0.105631

Table 1(**Variations in arrival rate λ**)

$\lambda=2.0;\ \mu=2.0;\ \theta=0.8;\ \gamma=0.2;\ \delta=0.3;\ N=26;$
$s=1;S=3;\ c_1=1;c_2=4;c_3=1;c_4=6.9;c_5=1.5;c_6=1.$

β	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
1.1	0.301123	0.102397	0.311810	0.148026	0.173137	0.080878
1.2	0.299326	0.101979	0.299024	0.147131	0.171667	0.080406
1.3	0.297566	0.101495	0.287624	0.146056	0.170032	0.079872
1.4	0.295846	0.100938	0.277451	0.144765	0.168214	0.079267
1.5	0.294167	0.100303	0.268361	0.143215	0.166195	0.078581
1.6	0.292534	0.099583	0.260224	0.141353	0.163951	0.077804
1.7	0.290948	0.098775	0.252925	0.139115	0.161457	0.076922
1.8	0.289413	0.097873	0.246362	0.136423	0.158684	0.075919
1.9	0.287933	0.096876	0.240447	0.133186	0.155597	0.074776
2.0	0.286508	0.095781	0.235101	0.129293	0.152158	0.073472

Table 2 (**Variations in replenishment rate β**)

λ=2.0; β=1.0; θ=0.8;.γ=0.2; δ=0.3; N=23;
s=1;S=3; c_1=0.1;c_2=0.1;c_3=0.1;c_4=0.1;c_5=11;c_6=1.

μ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
2.1	0.299381	0.101859	0.324034	0.122078	0.145082	0.073800
2.2	0.296001	0.101386	0.322144	0.119890	0.142226	0.073517
2.3	0.292798	0.100869	0.320449	0.117996	0.139700	0.073251
2.4	0.289756	0.100305	0.318923	0.116349	0.137459	0.073002
2.5	0.286860	0.099690	0.317545	0.114910	0.135468	0.072775
2.6	0.284100	0.099018	0.316297	0.113649	0.133695	0.072570
2.7	0.281464	0.098284	0.315162	0.112539	0.132114	0.072391
2.8	0.278944	0.097481	0.314128	0.111559	0.130704	0.072240
2.9	0.276530	0.096604	0.313182	0.110691	0.129443	0.072119
3.0	0.274216	0.095643	0.312315	0.109919	0.128317	0.072032

Table 3(**Variations in service rate μ)**

λ=2.0;β=1.0;μ=2.0; θ=0.8; δ=0.3; N=42;s=1;
S=3; c_1=0.1;c_2=30.3;c_3=0.1;c_4=0.1;c_5=2.1;c_6=1

γ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
0.22	0.336571	0.104408	0.361051	0.137143	0.167166	0.080645
0.24	0.370784	0.114273	0.396361	0.149678	0.186653	0.089509
0.26	0.405588	0.124182	0.432078	0.162216	0.206760	0.098558
0.28	0.440977	0.134128	0.468199	0.174757	0.227467	0.107778
0.30	0.476944	0.144109	0.504721	0.187299	0.248752	0.117154
0.32	0.513482	0.154117	0.541640	0.199841	0.270597	0.126675
0.34	0.550581	0.164148	0.578955	0.212380	0.292983	0.136328
0.36	0.588233	0.174198	0.616661	0.224916	0.315890	0.146101
0.38	0.626428	0.184260	0.654756	0.237447	0.339300	0.155983
0.40	0.665154	0.194329	0.693235	0.249971	0.363196	0.165964

Table 4 (**Variations in probability γ of primary arrivals joining the orbit)**

λ=2.0; β=1.0; μ=2.0; θ=0.8; γ =0.2; N=30 s=1;
S=3; c_1=1;c_2=9.1;c_3=1;c_4=1;c_5=2;c_6=1

δ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
0.24	0.283859	0.089081	0.307631	0.118082	0.141981	0.068549
0.28	0.296300	0.092677	0.319726	0.122355	0.146140	0.070800
0.32	0.309927	0.096591	0.332843	0.126964	0.150571	0.073200
0.36	0.324920	0.100868	0.347124	0.131952	0.155304	0.075766
0.40	0.341500	0.105561	0.362737	0.137370	0.160372	0.078516
0.44	0.359939	0.110737	0.379886	0.143279	0.165812	0.081470
0.48	0.380574	0.116476	0.398821	0.149754	0.171670	0.084653
0.52	0.403832	0.122876	0.419852	0.156885	0.177997	0.088093
0.56	0.430263	0.130065	0.443368	0.164785	0.184856	0.091824
0.60	0.460583	0.138200	0.469867	0.173596	0.192320	0.095886

Table 5(**Variations in return probability δ of retrial customers**)

λ=2.0; β=1.0; μ=2.0; γ =0.2; δ=0.3; N=34; s=1;
S=3; c_1=1; c2=100;c_3=0.1;c_4=10;c_5=2;c_6=1.

θ	Model I		Model II		Model III	
	ORR	SRR	ORR	SRR	ORR	SRR
0.55	0.299169	0.089674	0.324700	0.095944	0.147219	0.068509
0.60	0.300013	0.090836	0.325008	0.101924	0.147444	0.069338
0.65	0.300921	0.092040	0.325307	0.108606	0.147666	0.070192
0.70	0.301898	0.093291	0.325596	0.116117	0.147887	0.071072
0.75	0.302955	0.094592	0.325877	0.124615	0.148104	0.071980
0.80	0.304102	0.095948	0.326148	0.134301	0.148320	0.072917
0.85	0.305354	0.097365	0.326412	0.145432	0.148533	0.073885
0.90	0.306727	0.098851	0.326668	0.158348	0.148743	0.074886
0.95	0.308244	0.100414	0.326916	0.173500	0.148951	0.075922
1.00	0.309932	0.102065	0.327156	0.191504	0.149157	0.076994

Table 6(**Variations in retrial rate θ**)

Interpretations of the Numerical Results in the Tables

Table 1 shows that as the arrival rate λ increases, the number of customers in the orbit becomes larger so that the overall and successful rates of retrials from the orbit increase. As the production rate β or service rate μ increases the arriving customers will get the inventory more rapidly thereby the number of customers in the orbit gets decreased. In that case, the overall and successful rates of retrials will decrease (see tables 2 and 3). With the increase in probability γ of primary arrivals joining the orbit or increase in return probability δ of retrial customers, the orbit size increases. Here also, overall and successful rates of retrials increase (see tables 4 and 5). Table 6 indicates that as the retrial rate θ of customers in the orbit increases, the overall and successful rates of retrials from the orbit will increase.

Next we provide graphical illustrations of the Performance measures of the above described models.

