STUDIES IN FINITE TEMPERATURE QUANTUM FIELD THEORY

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Mr.K.P.Satheesh, under my guidance in the Department of Physics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

Cochin–22
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DECLARATION

Certified that the work presented in this thesis is based on the original work done by me under the guidance of Prof.K.Babu Joseph in the Department of Physics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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Preface

The present thesis deals with the behavior of scalar quantum fields at finite temperature. The analysis is most relevant to the study of early universe and high energy particle physics. The effective potential of a quantum field which is quantum corrected classical potential, determines the spontaneous symmetry breaking. Finite temperature correction to the effective potential is used to find the critical temperature at which symmetry is restored. In the thesis, calculations are performed for Φ^4 theory Φ^6 theory and charged scalar field with a Φ^4 interaction. Φ^4 theory is renormalizable (3+1) dimensions and Φ^6 theory in (2+1) dimensions. The main theoretical tools employed are CJT (Cornwall,Jackiw and Tomboulis) formalism and Functional Schrödinger picture formalism. These methods are more efficient than ordinary loop expansion for determining the effective potential and analysing the phase structure of a quantum field. Variational methods are used throughout so that non perturbative results are obtained. All calculations in Heisenberg picture use Hartee-Fock approximation.

Chapter 1 is introductory in nature. It develops all the theoretical tools required for the analysis. Chapter 2 deals with the applications of CJT formalism for a Φ^6 model. Chapter 3 uses the CJT method for an O(N) symmetric theory for studying a charged scalar field at finite chemical potential. Chapter 4 establishes the connection between CJT formalism and Schrődinger picture formalism both at zero and finite temperature.

Contents

Chapter 1

Basic Formalism

1.1 Introduction

Study of symmetry changing phase transitions of a quantum field in the presence of of a surrounding thermal bath is very important in the study of the evolution of the universe and in the analysis of very high energy collisions where very high matter and radiation density exist. Detailed study of these phase transitions has been extensively done by various authors [1-4]. The effective potential method is very useful in studying spontaneous symmetry breaking (SSB) at zero temperature [5,6]. Estimation of the critical temperature of phase transitions can be done by extending this approach to finite temperature [7]. These studies mainly using loop expansion techniques have played a pivotal role in framing our understanding about the early universe, unified theories, quark gluon plasma etc.

Recently there is a revival of interest in finite temperature quantum field theory, apparently caused by the recognition of the importance of symmetry breaking phase transitions and the problem of precise determination of critical temperature. An accurate analysis of phase transitions (both analytical and numerical) becomes necessary because most of the cosmological models critically depend on it. For example Baryon asymmetry may be generated at the electro-weak level if the phase transition is of first order [8-11].

For the calculation of critical temperature, various perturbative and non-perturbative techniques have been suggested [12-20]. Field-theoretic quantum dynamics involves too many degrees of freedom which are to be reduced for performing numerical computations. One way to do this is to use a variational approximation that yields an approximate wave functional in the Schrödinger picture [34-43]. A related procedure in the Heisenberg picture is to obtain a second Legendre transform and keep the lowest order in the coupling constant (Hartee-Fock approximation) for two particle irreducible (2PI) graphs [21,22,26-28,44]. In this thesis we use these two methods for analysing scalar fields at finite temperature and finite chemical potential.

Effective potential defined as single Legendre transform provides an efficient way to obtain quantum corrections to the classical potential. But this popular method suffers from a serious shortcoming. Since it is defined as a single Legendre transform , it is always a convex function. This forbids the double well shape for the exact effective potential and implies the absence of local maximum at the symmetric origin. But in a theory possessing SSB classical analysis predicts a maximum at the origin. Various procedures to avoid this difficulty have been suggested earlier [3,21,22]. One of the most efficient of these approaches is to define an effective action by including a source $K(x,y)$ coupled to a term which is quadratic in the field variable. By this procedure an effective potential with a proper loop expansion to each order which is not convex, is obtained. This idea was first put forward by Hawking and Moss in the context of quantum field theory in the early universe [23]. A self-consistent improvement for the finite temperature effective potential has also been suggested [24]. In this formalism it is possible to sum a large class of ordinary perturbation theory diagrams that contribute to the effective action and the gap equation which determines the form of the propagator, is obtained by variational method. Extension of this idea to non-equilibrium quantum fields has also been performed [25].

All the above improvements developed for conventional finite temperature effective potential are based on an important contribution by Cornwall, Jackiw and Tomboulis (CJT). They defined the effective action for composite fields in flat space and zero temperature as a double Legendre transform with two sources $J(x)$ and $K(x,y)$. These two sources are coupled respectively to $\phi(x)$ and $\phi(y)$ [26]. The CJT formalism is considered to be best suited for studying phase transitions because it uses a generalised effective action in which not only the mean field but also the correlation functions appear as independent variables. In ref. [26] a simple series expansion has been developed for the improved effective action.

The CJT formalism has recently been used to resolve various difficulties in quantum field theory [29,30]. For example, it has been applied to the triviality problem in Φ^4 theory [31]. An improved effective potential based on this formalism is discussed in ref. [32]. The self-consistent improvement of finite temperature effective potential (used in this thesis) involves the summation of daisy and superdaisy diagrams, for which a novel re-summation procedure has been recently proposed [33].

The functional Schrödinger picture formalism for quantum field theory is a generalisation from ordinary quantum mechanics to infinite number of degrees of freedom that comprise a field [34]. The method is suitable for both static and time-dependent problems at zero temperature and finite temperature and for quantum fields far from equilibrium. It has also been shown that renormalization in this model does not pose any special difficulties for static or time-dependent problems [34-41]. Among the applications of this approach are scalar QED, quantum mechanics of inflation, quantum roll process and quantum processes in non-euclidean space-time [34,35,36-41,42,43,45].

1.2 CJT Formalism

The CJT method provides a generalization of the ordinary effective action $\Gamma(\phi)$ (the generating functional for single particle irreducible n-point functions). This generalized effective action $\Gamma(\phi, G)$ depends both on $\phi(x)$ the expectation value of quantum field $\Phi(x)$, and $G(x,y)$ the expectation value of time ordered product $T\langle\Phi(x)\Phi(y)\rangle$. Physical (on-shell) solutions require the following variational equations:

$$\frac{\delta\Gamma(\phi, G)}{\delta\phi(x)} = 0 \tag{1.1}$$

$$\frac{\delta\Gamma(\phi, G)}{\delta G(x,y)} = 0 \tag{1.2}$$

Consider the vacuum persistence amplitude Z(J,K) in the presence of two source terms $J(x)\phi(x)$ and $\frac{1}{2}\phi(x)\phi(y)K(x,y)$:

$$Z(J,K) \equiv N \int \mathcal{D}\Phi \, exp\left[i \int d^4x \left[\mathcal{L}[\Phi(x)] + J(x)\Phi(x) + \frac{1}{2}\Phi(x)K(x,y)\Phi(y)\right]\right] \quad (1.3)$$

$W(J,K)$ the generating functional for connected diagrams is defined as

$$Z(J,K) \equiv exp[iW(J,K)] \quad (1.4)$$

The classical action $I(\Phi) = \int d^4x \mathcal{L}(x)$ may be written as

$$I(\phi) = \int d^4x d^4y \phi(x) D_0^{-1}(x-y)\phi(y) + I_{int}(\phi) \quad (1.5)$$

$$I_{int}(\phi) = \int d^4x \mathcal{L}_{int}(x) \quad (1.6)$$

$D_0(x-y)$ is the free propagator that satisfies

$$D_0^{-1}(x-y) = -(\Box + m^2)\delta^4(x-y) \quad (1.7)$$

The generalized effective action $\Gamma(\phi, G)$ is the double Legendre transform: of $W(J,K)$

$$\Gamma(\Phi, G) = W(J,K) - \int d^4(x)\Phi(x)J(x) - \frac{1}{2}\int d^4x d^4y [\Phi(x)K(x,y)\Phi(y)] - \frac{1}{2}\int d^4x d^4y G(x,y)K(x,y) \quad (1.8)$$

Where $J(x)$ and $K(x,y)$ are determined by

$$\frac{\delta W(J,K)}{\delta(J(x)} = \phi(x) \tag{1.9}$$

$$\frac{\delta W(J,K)}{\delta K(x,y)} = \frac{1}{2}[\phi(x)\phi(y) + G(x,y)] \tag{1.10}$$

By actually performing functional differentiation on eqn(1.8) we find

$$\frac{\delta\Gamma(\phi,G)}{\delta\phi(x)} = -J(x) - \int d^4y K(x,y)\phi(y) \tag{1.11}$$

$$\frac{\delta\Gamma(\phi,G)}{\delta G(x,y)} = -\frac{1}{2}K(x,y) \tag{1.12}$$

In the absence of sources, eqn.(1.1) and (1.2) are regained which permit a variational solution. The conventional effective action $\Gamma(\phi) = \Gamma(\phi, G_0)$ where G_0 is the solution of eq.(1.2).Generalized effective action $\Gamma(\phi, G)$ is the generating functional for the two particle irreducible (2PI) Greens functions expressed in terms of the full propagator. The series expansion for $\Gamma(\phi, G)$ is shown to be [26]:

$$\Gamma(\phi,G) = I_{class.}(\phi) + \frac{1}{2}tr\ln D_0 G^{-1} + \frac{1}{2}tr\left[D^{-1}G - 1\right] + \Gamma^{(2)}(\phi,G) \tag{1.13}$$

where tr is the functional trace, ln is the functional logarithm and $D^{-1}G$ is the functional product.

