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**LETTER TO THE EDITOR**

# Invariant density for a class of initial distributions under quadratic mapping

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**Abstract.** For the discrete-time quadratic map $x_{t+1}=4x_t(1-x_t)$ the evolution equation for a class of non-uniform initial densities is obtained. It is shown that in the $t\to\infty$ limit all of them approach the invariant density for the map.

Recently Falk (1984) has studied the evolution of a uniform probability density distribution towards an invariant density for a discrete-time quadratic map. He considered an initial density $r_0$ which is uniform over the interval (0, 1) and showed that under the quadratic map

$$x_{t+1}=4x_t(1-x_t) \tag{1}$$

$r_0$ approaches the invariant density

$$r(x)=1/\pi[x(1-x)]^{1/2} \tag{2}$$

(Ulam and von Neumann 1947) associated with the map. That is

$$\lim_{t\to\infty} r_t(x)=r(x).$$

In this letter we show that for the above quadratic map there exists a class of initial non-uniform densities all converging towards the invariant density (2) in the limit $t\to\infty$.

We consider a non-uniform initial density of the form

$$r_0(x)=(1/\beta(n+1,n+1))x^n(1-x)^n, \qquad 0<x<1 \tag{3}$$

where

$$\beta(n+1,n+1)=\int_0^1 x^n(1-x)^n\,\mathrm{d}x \tag{4}$$

is the $\beta$ function.

Equation (1) can be considered as defining a transformation between two random variables $x_t$ and $x_{t+1}$. One can then study, using standard methods (Papoulis 1965), how the probability distribution changes under the transformation. It can easily be shown that $r_t(x)$, the distribution at time $t$ satisfies an evolution equation of the form

$$r_{t+1}(x)=[1/4(1-x)^{1/2}](r_t(r_+)+r_t(r_-)) \tag{5}$$

where

$$r_\pm=\tfrac{1}{2}[1\pm(1-x)^{1/2}]. \tag{6}$$

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From (5) we can obtain the following set of equations:

$$r_1(x)=\frac{x^n}{(1-x)^{1/2}2^{2n+1}\beta(n+1,\,n+1)} \tag{7}$$

$$r_2(x)=\frac{1}{[x(1-x)]^{1/2}2^{2n+2}\beta(n+1,\,n+1)}[(r_+(x))^{n+1/2}+(r_-(x))^{n+1/2}] \tag{8}$$

$$r_3(x)=\frac{1}{[x(1-x)]^{1/2}2^{2n+3}\beta(n+1,\,n+1)}[(r_+r_+(x))^{n+1/2}+(r_+r_-(x))^{n+1/2}$$
$$+(r_-r_+(x))^{n+1/2}+(r_-r_-(x))^{n+1/2}]. \tag{9}$$

For general $t$,

$$r_t(x)=\frac{1}{[x(1-x)]^{1/2}2^{2n+t}\beta(n+1,\,n+1)}\sum_{s_1,s_2\ldots s_t=\pm}(r_{s_1}r_{s_2}\ldots r_{s_t}(x))^{n+1/2} \tag{10}$$

where

$$r_s(x)=\tfrac{1}{2}[1+s(1-x)^{1/2}] \tag{11}$$

with $s=\pm 1$.

Setting $x=\sin^2\theta$ in (10) one obtains

$$(r_{s_1}r_{s_2}\ldots r_{s_t}(\sin^2\theta))^{n+1/2}=(\sin\Phi)^{2n+1} \tag{12}$$

where

$$\Phi=\frac{\theta}{2^{t-1}}+\sum_{j=1}^{t-1}\frac{1}{2}(1+s_j)\frac{\pi}{2^j}. \tag{13}$$

Now

$$(\sin\Phi)^{2n+1}=\frac{(-1)^n}{2^{2n}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\sin(2n-2k+1)\Phi \tag{14}$$

(Gradshteyn and Ryzhik 1965). Using (14) in (10) we obtain the evolution equation for $r_t(x)$

$$r_t(x)=\frac{(-1)^n}{[x(1-x)]^{1/2}\beta(n+1,\,n+1)2^{4n+1}}\sum_{k=0}^{n}\left\{(-1)^k\binom{2n+1}{k}\prod_{j=1}^{t-1}\cos\frac{(2n-2k+1)\pi}{2^{j+1}}\right.$$
$$\left.\times\sin\left[\frac{(2n-2k+1)\theta}{2^{t-1}}+\frac{(2n-2k+1)\pi}{2}\left(1-\frac{1}{2^{t-1}}\right)\right]\right\}. \tag{15}$$

Now we can consider the limit of $r_t(x)$ as $t\to\infty$

$$\lim_{t\to\infty}r_t(x)=\frac{1}{\pi[x(1-x)]^{1/2}2^{4n}}\left[\frac{(-1)^n}{\beta(n+1,\,n+1)}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\frac{1}{(2n-2k+1)}\right]. \tag{16}$$

In obtaining (16) we have used the relation

$$\prod_{j=1}^{\infty}\cos\left(\frac{x}{2^j}\right)=\frac{\sin x}{x},\qquad -\infty<x<\infty. \tag{17}$$

From (16) the result of Falk can be recovered by setting $n=0$. When $n=1$, the term within the large square brackets becomes $2^4$ so that

$$\lim_{t\to\infty} r_t(x) = 1/\pi[x(1-x)]^{1/2}.$$

The special case for $n = 1$ has been previously considered by the author (1985).

When $n = 2, 3, 4\ldots$ the term in the large square brackets becomes $2^8$, $2^{12}$, $2^{16}\ldots$ respectively. For general $n$ it becomes $2^{4n}$. Therefore for all integer values of $n$

$$\lim_{t\to\infty} r_t(x) = \frac{1}{\pi[x(1-x)]^{1/2}}.$$

In summary, we have shown that for the quadratic map (1) there is a class of initial distributions all evolving towards the same invariant density. This invariant density represents an 'equilibrium state' which all other 'states' of the form (3) approach asymptotically.

## References

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