μ=2.0; β=1.0; θ=0.8; γ=0.2; δ=0.3; N=34;
s=1; S=3; c_1=1;c_2=10.11;c_3=1;c_4=1;c_5=2;c_6=8.2.

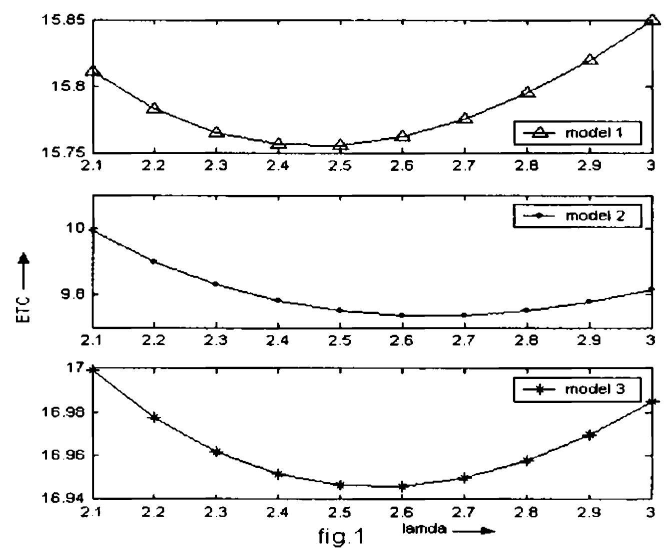

fig.1

(Arrival rate vs. ETC)

$\lambda=2.0$; $\mu=2.0$; $\theta=0.8$; $\gamma=0.2$; $\delta=0.3$; $N=25$; $s=1$; $S=3$; $c_1=1$;$c_2=4$;$c_3=1$;$c_4=6.9$;$c_5=1.5$;$c_6=1$.

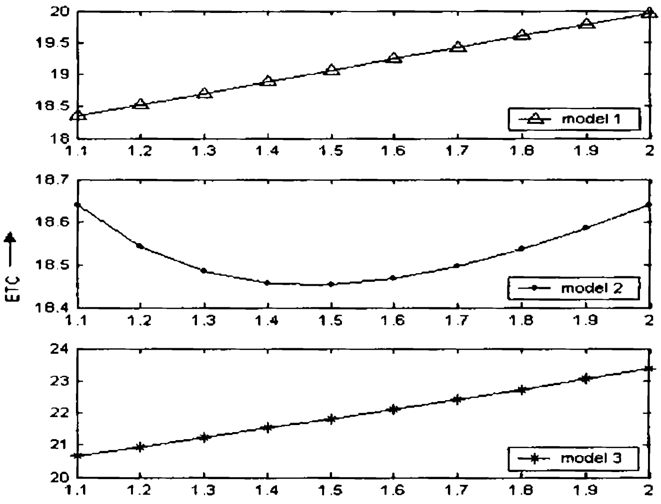

fig. 2

(Replenishment rate vs. ETC)

$\lambda=2.0$; $\beta=1.0$; $\theta=0.8$; $\gamma=0.2$; $\delta=0.3$; $N=25$; $s=1$; $S=3$; $c_1=0.1$; $c_2=0.1$; $c_3=0.1$;$c_4=0.1$;$c_5=11$;$c_6=1$

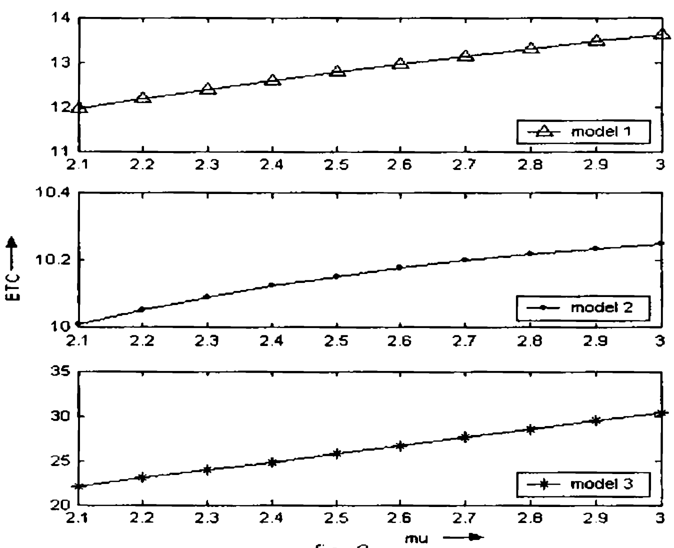

fig 3

(Service rate vs. ETC)

λ=2.0; β=1.0; μ=2.0; θ=0.8; δ=0.3; N=42;
s=1; S=3; c_1=0.1; c_2=30.3; c_3=0.1; c_4=0.1;c_5=2.1;c_6=1.

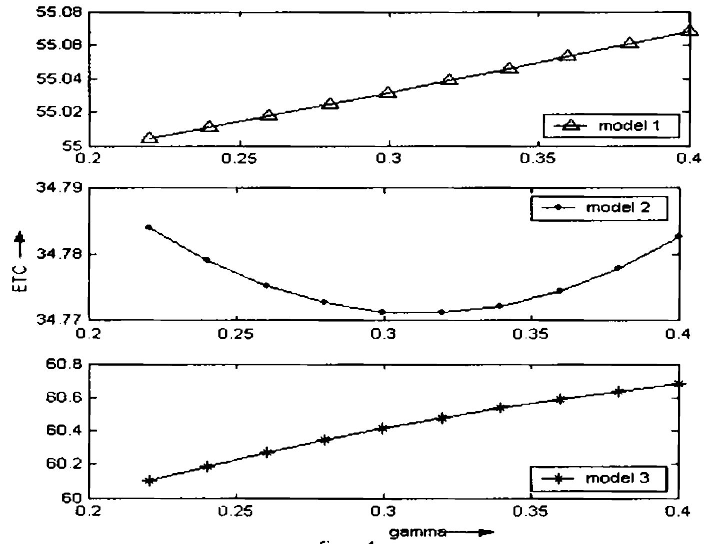

fig. 4

(Gamma vs. ETC)

λ=2.0; β=1.0; μ=2.0; θ=0.8; γ =0.2; N=32;
s=1; S=3; c_1=1;c_2=9.1;c_3=1;c_4=1;c_5=2;c_6=1.