The inverse propagator is defined by

$$D^{-1}(\phi; x, y) = \frac{\delta^2 I(\phi)}{\delta\phi(x)\delta\phi(y)} = D^{-1}(x-y) + \frac{\delta^2 I_{int}(\phi)}{\delta\phi(x)\delta\phi(y)} \tag{1.14}$$

1.3 Computation of $\Gamma^{(2)}(\phi, G)$

Computation of the quantity $\Gamma^{(2)}(\phi, G)$ is done as follows. In the classical action $I(\Phi)$, the field Φ is shifted by $\phi(x)$. The shifted action $I(\Phi + \phi)$, possesses terms cubic and higher in Φ. These define an interaction part I_{int} with vertices depending on $\phi(x)$. $\Gamma^{(2)}(\phi, G)$ is then given by all the 2PI vacuum graphs and the propagator is set equal to $G(x, y)$. The theory in its full generality is not translationally invariant since vertices depend on $\phi(x)$ and G is not a function of $(x - y)$ alone. The propagator of the theory is determined by finding the gap equation for G using the variational equations [eqns. 1.1 and 1.2].

1.4 Effective Potential

Translationally invariant solutions are obtained by imposing the following conditions (homogeneous states):

$$\phi(x) = Constant \tag{1.15}$$

$$G(x,y) = G(x-y) \tag{1.16}$$

$$\Gamma(\phi, G) = -E(\phi, G) \int dt \tag{1.17}$$

where $E(\phi, G)$ is the minimum of the energy when varying over all the normalized states with constraints:

$$\langle\Phi(x)\rangle = \phi(x) \tag{1.18}$$

$$\langle\Phi(x)\Phi(y)\rangle = \phi(x)\phi(y) + G(x,y) \tag{1.19}$$

$$E(\phi, G) = V(\phi, G) \int d^{\nu-1}x \tag{1.20}$$

where ν is the space-time dimension of the theory . Thus the effective potential is given by

$$V(\phi,G) = -\frac{\Gamma(\phi,G)}{\int d^\nu x} \tag{1.21}$$

A series expansion for the effective potential is obtained by defining the following fourier transformed propagators.

$$G(k) = \int d^\nu x e^{ik(x-y)} G(x-y) \tag{1.22}$$

$$D(\phi,k) = \int d^\nu x e^{ik(x-y)} D(\phi;x-y) \tag{1.23}$$

$$D_0(k) = \int d^\nu x e^{ik(x-y)} D_0(x-y) \tag{1.24}$$

$$\begin{aligned} V(\phi,G) &= U(\phi) + \frac{1}{2}\int \frac{d^\nu x}{2\pi} \ln det D_0(k) G^{-1}(k) \\ &+ \frac{1}{2}\int \frac{d^\nu k}{(2\pi)^\nu} tr\left[D^{-1}(\phi,k) G^{-1} k - 1\right] + V_{(2)}(\phi,G) \end{aligned} \tag{1.25}$$

where $U(\phi)$ is the classical potential, $V_{(2)}(\phi,G)$ is the sum of 2PI vacuum graphs, with vertices given by $I_{int}(\phi,\Phi)$ and the propagator is set equal to $G(k)$. The field ϕ on which vertices depend is now a constant parameter. Trace and logarithms apply to component fields and determinants are no more functional.

1.5 Extension to Finite Temperature

To describe the theory at finite temperature we use the Euclidean time τ satisfying the boundary conditions $0 \leq \tau \leq \beta \equiv \frac{1}{T}$. All the Feynman diagrams (2PI diagrams) developed at zero temperature are valid here also. The Feynman rules for writing the algebra of the diagrams are different at finite temperature [22]. They are

$$\omega_n = \frac{2\pi n}{\beta} \tag{1.26}$$

$$\textit{loop integral} \longrightarrow \frac{1}{\beta}\sum_n \int \frac{d^{\nu-1}k}{(2\pi)^{\nu-1}} \tag{1.27}$$

$$\textit{vertex delta function} \longrightarrow \beta(2\pi)^{\nu-1}\delta\sum_{\omega_n}\delta^{\nu-1}(\sum_i k_i) \tag{1.28}$$

Field Φ satisfies the periodic boundary conditions

$$\Phi(-\frac{\beta}{2}, x) = \Phi(\frac{\beta}{2}, x) \tag{1.29}$$

With these modifications we can write the series expansion for finite temperature CJT effective potential (analog of Gibb's potential) with time integration suppressed and a summation performed.

$$\int \frac{d^\nu k}{(2\pi)^\nu} \longrightarrow \frac{1}{\beta}\sum_n \int \frac{d^{\nu-1}k}{(2\pi)^{\nu-1}} \tag{1.30}$$

1.6 Functional Schrödinger Picture Formalism

The functional Schrödinger Picture Formalism for quantum field theory has been shown to be superior to conventional Fock space methods for analyzing detailed structural properties of a quantum field [59,70]. In this method one need not choose a vacuum and normal ordering is not required. Schrödinger Picture method has been extensively used in studying solitons and other collective phenomena, topological defects in Gauge theories and confinement. Time dependent problems relevant to early universe studies and non-equilibrium thermal physics cannot be studied using conventional Green's function methods which require an initial condition for the solution. In these areas time-dependent Schrödinger Picture has been profitably used. Also for analyzing representations of transformation groups the method is very useful [35]. One can achieve intrinsic regularisation and renormalization without any reference to vacuum state and a unique representation is obtained.

In this method a quantum mechanical state $|\Psi(t)\rangle$ is replaced by a functional of the c-number field $\phi(x)$

$$|\Psi(t)\rangle \longrightarrow \Psi[\phi, t] \tag{1.31}$$

The action of an operator can be realized as a product and that of a canonical momentum as a functional differentiation.

$$\Phi(x)|\Psi(t)\rangle \longrightarrow \Psi(x)\Psi(\phi, t) \tag{1.32}$$

$$\Pi(x)|\Psi(t)\rangle \longrightarrow -i\frac{\delta}{\delta\phi(x)}\Psi(\phi, t) \tag{1.33}$$

The dynamical evolution of a given initial state is described by the functional Schrödinger equation. This equation can be derived using variational principles if we define a time-dependent effective action [38]:

$$\Gamma = \int dt \langle\Psi(t)|i\partial_t - H|\psi(t)\rangle \tag{1.34}$$

and impose the condition that $|\Psi(t)\rangle$ is stationary against arbitrary variations we get

$$i\frac{\partial\Psi(\phi,t)}{\partial t} = H\Psi(\phi,t) =$$
$$\int_x \left[-\frac{1}{2}\frac{\delta^2}{\delta\phi^2(x)} + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]\Psi(\phi,t) \tag{1.35}$$

In the Gaussian approximation we assume a Gaussian trial state:

$$\Psi(\phi,t) = exp\left[-\int_{x,y}(\phi(x)-\hat{\phi}(x,t))\left[\frac{G^{-1}(x,y,t)}{4} - i\Sigma(x,y,t)\right]\right.$$
$$\left.\times(\phi(y)-\hat{\phi}(y,t)) + i\int_x \hat{\Pi}(x,t)\left[\phi(x)-\hat{\phi}(x,t)\right]\right] \tag{1.36}$$

It can be seen that the Gaussian is centered at $\hat{\phi}$ and the width is given by G. Σ plays the role of the conjugate momentum of G and $\hat{\Pi}$ that of $\hat{\phi}$. $\hat{\phi}, \hat{\Pi}, G$, and Σ are the variational parameters as well as the expectation values:

$$\langle\phi(x)\rangle = \hat{\phi}(x,t) \tag{1.37}$$

$$\langle -i\frac{\delta}{\delta\phi(x)}\rangle = \hat{\Pi}(x,t) \tag{1.38}$$

$$\langle\phi(x)\phi(y)\rangle = \hat{\phi}(x,t)\hat{\phi}(y,t) + G(x,y,t) \tag{1.39}$$

$$\langle i\frac{\partial}{\partial\phi(x)}\rangle = \int_x \hat{\Pi}(x,t)\dot{\hat{\phi}}(x,t) + \int_{x,y}\Sigma(x,y,t)\dot{G}(y,x,t) \tag{1.40}$$

$$V^{(n)}(\hat{\phi}) \equiv \frac{d^n V(\hat{\phi})}{d\hat{\phi}^n} \tag{1.41}$$

For applying the formalism to a Φ^4 model, the following expression for effective action can be written up to two loop level [36,37]:

$$\begin{aligned}\Gamma = \int dt \Big[\int_x \hat{\Pi}\dot{\hat{\phi}} - \frac{1}{2}(\nabla\hat{\phi})^2 - V(\hat{\phi}) + \int_{x,y}\Sigma\dot{G} - \\ 2\int_{x,y,z}\Sigma G\Sigma - \int_x \frac{1}{8}G^{-1}(x,x,t) \\ -\frac{1}{2}\nabla_x^2 G(x,y,t)|_{x=y} + \frac{1}{2}V^{(2)}(\hat{\phi})G(x,x,t)\Big] - \\ \frac{1}{8}V^{(4)}(\hat{\phi})\int_x G(x,x,t)^2\end{aligned} \tag{1.42}$$

Identifying the first term as the classical action and performing variations we get

$$\frac{\delta\Gamma}{\delta\dot{\hat{\phi}}(x,t)} = 0 \longrightarrow \dot{\hat{\Pi}}(x,t) = \nabla_x^2\hat{\phi}(x,t) - V^{(1)}(\hat{\phi})$$
$$-\frac{1}{2}V^{(3)}(\hat{\phi})G(x,x,t) \quad (1.43)$$
$$\frac{\delta\Gamma}{\delta\dot{\hat{\Pi}}(x,t)} = 0 \longrightarrow \dot{\Sigma}(x,y,t) + 2\int_z \Sigma(x,z,t)\Sigma(x,y,t)$$
$$= \frac{1}{8}G^{-2}(x,y,t) + \left[\frac{1}{2}\nabla_x^2 - \frac{1}{2}V^{(2)}(\hat{\phi}) - \frac{1}{4}V^{(4)}(\hat{\phi})G(x,x,t)\right]\delta^{r}(x-y) \quad (1.44)$$
$$\frac{\delta\Gamma}{\delta\Sigma(x,y,t)} = 0 \longrightarrow \dot{G}(x,y,t) = 2\left[\int_z G(x,z,t)\Sigma(x,y,t)\right.$$
$$\left.+\sigma(x,z,t)G(z,y,t)\right] \quad (1.45)$$

The static effective potential can be obtained by taking $\hat{\phi}$ to be x- independent and by putting $\Sigma = 0$. By performing variation for G, a gap equation for G could be written. A slightly different but equivalent method has been used in (2+1) dimensional Liouville model [46].