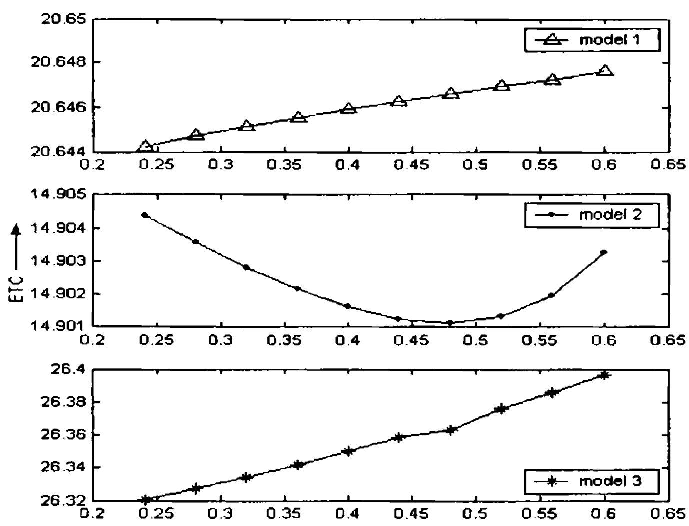

fig. 5

(Delta vs. ETC)

λ=2.0; β=1.0; μ=2.0; γ =0.2; δ=0.3; N=34;
s=1; S=3; c_1=1; c2=100; c_3=0.1;c_4=10;c_5=2;c_6=1.

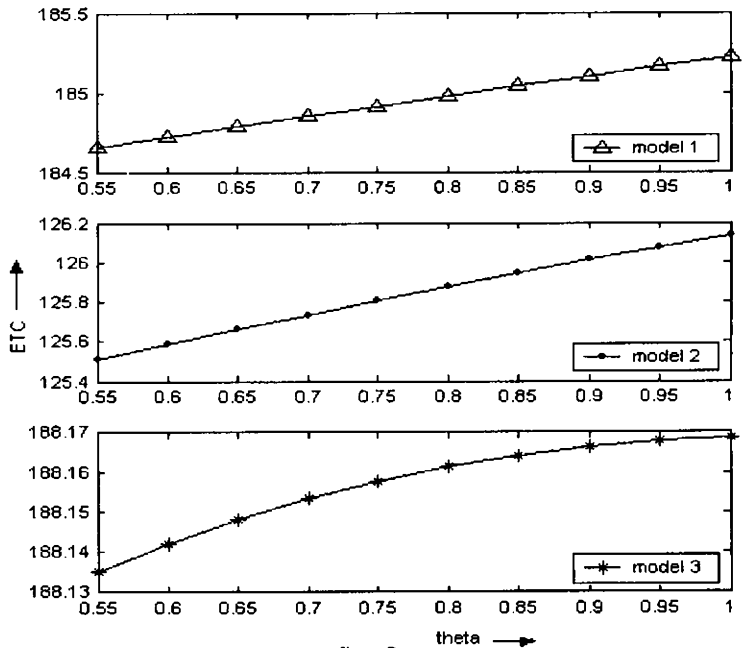

fig. 6

(Theta vs. ETC)

Interpretations of the Graphs

In order to find the best profitable model, we compute the expected total cost per unit time for each model by varying the parameters one at a time keeping others fixed. By fixing all the parameters except the arrival rate λ, it is clear from the fig.1 that the cost function is convex in λ for all the models I, II and II; for given parameter values this function attains the following minimum values; (a)15.7551 at $\lambda = 2.5$ for model I (b) 9.7366 at $\lambda = 2.6$ for model II and (c) 16.9459 at $\lambda = 2.6$ for model III. Therefore model II is the best for minimum cost in the range of λ between 2.1 and 3.0 (see fig.1). One can observe the minimum value of the objective function by changing other parameters $\beta, \mu, \gamma, \delta$ and θ

(see fig.2, fig.3, fig.4, fig.5 and fig.6). In all examples considered here, the cost function has the least minimum value for model II. Therefore model II (model with buffer size equal to the inventoried items) can be considered as the best model for practical applications in the ranges specified for the parameter values.

Chapter 6

PH/PH/1 Inventory System with Service Time, Reneging of Customers and Shortage

6.1 Introduction

Phase type (*PH*) distribution, which was introduced by Neuts [52], generalizes the conventional approach employing only exponential distributions and other distributions such as Erlang, Generalized Erlang, and Coxian. In contrast to exponential distribution (which is closed under minimum only), the class of phase-type distributions has very strong closure properties: they are closed under maximum, minimum and convolution (see Neuts [53]).Erlang (i.e. convolutions of identical exponential distributions), hyper-exponential, and Coxian distributions are examples of phase type distributions. Some works related to *PH* distributions include Alfa [1], Chakravarthy ([17], [18]).

This chapter analyses a $PH/PH/1$ inventory model with reneging of customers and finite shortage of items. We assume that arrivals occur according to a phase type renewal process. The inter-arrival distribution is of phase type with representation $(\boldsymbol{\alpha}, U)$. The service times have common phase type distribution with representation $(\boldsymbol{\beta}, V)$. The lead-time is zero. The inter-reneging times of customers from the system have exponential distribution with constant rate γ. Shortage is permitted and shortage cost is finite. We perform the steady state analysis of the inventory model using Matrix Analytic Method. Some measures of the system performance in the steady state are derived. A suitable cost function is defined and analyzed numerically.

This chapter is organized as follows. Section 6.2 deals mathematical model description. Section 6.3 presents stability condition. Section 6.4 describes algorithmic analysis. Section 6.5 gives performance measures of the system. Finally Cost analysis and numerical examples are included in section 6.6.

6.2 Mathematical Model Description

The following Assumptions and Notations are used for the analysis of the model.

Assumptions

(i) Maximum inventory level is S.

(ii) Inter-arrival distribution is phase type with representation $(\boldsymbol{\alpha}, U)$.

(iii) Lead-time is zero.

(iv) Service times have phase type distribution with representation $(\boldsymbol{\beta}, V)$.

(v) The reneging rate is constant with value γ, when there are $i\ (\geq 2)$ customers in the system.

(vi) A maximum of $K(>0)$ shortages is allowed in the system.

Notations

$N(t)$: Number of customers in the system at time t.

$I(t)$: Inventory level at the time t.

$J_1(t)$: Phase of the arrival process at time t.

$J_2(t)$: Phase of the service process at time t.

$\mathbf{e}$: $(1,1,1,.....1)'$, column vector of 1's of appropriate order.

Let $I(t)$, $N(t)$, $J_1(t)$ and $J_2(t)$ be respectively the inventory level , number of customers in the system, phase of the arrival process and phase of the service process at time t . Let $X(t) = \{(N(t), I(t), J_1(t), J_2(t)); t \geq 0\}$. Now $\{X(t), t \geq 0\}$ is a Level Independent Quasi-Birth Process (LIQBD) on the state space $\{(0, j, k, 0); 0 \leq j \leq S-1, 1 \leq k \leq m_1\} \cup \{(i, 0, k, 0); i \geq 1, 1 \leq k \leq m_1\} \cup \{(i, j, k, l); i \geq 1, 1 \leq j \leq S, 1 \leq k \leq m_1, 1 \leq l \leq m_1\}$. Here the value 0 in the last coordinate indicates that no service is going due to either the absence of customers or zero inventory level or both. The infinitesimal generator Q of the process is a block tri-diagonal matrix given by

$$Q = \begin{pmatrix} B_0 & A_{0,0} & & & & & & & \\ A_{2,1} & A_{1,1} & A_{0,1} & & & & & & \\ & A_{2,2} & A_{1,2} & A_{0,2} & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & A_{2,K-1} & A_{1,K-1} & A_{0,K-1} & & & \\ & & & & A_{2,K} & A_1 & A_0 & & \\ & & & & & A_2 & A_1 & A_0 & \\ & & & & & & A_2 & A_1 & A_0 \\ & & & & & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{6.1}$$

where

$$(B_0)_{m_1 S \times m_1 S} = I_S \otimes U \ , \quad (A_{0,0})_{m_1 S \times (m_1 m_2 S + m_1)} = \begin{pmatrix} \mathbf{U}^0 \alpha & 0 \\ 0 & I_{S-1} \otimes (\mathbf{U}^0 \alpha \otimes \beta) \end{pmatrix},$$

$$\left(A_{2,1}\right)_{(m_1 m_2 S+m_1)\times m_1 S} = \begin{pmatrix} \gamma I_{m_1} & 0 \\ I_{m_1} \otimes \mathbf{V}^0 & 0 \\ 0 & I_{S-1} \otimes \left(I_{m_1} \otimes \mathbf{V}^0\right) \end{pmatrix},$$

$$A_{1,1} = \begin{pmatrix} U-\gamma I_{m_1} & 0 \\ 0 & I_S \otimes (U \oplus V) \end{pmatrix}, \quad A_{0,1} = \begin{pmatrix} \mathbf{U}^0 \alpha & 0 \\ 0 & I_S \otimes (U \otimes I_{m_2}) \end{pmatrix},$$

$$A_{2,2} = \begin{pmatrix} \gamma I_{m_1} & & & & \\ I_{m_1} \otimes \mathbf{V}^0 & \gamma I_{m_1 m_2} & & & \\ & I_{m_1} \otimes \mathbf{V}^0 \beta & \gamma I_{m_1 m_2} & & \\ & & \ddots & \ddots & \\ & & & I_{m_1} \otimes \mathbf{V}^0 \beta & \gamma I_{m_1 m_2} \end{pmatrix},$$

$$A_{1,2} = \begin{pmatrix} (U-\gamma I_{m_1}) & 0 \\ 0 & I_S \otimes (U \oplus V - \gamma I_{m_1 m_2}) \end{pmatrix},$$

Here the above notations $\otimes$, $\oplus$ stand for Kronecker product and sum, respectively. For more details about the Kronecker operations on matrices, we refer the reader to Bellman [12]. Note that

$$A_{0,2} = A_{0,3} = \dots\dots\dots = A_{0,K-3} = A_{0,K-2} = A_{0,1},$$

$$A_{1,3} = A_{1,4} = \dots\dots\dots = A_{1,K-2} = A_{1,K-1} = A_{1,2},$$

$$A_{2,3} = A_{2,4} = \dots\dots\dots = A_{2,K-2} = A_{2,K-1} = A_{2,2}, \text{ where}$$

$A_{0,i} (1 \le i \le K-2)$, $A_{1,i} (2 \le i \le K-1)$ and $A_{2,i} (2 \le i \le K-1)$ are square matrices of the same order $(m_1 m_2 S + m_1)$. Also we have

$$\left(A_{0,K-1}\right)_{(m_1m_2S+m_1)\times m_1S}=\begin{pmatrix} 0 & (\mathbf{U}^0\alpha)\otimes\beta \\ I_{S-1}\otimes\left((\mathbf{U}^0\alpha)\otimes I_{m_2}\right) & 0 \\ 0 & (\mathbf{U}^0\alpha)\otimes I_{m_2} \end{pmatrix},$$

$$\left(A_{2,K}\right)_{(m_1m_2S)\times(m_1m_2S+m_1)}=\begin{pmatrix} I_{m_1}\otimes\mathbf{V}^0 & \gamma I_{m_1m_2} & & & & \\ & I_{m_1}\otimes\mathbf{V}^0\beta & \gamma I_{m_1m_2} & & & \\ & & I_{m_1}\otimes\mathbf{V}^0\beta & \gamma I_{m_1m_2} & & \\ & & & \ddots & \ddots & \\ & & & & I_{m_1}\otimes\mathbf{V}^0\beta & \gamma I_{m_1m_2} \end{pmatrix},$$

$$A_0=I_S\otimes\left((\mathbf{U}^0\alpha)\otimes I_{m_2}\right),\quad A_1=I_S\otimes\left((U\oplus V)-\gamma I_{m_1m_2}\right),$$

$$A_2=\begin{pmatrix} \gamma I_{m_1m_2} & & & I_{m_1}\otimes\mathbf{V}^0\beta \\ I_{m_1}\otimes\mathbf{V}^0\beta & \gamma I_{m_1m_2} & & \\ & \ddots & \ddots & \\ & & I_{m_1}\otimes\mathbf{V}^0\beta & \gamma I_{m_1m_2} \end{pmatrix}, \text{ where}$$

A_0, A_1 and A_2 are square matrices of the same order m_1m_2S.

6.3 Stability Condition

Define the generator A as $A = A_0 + A_1 + A_2$ .Then

$$A=\begin{pmatrix} U\oplus V+(\mathbf{U}^0\alpha)\otimes Im_2 & & & I_{m_1}\otimes\mathbf{V}^0\beta \\ I_{m_1}\otimes\mathbf{V}^0\beta & U\oplus V+(\mathbf{U}^0\alpha)\otimes Im_2 & & \\ & I_{m_1}\otimes\mathbf{V}^0\beta & & \\ & & \ddots & \\ & & I_{m_1}\otimes\mathbf{V}^0\beta & U\oplus V+(\mathbf{U}^0\alpha)\otimes Im_2 \end{pmatrix}$$

Now we can represent A as $A = A_U + A_V$, where

$$A_U = \begin{pmatrix} (U+\mathbf{U}^0\alpha)\otimes Im_2 & & \\ & (U+\mathbf{U}^0\alpha)\otimes Im_2 & \\ & & \ddots \\ & & & (U+\mathbf{U}^0\alpha)\otimes Im_2 \end{pmatrix} \quad \text{and}$$

$$A_V = \begin{pmatrix} I_{m_1}\otimes V & & & & I_{m_1}\otimes \mathbf{V}^0\beta \\ I_{m_1}\otimes \mathbf{V}^0\beta & I_{m_1}\otimes V & & & \\ & I_{m_1}\otimes \mathbf{V}^0\beta & I_{m_1}\otimes V & & \\ & & \ddots & \ddots & \\ & & & I_{m_1}\otimes \mathbf{V}^0\beta & I_{m_1}\otimes V \end{pmatrix}$$