1.7 Time Dependence

One of the major advantages of the Schrödinger (S) picture formalism is its power to analyze time-dependent evolution. Starting from the variational equations it can be seen that for the free theory ($\lambda = 0$) with $\hat{\phi}(x) = 0$ $G_0(k,t)$ oscillates with a frequency given by

$$\frac{1}{2\omega_k} = \frac{1}{2(k^2+\mu^2)^{\frac{1}{2}}} \tag{1.46}$$

The most general solution of the free equation is given by

$$G_0(k,t) = \frac{1}{2\omega_k}\left[1 + 2n_k - \left[(1+2n_k)^2 - 1\right]^{\frac{1}{2}} \cos 2\left[\omega_k(t) - \delta_0(k)\right]\right] \tag{1.47}$$

where the average energy E_k of the k^{th} mode is given by

$$E_k = (n_k + \frac{1}{2})\omega_k = \frac{\dot{G}^2(k,0)}{\delta G(k,0)} + \frac{1}{8}G^{-1}(k,0) + \frac{1}{2}\omega_k^2 G(k,0) \tag{1.48}$$

n_k is the k^{th} mode particle number.

For obtaining a solution we have to specify the initial data that is we have to select the initial Gaussian state. For the equations to be renormalisable G and Σ cannot be arbitrary. Detailed discussion of the criteria for selecting initial Gaussian state are given in ref[17].

Considering the simple case $\hat{\phi}(x) = 0$ the initial value of G is chosen as

$$G_{(k,0)} = \frac{1}{2(k^2 + m^2)^{\frac{1}{2}}}[1 + f(k)] \tag{1.49}$$

where

$$\lim_{k \to \infty} G(k,0) = \frac{1}{2k}\left[1 - \frac{\bar{m}^2}{2k^2} + \frac{g\cos\alpha(k)}{k^2}\right] \tag{1.50}$$

$$\lim_{k \to \infty} \dot{G}(k,0) = \frac{A + B\cos\beta(k)}{k^2} \tag{1.51}$$

α and β are non oscillatory and g,A,B and $\bar{m}$ are independent of k. The variational equation in terms of G alone is

$$\ddot{G} = \frac{1}{2}G^{-1} + \frac{1}{2}G^{-1}\dot{G}^2 - 2\left[k^2 + \mu^2 + \frac{\lambda}{2}\int_k G(k,t)\right]G \tag{1.52}$$

A divergence is present in the $\int_k G(k,t)$ term.It has been shown that static renormalization conditions can be used to renormalize the time-dependent equations if the initial Gaussian is correctly chosen. The condition $\phi(x) \neq 0$ also will not affect the renormalizability.

1.8 Finite Temperature Extension

Several attempts have been made to extend functional S.picture approach to finite temperature [33,34]. In reference [34] S.picture method is extended to finite temperature both for equilibrium and non-equilibrium situations. In s picture formalism effect of temperature is considered as an external influence. System starts as a pure state and gradually evolves into a mixed state governed by the density matrix. They propose a variational solution of the Liouville equation through the introduction of a Gaussian density matrix for a mixed state. The Gaussian density matrix for one quantum mechanical degree of freedom is taken as [34].

$$\rho(x_1, x_2) - e^{-\gamma}\left[-\frac{1}{4G}(x_1^2 + x_2^2 - 2x_1x_2\zeta)\right]\exp[i\Pi(x_1^2 - x_2^2)] \tag{1.53}$$

ζ is defined to be the degree of mixing between the states. Gaussian density operator can be defined as follows.

$$\rho(x_1, x_2) = < x_1|\hat{\rho}|x_2 > \tag{1.54}$$

$$\hat{\rho} = \left[\frac{2\pi}{\omega}\sinh b\omega\right]^{1/2} e^{-\gamma} exp\left[-\frac{b}{2}(p^2 - 2\Pi(xp + px) + ax^2)\right] \tag{1.55}$$

After proper normalizations the equations are shown to be

$$\hat{\rho} = 2\sinh\frac{b\omega}{2} exp\left[-\frac{b}{2}(p^2 - 2\Pi(xp + px) + ax^2)\right] \quad (1.56)$$

$$\rho(x_1, x_2) = \left[\frac{\omega}{\pi}\tanh\frac{b\omega}{2}\right]^1 /2\exp\left[-\frac{\omega}{2\sinh b\omega}[(x_1^2 + x_2^2)\cosh b\omega - 2x_1x_2]\right]\exp[i\Pi(x_1^2 - x_2^2)] \quad (1.57)$$

Using the above Gaussian density matrix the entropy and free energy can be calculated. We use the restricted density matrix [34] in which all linear averages vanish and bilinear averages survive. More general anzatz for the density matrix has also been suggested.

Chapter 2

Finite Temperature CJT Formalism - Φ^6 Model

2.1 Introduction

Recently there has been considerable interest in the application of functional techniques in field theories with dimensions less than (3+1), mainly because some of the problems afflicting 4-dimensional theories are absent there [49-51]. According to Coleman's theorem, spontaneous symmetry breaking can occur only when the dimensions are higher than (1+1) [53]. In (2+1) dimensions the most general renormalizable theory is for a Φ^6 model. This model has been studied earlier by using various methods and it has been shown that it possesses an ultraviolet fixed point in $1/N$ expansion and Gaussian approximation [55,56,58]. Finite temperature field theory has also been analyzed [18,19,57]. It is well known that in finite temperature CJT analysis of Φ^4 theory ,the effective potential shows a cut-off dependence due to the presence of a $(\lambda\phi^4/12)$ term. It is natural to think that in lower dimensions where coupling constant renormalization is not required this difficulty will be absent. Even though this is found to be true for Φ^4 theory in (2+1) dimensions, an unrenormalized mass term appears in the expression for effective potential for Φ^6 theory. Thus the difficulty persists in Φ^6 theory in a disguised form.

2.2 Φ^6 Theory

The classical potential of the theory is given by

$$U(\Phi) = \frac{1}{2} m_B^2 \Phi^2 + \frac{\lambda_B}{4!} \Phi^4 + \frac{\xi_B}{6!} \Phi^6 \tag{2.1}$$

where the subscript B indicates bare parameters. The functional operator D^{-1} is given by

$$D^{-1}(\Phi; (x,y)) = -\left[\Box + m_B^2 + \frac{\lambda_B}{2}\Phi^2 + \frac{\xi_B}{24}\right] \delta^\nu(x-y) \tag{2.2}$$

After shifting the interaction Lagrangian takes the form

$$L_{int}(\Phi,\phi) = \left[\frac{\lambda_B}{6}\phi\Phi^3 + \frac{\lambda_B}{4!}\Phi^6 + \frac{\xi_B}{6!}\Phi^6 + \frac{\xi_B}{5!}\phi\Phi^5 + \frac{\xi_B}{48}\phi^2\Phi^4 + \frac{\xi_B}{36}\phi^3\Phi^5\right] \tag{2.3}$$

A few of the 2PI vacuum graphs up to three loops are shown in fig.1. We introduce the following approximation to obtain tractable equations. Only those graphs with vertices depending on first order in the coupling constant are selected. This approximation corresponds to a systematic variational procedure and is superior to commonly used one loop approximation. No graphs with internal lines appear.(Note that daisy and superdaisy graphs shown in fig.3 are of this type). This approximation is called Hartee-Fock approximation according to which only the graphs shown in Fig. 2 need be summed. Thus the sum of the relevant 2PI graphs takes the form

$$\Gamma_{\beta}^{(2)}(\phi, G) = \frac{3}{4!}\lambda_B \int d^{\nu}x G(x,x)G(x,x) + \frac{10}{6!}\xi_B \int d^{\nu}x G(x,x)G(x,x)G(x,x) \tag{2.4}$$

The expression for the finite temperature non local composite operator effective action in Hartee-Fock approximation becomes

$$\Gamma_{\beta}(\phi, G) = I_{class.} + \frac{1}{2} tr \ln D_0 G^{-1} + \frac{1}{2} tr(D^{-1}G - 1) + \frac{3}{4!}\lambda_B \int d^{\nu}x G(x,x)G(x,x) + \frac{10}{6!}\xi_B \int d^{\nu}x G(x,x)G(x,x)G(x,x) \tag{2.5}$$

By performing variation with respect to G the modified gap equation is obtained.

$$G^{-1}(x,y) = D^{-1}(x,y) + \left[\frac{\lambda_B}{2}G(x,x) + \frac{\xi_B}{12}G(x,x)G(x,x)\right]\delta^3(x-y) \quad (2.6)$$

By iteration it can be seen that G generates all daisy and super daisy graphs of Fig.3, which reveals the fact that we have achieved a definite improvement in re-summation of diagrams.