Theorem 6.3.1: The stability condition of the system under steady state is

$$\rho < 1 \tag{6.2}$$

where $\rho = \left(\tilde{\boldsymbol{\pi}}\mathbf{U}^0 / (\gamma + \tilde{\tilde{\boldsymbol{\pi}}}\mathbf{V}^0)\right)$ with $\tilde{\boldsymbol{\pi}}$ satisfying $\tilde{\boldsymbol{\pi}}(U+\mathbf{U}^0\boldsymbol{\alpha}) = 0$, $\tilde{\boldsymbol{\pi}}\mathbf{e} = 1$; and $\tilde{\tilde{\boldsymbol{\pi}}}$ satisfying $\tilde{\tilde{\boldsymbol{\pi}}}(V+\mathbf{V}^0\boldsymbol{\beta}) = 0$, $\tilde{\tilde{\boldsymbol{\pi}}}\mathbf{e} = 1$.

Proof: From the well-known result due to Neuts [61] we have Q is positive recurrent iff

$$\boldsymbol{\pi}A_0\mathbf{e} < \boldsymbol{\pi}A_2\mathbf{e} \tag{6.3}$$

where $\boldsymbol{\pi}$ is the steady state probability vector of A. That is

$$\boldsymbol{\pi}A = 0 \tag{6.4}$$

and

$$\boldsymbol{\pi}\mathbf{e} = 1. \tag{6.5}$$

Now $\boldsymbol{\pi} = \frac{1}{S}\left(\mathbf{e}'_S \otimes (\tilde{\boldsymbol{\pi}} \otimes \tilde{\tilde{\boldsymbol{\pi}}})\right)$ satisfies (6.4) and (6.5) where $\tilde{\boldsymbol{\pi}}$ is such that $\tilde{\boldsymbol{\pi}}(U + \mathbf{U}^0\boldsymbol{\alpha}) = 0$, $\tilde{\boldsymbol{\pi}}\mathbf{e} = 1$ and $\tilde{\tilde{\boldsymbol{\pi}}}$ is such that $\tilde{\tilde{\boldsymbol{\pi}}}(V + \mathbf{V}^0\boldsymbol{\beta}) = 0$, $\tilde{\tilde{\boldsymbol{\pi}}}\mathbf{e} = 1$.Substituting $\boldsymbol{\pi}$ in (6.3) we get (6.2). This completes the proof.

6.3.1 Steady State Probability Vector

Let $\mathbf{x} = (x_0, x_1, \ldots\ldots. x_K, x_{K+1}, \ldots\ldots..)$ be the steady state probability vector of Q. Under the stability condition (2), $x_i s'$ $(i \geq N)$ are given by

$$x_{K+r} = x_K R^r \, (r \geq 1)$$

where R is the unique non- negative solution of the equation

$$R^2 A_2 + RA_1 + A_0 = 0$$

in which the spectral radius is less than one and the vectors $x_0, x_1 \ldots\ldots, x_K$ are given by solving the following equations

$$\left.\begin{aligned} & x_0 B_0 + x_1 A_{2,1} = 0 \\ & x_{i-1} A_{0,i-1} + x_i A_{1,i} + x_{i+1} A_{2,i+1} = 0, (1 \leq i \leq K-1) \\ & x_{K-1}(A_{0,K-1} + RA_1 + R^2 A_2) = 0 \end{aligned}\right\} \tag{6.6}$$

subject to the normalizing condition

$$\left(\sum_{i=1}^{K-1} x_i + x_K (I - R)^{-1}\right)\mathbf{e} = 1 . \tag{6.7}$$

6.4 Algorithmic Analysis

6.4.1 Evaluation of the Rate Matrix R

To find the rate matrix R we use the relation

$$R = A_0(-A_1 - A_0 G)^{-1},$$

where the matrix G is the minimal nonnegative solution of the matrix quadratic equation $A_2 + A_1G + A_0G^2 = 0$. The matrix G will be stochastic if $sp(R) < 1$. The logarithmic Reduction Algorithm due to Ramaswami (see Latouche and Ramaswami [48]) can be used to evaluate R.

6.4.2 Computation of the Boundary Probabilities

Let $\mathbf{x}^*$ be the partitioned vector $(x_0, x_1 \ldots\ldots, x_K)$ corresponding to the boundary portion of Q as in (6.1) Then $\mathbf{x}^*$ is the stationary vector normalized by (6.7) of the infinitesimal generator T shown below

$$T = \begin{pmatrix} B_0 & A_{0,0} & & & & & \\ A_{2,1} & A_{1,1} & A_{0,1} & & & & \\ & A_{2,2} & A_{1,2} & A_{0,2} & & & \\ & & \ddots & \ddots & & & \\ & & & A_{2,K-1} & A_{1,K-1} & A_{0,K-1} \\ & & & & A_{2,K} & A_1 + RA_2 \end{pmatrix}.$$

Now the system (6.6) can be written as $\mathbf{x}^*T = 0$. To solve this system, we use the block Gauss-Seidel iterative scheme. The vectors $x_0, x_1, \ldots\ldots\ldots x_K$ in the $(n+1)$th iteration are given by

$$\begin{aligned} x_0(n+1) &= x_1(n)A_{2,1}B_0^{-1} \\ x_i(n+1) &= [x_{i+1}(n)A_{2,i+1} + x_{i-1}(n+1)A_{0,i-1}]A_{1,i}^{-1}, \quad (1 \le i \le (K-1)) \\ x_K(n+1) &= -x_{K-1}(n+1)A_{0,K-1}(A_1 + RA_2)^{-1}. \end{aligned}$$

After each iteration, the elements of $\mathbf{x}^*$ may be scaled to satisfy (6.7).

6.5 System Performance Measures

The components of the steady state probability vector $\mathbf{x}=(x_0,x_1,\ldots\ldots,x_{K-1},x_K,\ldots\ldots)$ can be partitioned as

$$x_0=\left(y_{0,j,k,0}\right),\quad 0\le j\le(S-1) \text{ and } 1\le k\le m_1;$$

for $1\le i\le K-1$, $x_i=\left(y_{i,j,k,l}\right), 0\le j\le S, 1\le k\le m_1, 1\le l\le m_2$, with $l=0$ when $j=0$;

for $i\ge K$, $\quad x_i=\left(y_{i,j,k,l}\right)$, $1\le j\le S$, $1\le k\le m_1$ and $1\le l\le m_2$. Then we have

(i) Expected re-order rate, ERO, is given by

$$ERO=\sum_{k=1}^{m_1}\left((y_{K-1,0,k,0})\left(\mathbf{U}^0(k)\right)\right)$$

(ii) Expected inventory level, EI, is given by

$$EI=\sum_{j=0}^{S-1}j\left(\sum_{k=1}^{m_1}y_{0,j,k,0}\right)+\sum_{j=1}^{S}j\left(\sum_{i=1}^{\infty}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}y_{i,j,k,l}\right)$$

(iii) Expected number of departures after receiving service/unit time, EDS, is given by

$$EDS=\sum_{i=1}^{\infty}\sum_{j=1}^{S}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\left((y_{i,j,k,l})(\mathbf{V}^0(l))\right)$$

(iv) Expected number of departures due to reneging of customers/unit time, EDR, is given by

$$EDR=\gamma\left(\left(\sum_{i=2}^{\infty}\sum_{j=1}^{S}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}(y_{i,j,k,l})\right)+\left(\sum_{i=1}^{K-1}\sum_{k=1}^{m_1}(y_{i,0,k,0})\right)\right)$$

(v) Expected number of customers, EC, in the system is given by

$$EC=\left(\sum_{i=1}^{\infty}ix_i\right)\mathbf{e}$$

$$=\left((\sum_{i=1}^{K-1}ix_i)+x_K\left(K(I-R)^{-1}+R(I-R)^{-2}\right)\right)\mathbf{e}$$

(vi) Expected shortages, ES, in the system is given by

$$ES=\left(\sum_{i=1}^{K-1}ix_i\right)\mathbf{e}$$

6.6 Cost Analysis and Numerical Examples

Cost Analysis

In order to construct a cost function explicitly, we define the following costs

C = fixed cost

c_1 = procurement cost/unit

c_2 = holding cost of inventory /unit /unit time

c_3 = service cost/unit/unit time

c_4 = loss due to reneging of customers/ unit /unit time

c_5 = holding cost of customers / unit /unit time

c_6 = shortage cost/unit/unit time

c_7 = revenue (profit) due to service / unit /unit time

The expected total cost (ETC) of the system/unit time is given by

$$ETC = (C + Sc_1)ERO + c_2EI + (c_3 - c_7)EDS + c_4EDR + c_5EC + c_6ES.$$

Numerical Examples

We plot graphs representing the expected total cost based on the following values and matrices.

$$m_1 = 2,\ m_2 = 2,\ \boldsymbol{\alpha} = (0.5, 0.5),\ \boldsymbol{\beta} = (0.5, 0.5),$$

$$\mathbf{U}^0 = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix},\ \mathbf{V}^0 = \begin{pmatrix} 4.5 \\ 3.5 \end{pmatrix},\ U = \begin{pmatrix} -2.0 & 0.5 \\ 0.5 & -2.0 \end{pmatrix},\ V = \begin{pmatrix} -5.0 & 0.5 \\ 0.5 & -4.0 \end{pmatrix}.$$

(i) Variation in number of shortages (K)

$$C=100; S=20;\ \gamma=1;\ c_1=50;\ c_2=20;$$
$$c_3=10;\ c_4=30;\ c_5=10;\ c_6=10;\ c_7=20.$$

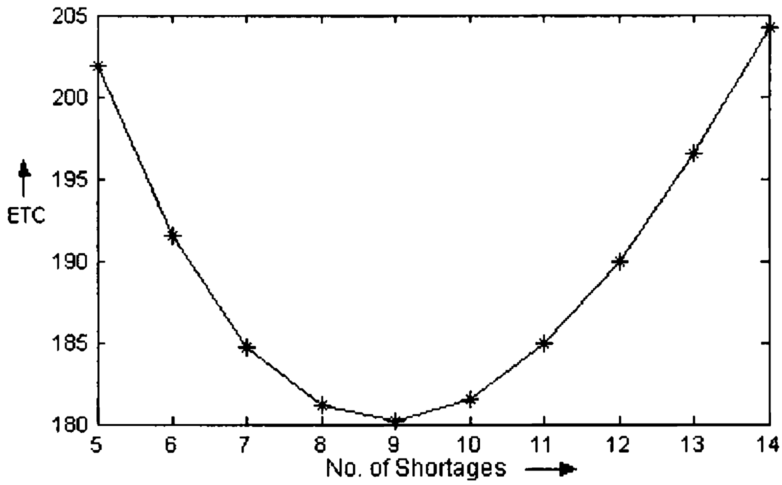

Fig.1 **Number of shortages vs. Expected total cost**

(ii) Variation in Maximum inventory (S)

$$C=100;\ K=10; \gamma=1;\ c_1=50;\ c_2=20;$$
$$c_3=10;\ c_4=30;\ c_5=10;\ c_6=10;\ c_7=200.$$

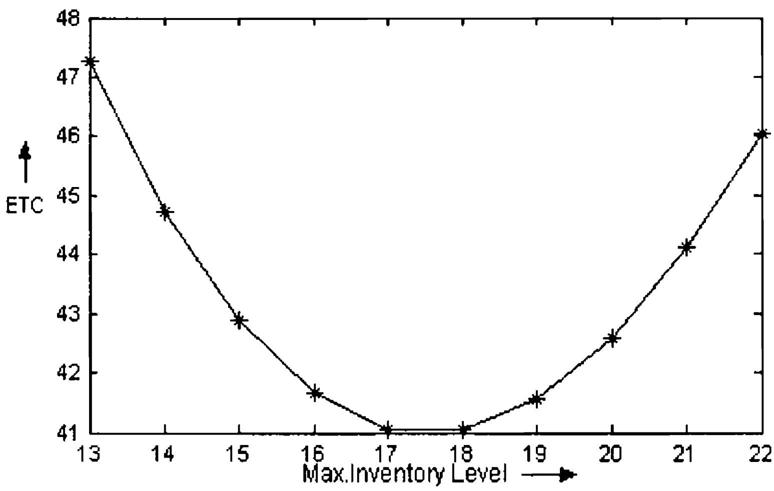

Fig.2 **Maximum inventory level vs. Expected total cost**

(iii) Variation in the reneging rate (γ)

$$C = 100;\ K = 10;\ S = 10;\ c_1 = 50;\ c_2 = 20;$$
$$c_3 = 10;\ c_4 = 100;\ c_5 = 10;\ c_6 = 10;\ c_7 = 200.$$