Since we are interested only in translation invariant theories we fix an anzatz for G and define the fourier transformed propagators [eqns 1.22,1.23 and 1.24].

$$G(k) \equiv \int \frac{d^\nu x}{(2\pi)^\nu} G(x-y) e^{ik(x-y)} = \frac{1}{k^2+M^2} \quad (2.7)$$

$$D(k) \equiv \int \frac{d^\nu x}{(2\pi)^\nu} D(x-y) e^{ik(x-y)} = \frac{1}{k^2+m_B^2+\frac{\lambda_B}{2}\phi^2+\frac{\xi_B}{24}\phi^4} \quad (2.8)$$

Here the propagator is chosen in terms of an effective mass M which acts as a variational parameter. Effective potential in terms of M^2 and ϕ can be written by using static configuration and constant background field:

$$V_B(\phi, M) = \left[\frac{1}{2}m_B^2\phi^2 + \frac{\lambda_B}{4!}\phi^4 + \frac{\xi_B}{6!}\phi^6\right] + \frac{1}{2}\int\frac{d^3x}{(2\pi)^3}ln(k^2 + M^2) -$$
$$\frac{1}{2}\left[M^2 - m_B^2 - \frac{\lambda_B}{2}\phi^2 - \frac{\xi_B}{24}\phi^4\right]G(x,x) + \frac{\lambda_B}{8}G(x,x)G(x,x) +$$
$$\frac{\xi_B}{48}G(x,x)G(x,x)G(x,x) \qquad (2.9)$$

Putting $\xi_B = 0$ we get Φ^4 theory in (2+1) dimensions. Comparing the expression obtained with the Gaussian effective potential studies of Stevenson [17] we see that both are identical in form. But as far as Φ^6 theory is concerned both are not identical because of the term $\xi\phi_0^2$. It is interesting to note that this factor is not contributed by the daisy or super daisy diagrams, but by a graph with vertex not proportional to ξ. This graph is not considered in Hartee-Fock approximation.

Since the effective potential is an ordinary function (not a functional) stationary requirements w.r.t ϕ and M^2 is obtained by ordinary differentiation.

$$\frac{\partial V}{\partial\phi} = \phi\left[m_B^2 + \frac{\lambda_B}{6}\phi^2 + \right.$$
$$\left.\frac{\xi_B}{120}\phi^4 + \frac{\lambda_B}{2}G(x,x) + \frac{\xi_B}{12}\Phi^2G(x,x)\right] = 0 \qquad (2.10)$$

$$\frac{\partial V}{\partial M^2} = -\frac{1}{2}\left[M^2 - m_B^2 - \frac{\lambda_B}{2}\phi^2 - \right.$$
$$\left. \frac{\xi_B}{24}\phi^4 - \frac{\lambda_B}{2}G(x,x) - \frac{\xi_B}{8}G(x,x)G(x,x)\right]\frac{\partial G(x,x)}{\partial M^2} = 0 \qquad (2.11)$$

Conventional effective potential is defined at the solution of [eqn(2.11)]. The effective mass is given by

$$M^2(\phi) = \left[m_B^2 + \frac{\lambda_B}{2}\phi^2 + \frac{\xi_B}{24}\phi^4 + \frac{\lambda_B}{2}G(x,x) + \frac{\xi_B}{8}G(x,x)G(x,x)\right] \qquad (2.12)$$

Required expression for the effective potential is obtained by replacing the effective mass M by $M(\phi)$ in [eqn.(2.28)]. [equation (2.9)] shows certain very important peculiarities of Φ^6 and Φ^4 theories relevant at zero temperature.

$$V(\phi) = \phi\left[M^2(\phi) - \left(\frac{\lambda_B}{3}\phi^2\right)\right] \qquad (2.13)$$

For Φ^4 theory $M^2(\phi)$ is intrinsically positive. Hence if $\lambda_B < 0$ only solution to the above equation is $\phi = 0$ (or potential is unbounded from below). That is for negative λ_B non zero turning points do not exist. In the case of Φ^6 theory

$$V(\phi) = \phi\left[M^2(\phi) - \left(\frac{\lambda_B}{3}\phi^2 + \frac{\xi_B}{30}\phi^4\right)\right] - \frac{\xi}{4}\left[1 - \frac{\phi^2}{3}\right]G^2 \qquad (2.14)$$

non zero turning points are possible also for $\lambda_B < 0$

Φ^6 theory in Hartee-Fock approximation requires up to three loops for obtaining the effects of ξ_6 coupling. We have four parts for the effective potential.

$$V_B(\phi, M(\phi)) = V^0 + V^1 + V^2 + V^3 \tag{2.15}$$

$$V^0 = \left[\frac{1}{2}m_B^2\phi^2 + \frac{\lambda_B}{4!}\phi^4 + \frac{\xi_B}{6!}\phi^6\right] \tag{2.16}$$

$$V^1 = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3} ln\left[k^2 + M^2(\phi)\right] \tag{2.17}$$

$$V^2 = -\frac{\lambda_B}{8}G(x,x)G(x,x) \tag{2.18}$$

$$V^3 = -\frac{\xi_B}{24}G(x,x)G(x,x)G(x,x) \tag{2.19}$$

which are obtained by substituting $M(\phi)$. Effective potential for both Φ^4 and Φ^6 theories can be obtained from this equation.

2.3 Renormalization

The effective mass term $M(\phi)$ defining the effective potential is divergent mainly due to the presence of G(x,x). Following re-normalization prescription is employed in (2+1) dimensions to regularize $M(\phi)$ [7,24].

Define

$$G(M(\phi)) \equiv -\frac{M(\phi)}{4\pi} + \frac{1}{\beta}\int \frac{d^2k}{(2\pi)^2} \frac{1}{E(e^{\beta E} - 1)} \tag{2.20}$$

with

$$E \equiv \left[k^2 + M^2(\phi)\right]^{\frac{1}{2}} \tag{2.21}$$

In (2+1) dimensions coupling constant re-normalization is not required.Define

$$m_r^2 \equiv m_B^2 + \frac{1}{2}\lambda I_1 + \frac{\xi'}{4} I_1 + \frac{\xi}{8} I_1^2 \tag{2.22}$$

$$I_1 \equiv \int \frac{d^2k}{2\pi)^2} \frac{1}{2k} = \lim_{\Lambda \to 0} \left(\frac{\Lambda}{4\pi}\right) \tag{2.23}$$

$\xi' \equiv \xi G(M(\phi))$

Using the summation procedure developed by Dolan and Jackiw [2,24,69] the summation in time co-ordinate can be performed.

$$G(x, x) = \int \frac{d^2k}{(2\pi)^2} \frac{1}{2E} + \frac{1}{\beta}\int \frac{d^2k}{(2\pi)^2} \frac{1}{E(e^{\beta E} - 1)} \tag{2.24}$$

By actual evaluation, introducing a cut off parameter Λ

$$G(x,x) = G(M(\phi)) + I_1 \tag{2.25}$$

[eqn.(2.24)] shows that $G(M(\phi))$ is the finite part of the vacuum propagator. A finite expression for $M(\Phi)$ is obtained by expressing it in terms of the renormalized parameters:

$$M^2(\phi) = -m_r^2 + \frac{\lambda}{2}\phi^2 + \frac{\xi}{24}\phi^4 + \frac{\lambda}{2}G(M(\phi)) + \frac{\xi}{8}G(M(\phi))G(M(\phi)) \tag{2.26}$$

By switching off the ξ coupling, (2+1) dimensional Φ^4 theory is obtained. Second derivative of the tree level potential is defined as $m_{tree.}$ (tree level mass).

$$M^2(\phi) = m^2_{tree.}(\phi) + \frac{\lambda}{2}G(M(\phi)) + \frac{\xi}{8}G(M(\phi))G(M(\phi)) \tag{2.27}$$

$$V^1_\beta(M(\phi)) = -\frac{M^3}{6\pi} + \frac{1}{\beta}\int\frac{d^2k}{(2\pi)^2}\ln\left(1 - e^{\beta E}\right) + \frac{\Lambda^3}{6\pi} \tag{2.28}$$

At zero temperature the second term vanishes and the last term is cut off dependent. Cancellation of this divergence is obtained by combining V^0, V^2 and V^3:

$$V_\beta(M) = -\frac{M^3}{6\pi} + \frac{1}{\beta}\int\frac{d^2k}{(2\pi)^2}ln\left(1 - e^{\beta E}\right) + \frac{M^4}{2\lambda} - \frac{1}{2}M^2G(M) - \frac{M^2}{24\eta}G^2(M) - F(\phi) \tag{2.29}$$

$$with\ \ \eta^{-1} = \frac{\xi}{\lambda}\ and\ F(\phi) = \left[\frac{m^2}{24\eta} - \frac{\lambda}{12} - \frac{29}{6!}\xi\phi^2\right]\phi^4 \tag{2.30}$$

$\xi = 0$ reproduces the result of Camellia and Pi. Using unre-normalized gap equation we combine V^0, V^2 and V^3 and writing them in terms of renormalized parameters

$$V^0 + V^2 + V^3 = \frac{\lambda_r}{8}\left[\phi^2 - \frac{2m_r^2}{\lambda}\right]^2 - \frac{\eta}{24}m_r^2\phi^4 - \frac{\lambda_r}{8}G^2(M) - \frac{\xi_r}{48}G^3(M) - F(\phi) \qquad (2.31)$$

In the case of (2+1) dimensional Φ^4 theory $F(\phi) = \frac{\lambda}{12}$ which is finite. Thus unlike (3+1) dimensional Φ^4 theory the effective potential do not contain any unrenormalized parameters. But in the case of Φ^6 theory $F(\phi)$ contains 'm' which is an unrenormalized mass parameter. But Here we can make $F(\phi)$=0 by adjusting the parameters suitably and make the unrenormalized parameters vanish.