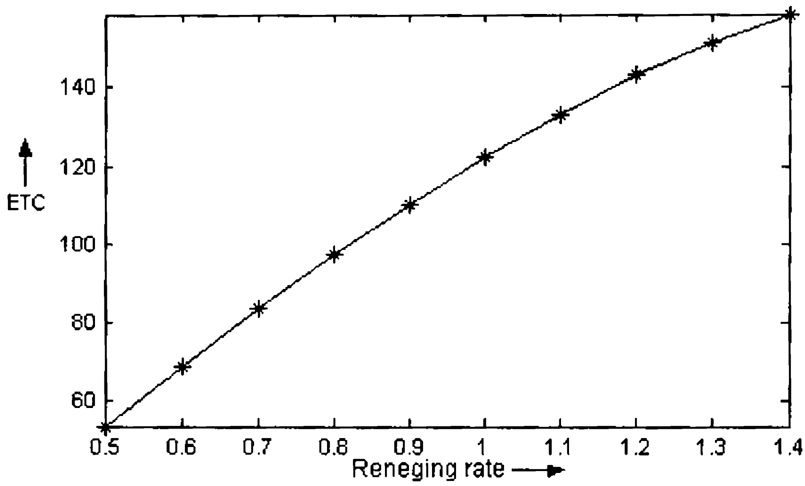

Fig.3 **Reneging rate vs. Expected total cost**

Interpretation of the graphs

Our aim is to minimize the total expected cost per unit time by varying the parameters one at a time and keeping other expressions fixed. By fixing the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{U}^0, \mathbf{V}^0$ and the matrices U, V and all parameters except the 'number of shortages K', it is clear from the fig.1 that the cost function is convex in K and attains its optimum (minimum) value 180.18780 at $K = 9$. As the maximum inventory level S increases (keeping other expressions fixed), the cost function is again convex. For given parameter values this function attains its optimum value 41.03711 at $S = 18$ (see fig.2). As the reneging rate γ increases (keeping other expressions fixed), the expected total cost increases monotonically. (see Fig.3).

References

[1] Alfa, A.S. (2002), *Discrete Time Queues and Matrix-analytic Methods*, TOP,10, 2, 147-210.

[2] Arivarignan, G., Elango, C.and Arumugam, N. (2002), *A continuous Review Perishable Inventory Control Systems at Service Facilities*, Advances in Stochastic Modeling, Eds. Artalejo, J.R. and Krishnamoorthy, A., Notable Publications, Inc New Jersey, 29-40.

[3] Arrow, K.J., Harri, T. and Marschak, J. (1951), *Optimal Inventory Policy*, Econometrica, 19, 3, 250-272.

[4] Arrow, K.J., Karlin, S. and Scraf, H. (1958), Studies in Applied Probability and Management Science, Standford University Press, Standford.

[5] Artalejo, J.R. (1996), *Stationary Analysis of Characteristics of the M/M/2 Queue with Constant Repeated Attempts*, OPSEARCH, 33, 83-95.

[6] Artalejo, J.R. (1999a), *Accessible Bibliography on Retrial Queues*, Mathematical and Computer Modelling, 30, 1-6.

[7] Artalejo, J.R.(1999b), *A Classified Bibliography of Research on Retrial Queues: Progress in 1990-1999*, Top 7, 187-211.

[8] Artalejo, J.R., Gomez-Corral, A. (1997), *Steady state Solution of a Single Server Queue with Linear Repeated Requests*, Journal of Applied Probability 34, 223-233.

[9] Artalejo, J.R., Gomez-Corral, A. (1998), *Analysis of Stochastic Clearing System with Repeated Attempts*. Commun. Statist. Models, 14 , 3, 623-645.

[10] Artalejo,J.R., Krishnamoorthy, A. and Lopez Herrero, M.J. (2006), *Numerical Analysis of (s, S) Inventory Systems with Repeated Attempts*, Annals of Operations Research, 141, 67-83.

[11] Azoury, K.S. and Brill, P.H. (1992), *Analysis of net Inventory in Continuous Review Models with Random Lead time*, European Journal of Operational Research, 59, 3, 383-392.

[12] Bellman, R. (1960), Introduction to Matrix Analysis, McGraw Hill Book Co., New York.

[13] Berman, O., Kim, E.H. and Shimshak, D.G. (1993), *Deterministic Approximation for Inventory Management for Service Facilities*, IIE Transaction, 25, 98-104.

[14] Berman, O. and Kim, E.H. (1999), *Stochastic Inventory Policies for Management for Service Facilities*, Stochastic Models, 15, 695-718.

[15] Berman, O. and Sapna, K.P. (2000), *Inventory Management at Service Facilities for Systems with Arbitrarily Distributed Service Times*, Stochastic Models, 16(3&4), 343-360.

[16] Bright, L and Taylor, P.G. (1995), *Equilibrium Distribution for Level Dependent Quasi-Birth-and-Death Processes*, Comm. Stat. Stochastic Models, 11, 497-525.

[17] Chakravarthy, S.R. (1993), *A finite capacity GI/PH/*1 queue with group services, Navel Research Logistics, 39, 345-357.

[18] Chakravarthy, S.R. (1996), *Analysis of the MAP/PH/*1*/K* queue with service control, Applied Stochastic Models and Data Analysis, 12, 179-181.

[19] Chakravarthy, S.R. (2000), *The batch Markovian arrival process: A review and future work.* Advances in Stochastic Modeling, Eds. Krishnamoorthy, A et al., Notable Publications, Inc. New Jersey, 21-49.

[20] Chikan, A. (1990), Inventory Models, Kluwer Academic Press, Dordrecht, Netherlands.

[21] Cinlar, E.(1975), Introduction to Stochastic Process, Prentice- Hall, New Jersey.

[22] Cooper, R.B. (1991), Introduction to Queueing Theory, 2nd edn., North-Holland, Amsterdam.

[23] Deepak, T.G., Joshua, V.C. and Krishnamoorthy, A. (2004), *Queues with Postponed Work*, TOP, 12, No. 3-4.

[24] Dudin, A.N, and Klimenock, V.I. (1999), *Queueing System BMAP/G/1 with Repeated calls*, Mathematical and Computer Modelling, 30, 115-128.

[25] Dudin, A.N, and Klimenock, V.I. (2000), *A Retrial BMAP/SM/1 System with Linear Repeated Requests*, Queueing Systems, 34, 47-66.

[26] Gross, D. and Harris, C.M.(1971), *On one-for-one Ordering Inventory Policies with State Dependent Lead Times*, Operations Research, 19, 735-760.

[27] Gross, D. and Harris, C.M. (2002), Fundamentals of Queueing Systems, 3rd edn., John Wiley and Sons (Asia).

[28] Falin, G.I.(1990),*A survey on Retrial Queues*, Queiueing Systems, 7, 127-168.

[29] Falin ,G.I. and Templeton, J.G.C. (1997), Retrial Queues, Chapman and Hall.

[30] Gomez Corral, A.(1999), *Stochastic Analysis of a Single Server Retrial Queue with General Retrial times*, Naval Research Logistics, 46, 561-581.

[31] Hadley, G. and Whitin, T.M. (1963), Analysis of Inventory Systems, Prentice Hall, Englewood Cliffs, N.J.