2.4 High Temperature Expansion.

Evaluation of the effective potential at high temperature up to one loop level has been done earlier in (3+1) and (1+1) dimensions [2,58]. Additional terms appearing in the expression for the effective potential can be obtained by evaluating G(M(ϕ)), for $\frac{M(\phi)}{T} << 1$. The relevant integral [eqn. 2.20] is of the form [69]

$$h_n(y) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}dx}{(x^2+y^2)^{1/2}} \frac{1}{e^{(x^2+y^2)^{1/2}}-1} \tag{2.32}$$

$$y = \frac{m}{T} \tag{2.33}$$

These integrals satisfy the differential equation

$$\frac{dh_{n+1}}{dy} = -\frac{yh_{n-1}}{n} \tag{2.34}$$

High temperature expansion for the integral is obtained by using the identity

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + 2\sum_{l=1}^{\infty} \frac{z}{z^2+(2\pi l)^2} \tag{2.35}$$

Multiplying the integrand by a factor $x^{-\epsilon}$ for convergence, performing term by term integration and letting $\epsilon \to 0$ at the end we get

$$h_1(y) = \frac{\pi}{2y} + \frac{1}{2}\ln\frac{y}{4\pi} + \frac{1}{2}\gamma + O(y^2) \tag{2.36}$$

$\gamma = 0.5772\cdots$ which is the Euler constant.

$$h_2(y) = -\ln(1 - e^{-y}) \tag{2.37}$$

Other can be found by using the differential equation [eqn.(2.34)]. We have:

$$\frac{1}{\beta}\int\frac{d^2k}{(2\pi)^2}\frac{1}{E(e^{\beta E}-1)} = \frac{1}{\pi}\int\frac{kdk}{(2\pi)^2}\frac{1}{E(e^{\beta E}-1)} \tag{2.38}$$

By using the formula [eqn.2.37] we get

$$G(M(\phi)) = -\frac{M(\phi)}{4\pi} + \frac{1}{\pi}\ln(1 - e^{-\frac{M}{T}}) \tag{2.39}$$

Which can be used to evaluate V_{eff} at high temperature using [eqn.(2.30)]. High temperature expansion for the effective mass can be obtained from [eqn. (2.26)].

2.5 Conclusions.

A self-consistent improvement for the finite temperature Φ^4 theory is obtained as an extension of CJT formalism. Certain peculiarities of the Φ^6 field theory in (2+1) dimensions are analyzed. Φ^4 theory in (2+1) dimensions does not contain any unrenormalized terms unlike its (3+1) dimensional counterpart. Φ^6 theory in (2+1) dimensions contains divergent terms in the form of an unrenormalized mass parameter in the expression for effective potential, but can be made to vanish. In this model physically meaningful stable theory is possible both for positive and negative λ, indicating the possibility of bound states. High temperature expansion for the effective potential is obtained.

Behavior of the effective mass can be clearly understood by numerically solving the equation for effective mass. Figures (4–7) show these graphs for certain relevant values at zero temperature. The straight region parallel to the ϕ axis indicates imaginary values of the effective mass. It is clear that this region indicates a broken symmetry phase. comparison between Φ^4 and Φ^6 theories show that they are identical in shape except for higher numerical values for Φ^6 model. A study of the behavior of effective mass for different couplings is also given (in weak coupling range since the reliability of the approximation in the strong coupling range is not well established). Shape of the graph is same for a reasonable range of couplings with a notable difference in the stretch of the straight region.

Graph for effective mass at finite temperature (fig. 8–15) shows that as temperature increases the straight region gradually decreases indicating an approach to symmetry restoration (critical temperature). At sufficiently high temperature symmetry is found to be restored. Graphs presented are for Φ^6 model, but the behavior is identical for Φ^4 model.

Using the numerically obtained value of the effective mass effective potential can be calculated (fig. 16 – 18). The graphs clearly indicate broken symmetry phase at zero temperature. Approach to symmetry restored phase is indicated in fig.19.

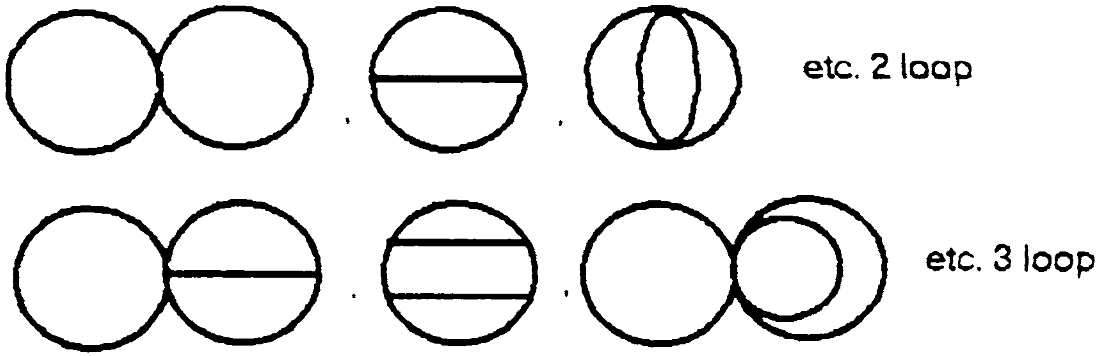


Fig. 1

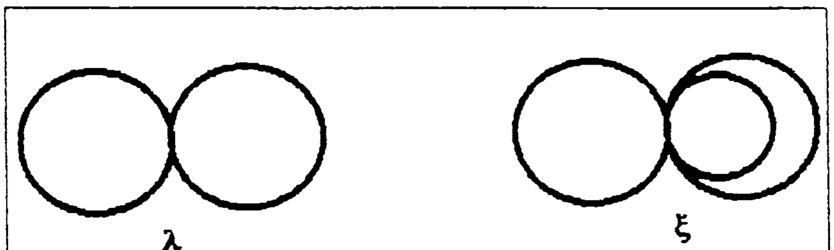


Fig.2

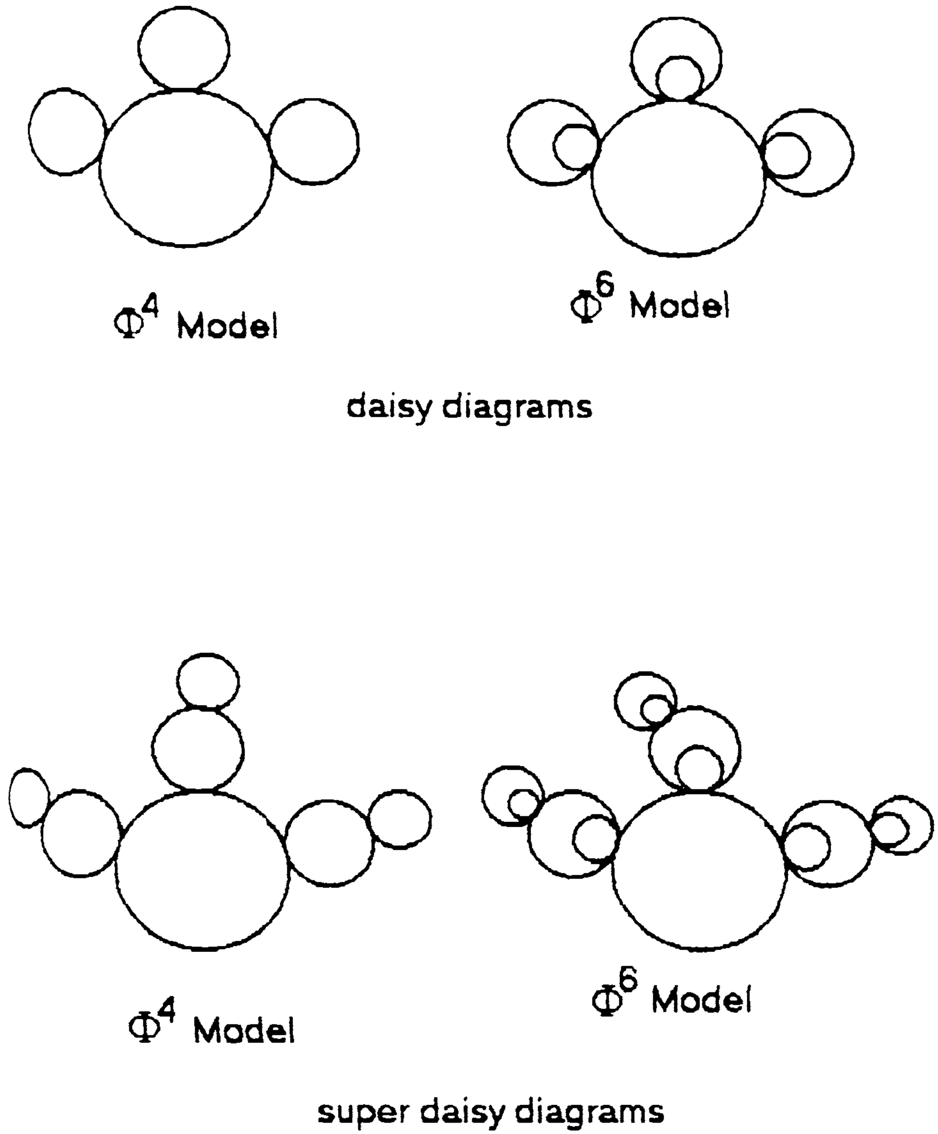

Φ^4 Model
Φ^6 Model
daisy diagrams
Φ^4 Model
Φ^6 Model
super daisy diagrams

Fig. 3

$$\phi := -20, -19.99 \,..\, 20$$

$$m := 1$$

$$M(\phi,\lambda) := \frac{-\lambda}{16\cdot\pi} + \left[\frac{\lambda^2}{32\cdot\pi^2} - 4\cdot\left[m^2 - \frac{\lambda}{2}\cdot\phi^2\right]\right]^{.5}$$

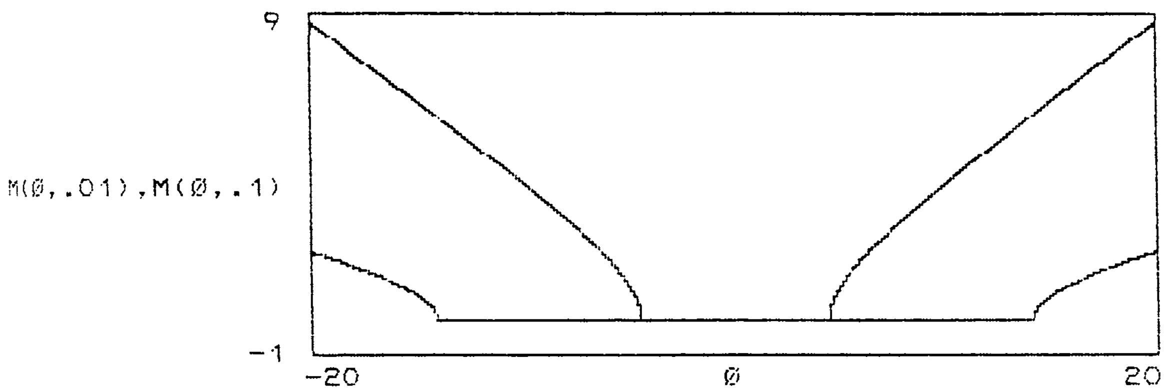


fig - 4

Graph connecting M(Ø) and Ø

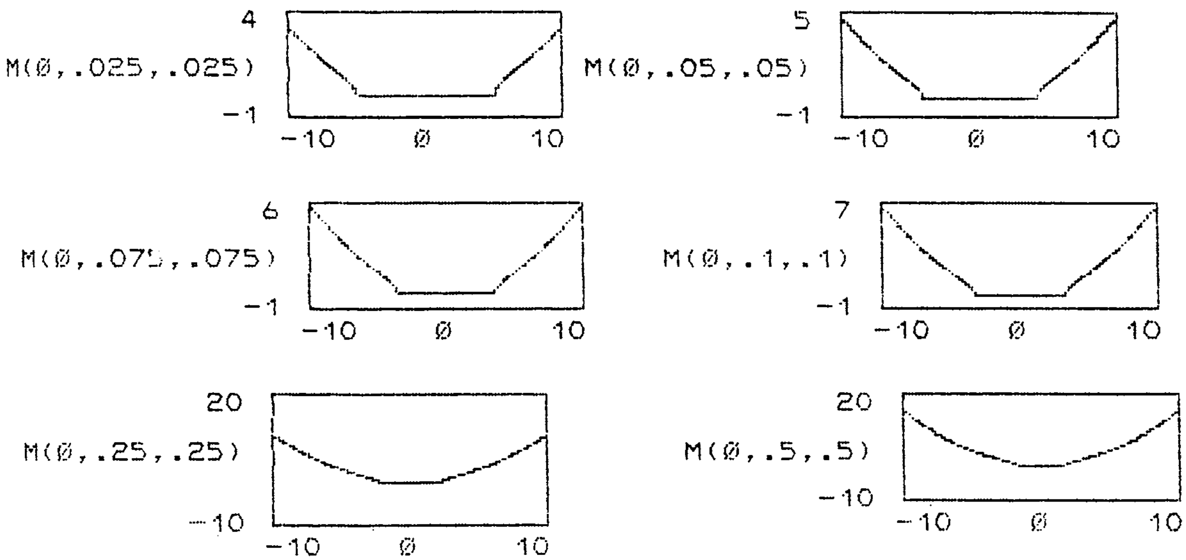
λ=.01 and λ=.1

$$\phi := -10, -9.9 \,..\, 10$$

$$m := 1$$

$$M := 3 + 2i$$

$$M(\phi,\lambda,\epsilon) := \mathrm{root}\left[M^2 + \lambda\cdot\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\phi^2}{2} - \epsilon\cdot\frac{\phi^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$



Graph for Ø6 theory

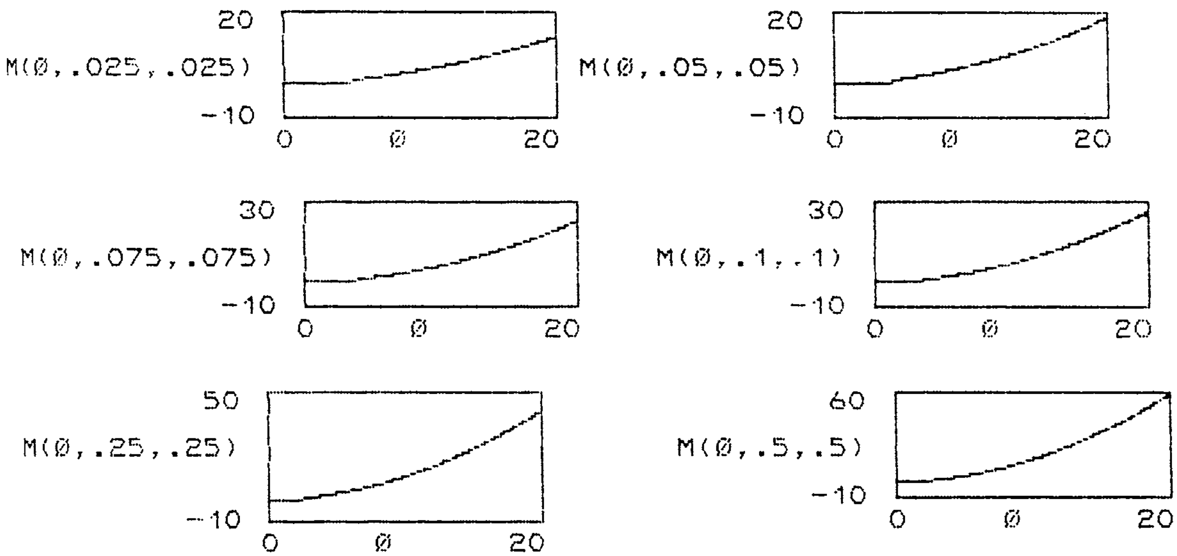
Fig 5

$$\phi := 0, .1 \,..20$$

$$m := 1$$

$$M := 3 + 2i$$

$$M(\phi,\lambda,\epsilon) := \mathrm{root}\left[M^2 + \lambda\cdot\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\phi^2}{2} - \epsilon\cdot\frac{\phi^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$



Graph for Ø6 theory

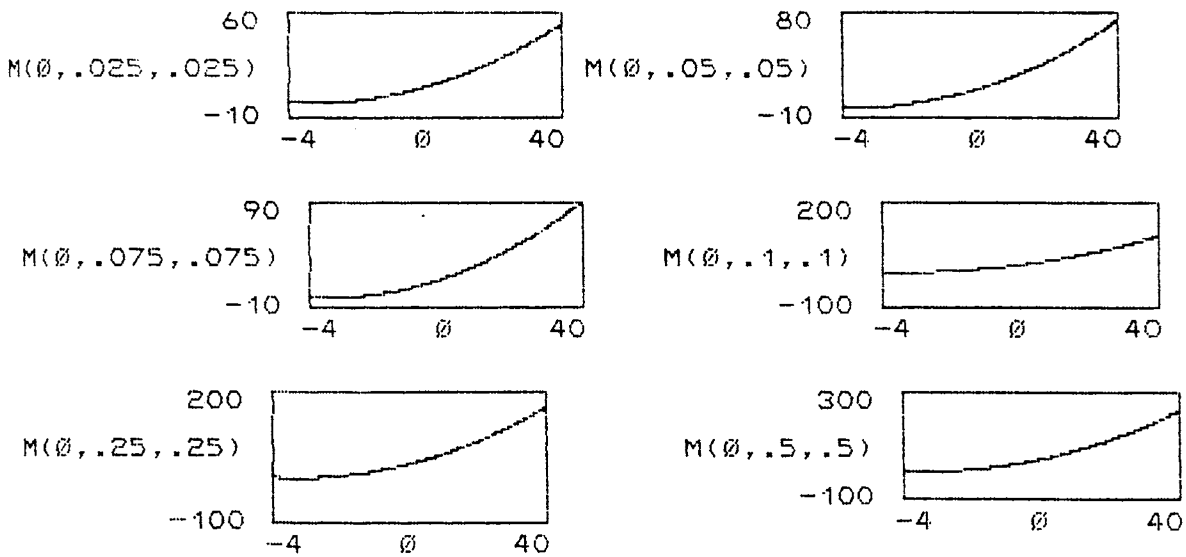
Fig 6

$$\phi := -4, -3.9\,..\,40$$

$$m := 1$$

$$M := 3 + 2i$$

$$M(\phi,\lambda,\epsilon) := \mathrm{root}\left[M^2 + \lambda\cdot\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\phi^2}{2} - \epsilon\cdot\frac{\phi^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$