[32] Harris, F.(1915), Operations and Costs, Factory Management Series, A.W. Shah CO., Chicago.

[33] Kalpakam, S. and Arivarignan,G. (1984), *Semi-Markov Models in Inventory Systems*, Journal of Mathematics and Physical Sciences, 18, S1-S17.

[34] Kalpakam, S. and Sapna, K.P. (1993), *A modified (s, S) Inventory System with Lost sales and Emergency orders*, OPSEARCH, 30, 4, 321-335.

[35] Kleinrock, L. (1975), Queueing Systems, Vol. I, Theory, John Wiley& Sons, Inc., New York.

[36] Kleinrock, L. (1976), Queueing Systems, Vol. II, Computer Applications, John Wiley& Sons, Inc., New York.

[37] Krishnamoorthy,A., Deepak, T.G., Viswanath C. Narayanan, and Vineetha, K.(2006), *Effective Utilization of Idle time in an* (s,S) *Inventory with Positive Service time,* Journal of Applied Mathematics and Stochastic Analysis.

[38] Krishnamoorthy, A. and Lakshmy, B. (1990), *An Inventory Model with Markov Dependent Reorder levels*, OPSEARCH, 27, 1.

[39] Krishnamoorthy, A. and Lakshmy, B. (1991), *An Inventory Model with Markov Dependent Demand quantities, Cahiers du* C.E.R.O., 33, 91-101.

[40] Krishnamoorthy, A. and Manoharan, M.(1991), *An Inventory System with unit Demand and Varying levels*, Optimization, 22, 2, 283-289.

[41] Krishnamoorthy,A.and Merlymole,J.K.(1997),*Analysis of General Correlated bulk Demand, Two Commodity Inventory problem*, International Journal of Information and Management Sciences,8, 2, 35-46.

[42] Krishnamoorthy, A. and Mohammad Ekramol Islam. (2004), (s,S) *Inventory System with Postponed Demands*, Stochastic Analysis and Applications, 22, 827-842.

[43] Krishnamoorthy, A. and Raju, N. (1998), (s,S) *Inventory Policy with Lead time- The* N *-policy.* International Journal of Information and Management Science, 9, 4, 45-52.

[44] Krishnamoorthy, A. and Rekha, A. (1999), (s,S) *Inventory system with lead time-The* T *-policy Approach with Modified Analytic Hierarchy Process.* International Journal of Information and Management Science, 10, 2, 25-36.

[45] Krishnamoorthy, A. and Ushakumari, P.V.(1999), *Reliability of a k-out-of n System with Repair and Retrial of failed units*, TOP, 7, 293-304.

[46] Krishnamoorthy, A. and Vargehese, T.V. (1995), *On a Two Commodity Inventory problem with Markov shift in Demand*, OPSEARCH, 32, 1, 44-56.

[47] Krishnamoorthy,A., Vishanath C. Narayanan, Deepak,T.G and Vineetha, K.(2006), *Control Policies for Inventory with Service time,* Stochastic Analysis and Applications, 24, 4, 889-899.

[48] Latouche, G. and Ramaswami, V. (1991), Introduction to Matrix Analytic Methods in Stochastic Modeling, SIAM, Pennsylvania.

[49] Maike Schwarz, Cornelia Sauer, Hans Daduna, Raful Kulik, Ryszard Szekli. (2006), *M/M/1 Queueing Systems with Inventory*, Queueing Systems, 1, 54, 55-78.

[50] Manoharan, M., Krishnamoorthy,A. and Madhusoodanan, T.P.(1987), *On the* (s,S) *Inventory Policy with Independent non-identically Distributed Inter-arrival Demand times and Lead Times*, Cahiers du C.E.R.O., 29, 239-248.

[51] Naddor, E. (1966), Inventory Systems, John Wiley & Sons, New York.

[52] Neuts, M.F. (1981), Matrix-Geometric Solutions in Stochastic Models: An algorithmic approach, The Johns Hopkins University Press, Baltimore, MD.

[53] Neuts, M.F. (1992), *Two further closure properties of Ph-distributions*, Asia-pacific Journal of Operational Research, 23, 241-260.

[54] Neuts, M.F. and Rao, B.M. (1990), *Numerical Investigation of a Multi Sever Retrial Model*, Queueing Systems, and 7, 169-190.

[55] Parthasarathy and Vijayalakshmi, V. (1996), *Transient Analysis of an* $(S-1,S)$ *Inventory Model: a numerical approach*, Intern. J. Computer Math., 59, 177-185.

[56] Qui-Ming He and Neuts, M.F.(2002), *Two M/M/1 Queues with Transfers of Customers*, Queueing Systems, 42, 377-400.

[57] Sahin, I. (1979), *On the stationary Analysis of Continuous Review* (s,S) *Inventory Systems with Constant Lead time*, Operations Research, 27, 719-729.

[58] Sahin, I.(1990),Regenerative Inventory Systems, Springer-Verlag, New York.

[59] Sivazlian, B.D. (1974), *A Continuous Review* (s,S) *Inventory with Arbitrary inter-arrival Distribution between units Demands*, Operations Research, 22, 65-71.

[60] Srinivasan, S.K. (1979), *General Analysis of* (s,S) *Inventory Systems with Random Lead times and unit Demands*, Journal of Applied Math. and Phy. Sciences, 13, 107-129.

[61] Takagi, H. (1991), Queueing Analysis: Vacations and Priority Systems,Vol. I, North Holland, Amsterdam.

[62] Tijms, H.C.(1972), Analysis of (s,S) inventory models, Mathematical Centre Tracts 10, Amsterdam.

[63] Tijms, H.C. (2003), A First Course in Stochastic Models, John Wiley & Sons, Ltd.

[64] Tweedie R.L. (1975), *Sufficient conditions for Regularity, Recurrence and Ergodicity of Markov Processes*, Proc. Camb. Phil. Soc., 78, Part I.

[65] Ushakumari, P.V. (2007), (s,S) *Inventory with Retrial of Customers*, Journal of Applied Mathematics and Stochastic Analysis.

[66] Veinott, A.F. Jr. (1966), *The Status of Mathematical Inventory Theory*, Management Science, 12, 745-777.

[67] Vishanath C.Narayanan, Deepak,T.G, Krishnamoorthy,A and Krishnakumar, B.(2007), *On an* (s,S) *Inventory Policy with Service Time, Vacation to Server and Correlated Lead time,* Quality Technology and Quality Management.

[68] Yang, T. and Templeton, J.G.C.(1987), *A survey on Retrial Queues*, Queueing Systems, 2, 201-233.

[69] Zhao,Y.Q., and Grassman, V.K. (1990), *The Shortest Queue Model with Jockeying*, Naval Research Logistics, 37, 773-787.