Graph for Ø6 theory

Fig 7

$$\emptyset := 25, 25.1 \,..\, 100$$

$$\lambda := .05$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .05$$

$$n := \frac{\lambda}{\epsilon}$$

$$M(\emptyset) := \mathrm{root}\left[M^2 + \lambda\cdot\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\emptyset^2}{2} - \epsilon\cdot\frac{\emptyset^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$

$$T := 1$$

$$y(\emptyset) := \frac{M(\emptyset)}{T}$$

$$G(\emptyset) := \frac{-M(\emptyset)}{4\cdot\pi} + \frac{1}{\pi}\cdot\ln(1 - \exp(-y(\emptyset)))$$

$$F(\emptyset) := m^2\cdot\frac{\emptyset^4}{24\cdot n} - \lambda\cdot\frac{\emptyset^4}{12} - 29\cdot\epsilon\cdot\frac{\emptyset^6}{6!}$$

$$tr(\emptyset) := -m^2 + \lambda\cdot\frac{\emptyset^2}{2}$$

$$mass(\emptyset) := tr(\emptyset)^2 + \lambda\cdot\frac{G(\emptyset)}{2} + n\cdot\frac{G(\emptyset)^2}{8}$$

Finite temperature Behaviour

Fig. 8

$$\phi := 25, 25.1 \,..\, 100$$

$$\lambda := .05$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .05$$

$$\eta := \frac{\lambda}{\epsilon}$$

$$M(\phi) := \text{root}\left[M^2 + \lambda \cdot \frac{M}{8 \cdot \pi} + m^2 - \lambda \cdot \frac{\phi^2}{2} - \epsilon \cdot \frac{\phi^4}{24} - \epsilon \cdot \frac{M^2}{128 \cdot \pi^2}, M\right]$$

$$T := 100$$

$$y(\phi) := \frac{M(\phi)}{T}$$

$$G(\phi) := \frac{-M(\phi)}{4 \cdot \pi} + \frac{1}{\pi} \cdot \ln(1 - \exp(-y(\phi)))$$

$$F(\phi) := m^2 \cdot \frac{\phi^4}{24 \cdot \eta} - \lambda \cdot \frac{\phi^4}{12} - 29 \cdot \epsilon \cdot \frac{\phi^6}{6!}$$

$$tr(\phi) := -m^2 + \lambda \cdot \frac{\phi^2}{2}$$

$$mass(\phi) := tr(\phi)^2 + \lambda \cdot \frac{G(\phi)}{2} + \eta \cdot \frac{G(\phi)^2}{8}$$

nite temperature
haviour

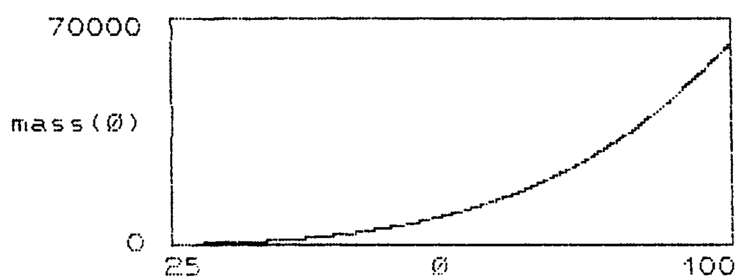


Fig. 9

$$\emptyset := 100, 100.1 \,..\, 200$$

$$\lambda := .05$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .05$$

$$\eta := \frac{\lambda}{\epsilon}$$

$$M(\emptyset) := \mathrm{root}\left[M^2 + \lambda\cdot\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\emptyset^2}{2} - \epsilon\cdot\frac{\emptyset^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$

$$T := 500$$

$$y(\emptyset) := \frac{M(\emptyset)}{T}$$

$$G(\emptyset) := \frac{-M(\emptyset)}{4\cdot\pi} + \frac{1}{\pi}\cdot\ln(1 - \exp(-y(\emptyset)))$$

$$F(\emptyset) := m^2\cdot\frac{\emptyset^4}{24\cdot\eta} - \lambda\cdot\frac{\emptyset^4}{12} - 29\cdot\epsilon\cdot\frac{\emptyset^6}{6!}$$

$$tr(\emptyset) := -m^2 + \lambda\cdot\frac{\emptyset^2}{2}$$

$$mass(\emptyset) := tr(\emptyset)^2 + \lambda\cdot\frac{G(\emptyset)}{2} + \eta\cdot\frac{G(\emptyset)^2}{8}$$

Finite temperature Behaviour

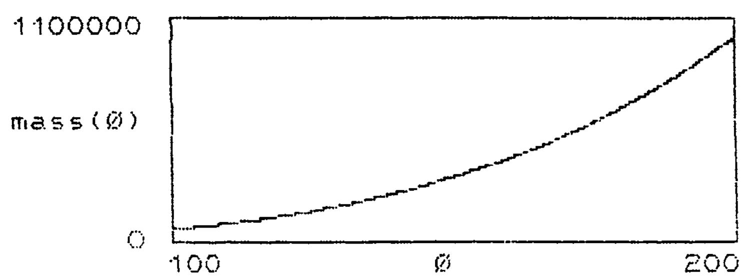


Fig. 10

$$\emptyset := 100\,..200$$

$$\lambda := .05$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .05$$

$$\eta := \frac{\lambda}{\epsilon}$$

$$M(\emptyset) := \mathrm{root}\left[M^2 + \lambda\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\emptyset^2}{2} - \epsilon\cdot\frac{\emptyset^4}{24} - \eta\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$

$$T := 1000$$

$$y(\emptyset) := \frac{M(\emptyset)}{T}$$

$$G(\emptyset) := \frac{-M(\emptyset)}{4\cdot\pi} + \frac{1}{\pi}\cdot\ln(1 - \exp(-y(\emptyset)))$$

$$F(\emptyset) := m^2\cdot\frac{\emptyset^4}{24\cdot\eta} - \lambda\cdot\frac{\emptyset^4}{12} - 29\cdot\epsilon\cdot\frac{\emptyset^6}{6!}$$

$$tr(\emptyset) := -m^2 + \lambda\cdot\frac{\emptyset^2}{2}$$

$$mass(\emptyset) := tr(\emptyset)^2 + \lambda\cdot\frac{G(\emptyset)}{2} + \eta\cdot\frac{G(\emptyset)^2}{8}$$

Finite temperature
Behaviour

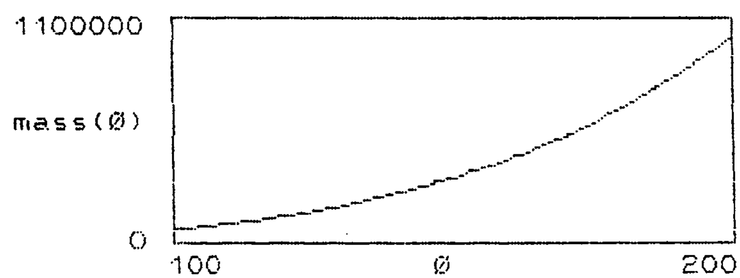


Fig. 11

$$\varnothing := 0\,..50$$

$$\lambda := .05$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .05$$

$$n := \frac{\lambda}{\epsilon}$$

$$M(\varnothing) := \mathrm{root}\left[M^2 + \lambda\cdot\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\varnothing^2}{2} - \epsilon\cdot\frac{\varnothing^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$

$$T := 10000$$

$$y(\varnothing) := \frac{M(\varnothing)}{T}$$

$$G(\varnothing) := \frac{-M(\varnothing)}{4\cdot\pi} + \frac{1}{\pi}\cdot\ln(1 - \exp(-y(\varnothing)))$$

$$F(\varnothing) := m^2\cdot\frac{\varnothing^4}{24\ n} - \lambda\cdot\frac{\varnothing^4}{12} - 29\cdot\epsilon\cdot\frac{\varnothing^6}{6!}$$

$$tr(\varnothing) := -m^2 + \lambda\cdot\frac{\varnothing^2}{2}$$

$$mass(\varnothing) := tr(\varnothing)^2 + \lambda\cdot\frac{G(\varnothing)}{2} + \epsilon\cdot\frac{G(\varnothing)^2}{8}$$

Finite temperature
Behaviour

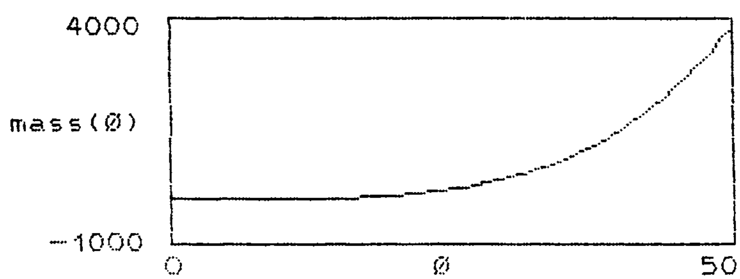


Fig. 12

$$\emptyset := 0 \,..50$$

$$\lambda := .05$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .05$$

$$n := \frac{\lambda}{\epsilon}$$

$$M(\emptyset) := \mathrm{root}\left[M^2 + \lambda\cdot\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\emptyset^2}{2} - \epsilon\cdot\frac{\emptyset^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$

$$T := 100000$$

$$y(\emptyset) := \frac{M(\emptyset)}{T}$$

$$G(\emptyset) := \frac{-M(\emptyset)}{4\cdot\pi} + \frac{1}{\pi}\cdot\ln(1 - \exp(-y(\emptyset)))$$

$$F(\emptyset) := m^2\cdot\frac{\emptyset^4}{24\cdot n} - \lambda\cdot\frac{\emptyset^4}{12} - 29\cdot\epsilon\cdot\frac{\emptyset^6}{6!}$$

$$tr(\emptyset) := -m^2 + \lambda\cdot\frac{\emptyset^2}{2}$$

$$mass(\emptyset) := tr(\emptyset)^2 + \lambda\cdot\frac{G(\emptyset)}{2} + \epsilon\cdot\frac{G(\emptyset)^2}{8}$$

Finite temperature Behaviour

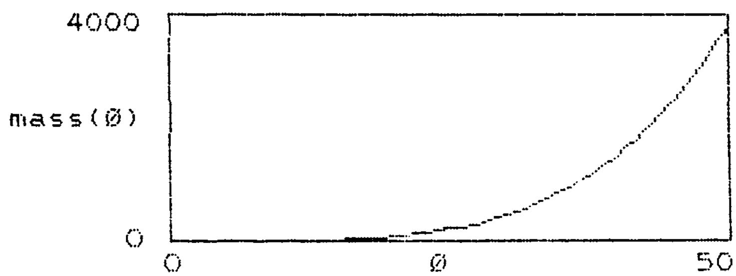


Fig. 13

$$\emptyset := 0 \,..\, 50$$

$$\lambda := .25$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .25$$

$$\eta := \frac{\lambda}{\epsilon}$$

$$M(\emptyset) := \mathrm{root}\left[M^2 + \lambda\cdot\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\emptyset^2}{2} - \epsilon\cdot\frac{\emptyset^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$

$$T := 100000$$

$$y(\emptyset) := \frac{M(\emptyset)}{T}$$

$$G(\emptyset) := \frac{-M(\emptyset)}{4\cdot\pi} + \frac{1}{\pi}\cdot\ln(1 - \exp(-y(\emptyset)))$$

$$F(\emptyset) := m^2\cdot\frac{\emptyset^4}{24\cdot\eta} - \lambda\cdot\frac{\emptyset^4}{12} - 29\cdot\epsilon\cdot\frac{\emptyset^6}{6!}$$

$$tr(\emptyset) := -m^2 + \lambda\cdot\frac{\emptyset^2}{2}$$

$$mass(\emptyset) := tr(\emptyset)^2 + \lambda\cdot\frac{G(\emptyset)}{2} + \epsilon\cdot\frac{G(\emptyset)^2}{8}$$

Finite temperature Behaviour

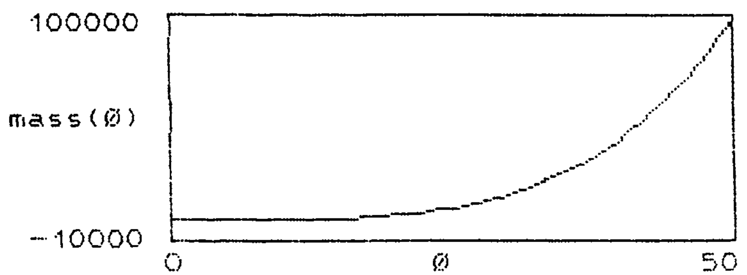


Fig. 14

$$\varnothing := 0\,..50$$

$$\lambda := .5$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .5$$

$$\eta := \frac{\lambda}{\epsilon}$$

$$M(\varnothing) := \mathrm{root}\left[M^2 + \lambda\cdot\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\varnothing^2}{2} - \epsilon\cdot\frac{\varnothing^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$

$$T := 100000$$

$$y(\varnothing) := \frac{M(\varnothing)}{T}$$

$$G(\varnothing) := \frac{-M(\varnothing)}{4\cdot\pi} + \frac{1}{\pi}\cdot\ln(1 - \exp(-y(\varnothing)))$$

$$F(\varnothing) := m^2\cdot\frac{\varnothing^4}{24\cdot\eta} - \lambda\cdot\frac{\varnothing^4}{12} - 29\cdot\epsilon\cdot\frac{\varnothing^6}{6!}$$

$$tr(\varnothing) := -m^2 + \lambda\cdot\frac{\varnothing^2}{2}$$

$$mass(\varnothing) := tr(\varnothing)^2 + \lambda\cdot\frac{G(\varnothing)}{2} + \epsilon\cdot\frac{G(\varnothing)^2}{8}$$

Finite temperature Behaviour

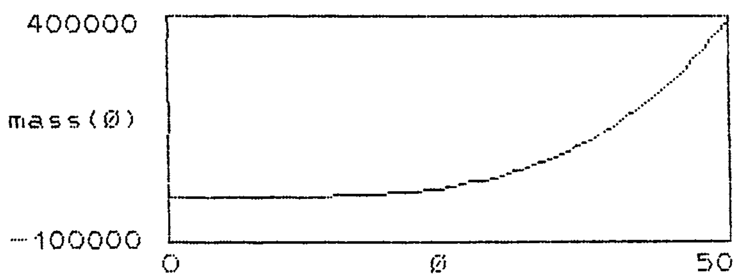


Fig. 15

$$\varnothing := 25 \,..\, 100$$

$$\lambda := .05$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .05$$

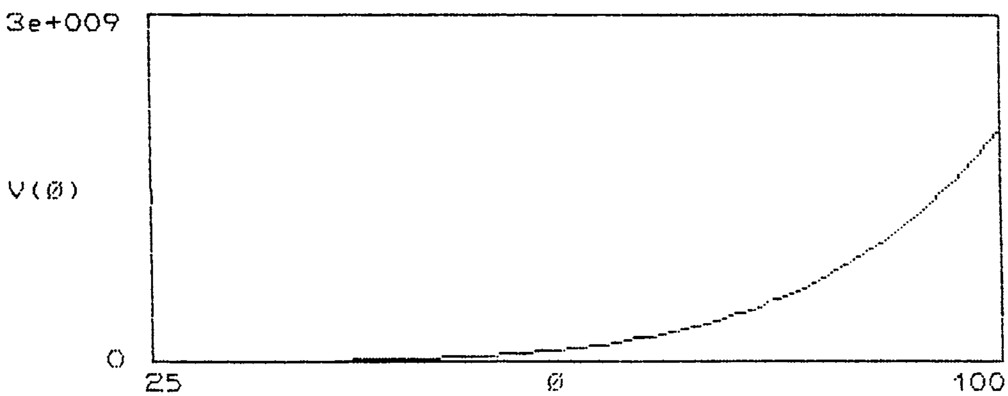
$$\eta := \frac{\lambda}{\epsilon}$$

$$M(\varnothing) := \mathrm{root}\left[M^2 + \lambda\cdot\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\varnothing^2}{2} - \epsilon\cdot\frac{\varnothing^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$

$$G(\varnothing) := \frac{-M(\varnothing)}{4\cdot\pi}$$

$$F(\varnothing) := m^2\cdot\frac{\varnothing^4}{24\cdot\eta} - \lambda\cdot\frac{\varnothing^4}{12} - 29\cdot\epsilon\cdot\frac{\varnothing^6}{6!}$$

$$(\varnothing) := \frac{-M(\varnothing)^3}{6\cdot\pi} + \frac{\lambda}{8}\cdot\left[\varnothing^2 - \left[2\cdot\frac{m^2}{\lambda}\right]^2\right] - \eta\cdot m^2\cdot\frac{\varnothing^4}{24} - \lambda\cdot\frac{G(\varnothing)^2}{8} - \eta\cdot\frac{G(\varnothing)^3}{48} - F(\varnothing)$$



Effective Potential Vs Ø

Fig. 16

$$Ø := -100 \,..\, 100$$

$$\lambda := .05$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .05$$

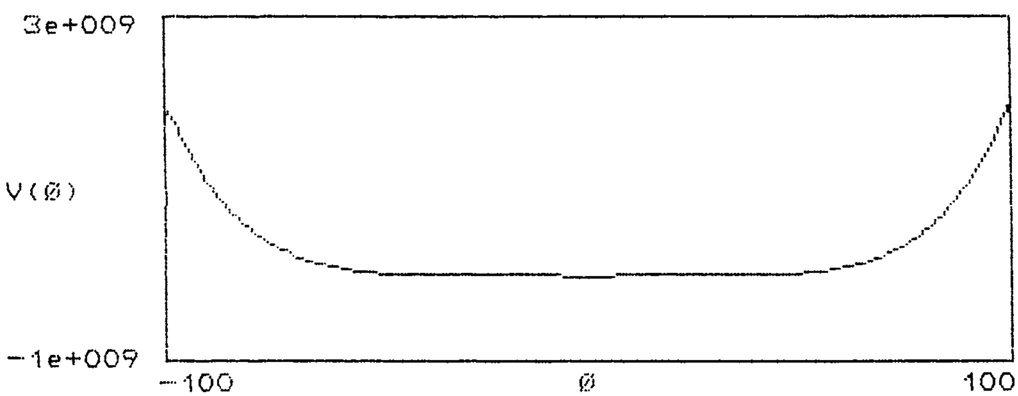
$$\eta := \frac{\lambda}{\epsilon}$$

$$M(Ø) := \mathrm{root}\left[M^2 + \lambda\cdot\frac{M}{8\ \pi} + m^2 - \lambda\cdot\frac{Ø^2}{2} - \epsilon\cdot\frac{Ø^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$

$$G(Ø) := \frac{-M(Ø)}{4\cdot\pi}$$

$$F(Ø) := m^2\cdot\frac{Ø^4}{24\cdot\eta} - \lambda\cdot\frac{Ø^4}{12} - 29\cdot\epsilon\cdot\frac{Ø^6}{6!}$$

$$Ø) := \frac{-M(Ø)^3}{6\cdot\pi} + \frac{\lambda}{8}\cdot\left[Ø^2 - \left[2\cdot\frac{m}{\lambda}\right]^2\right]^2 - \eta\cdot m^2\cdot\frac{Ø^4}{24} - \lambda\cdot\frac{G(Ø)^2}{8} - \eta\cdot\frac{G(Ø)^3}{48} - F(Ø)$$



Effective Potential Vs Ø

Fig. 17

$$\varnothing := 25 .. 100$$

$$\lambda := .5$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .5$$

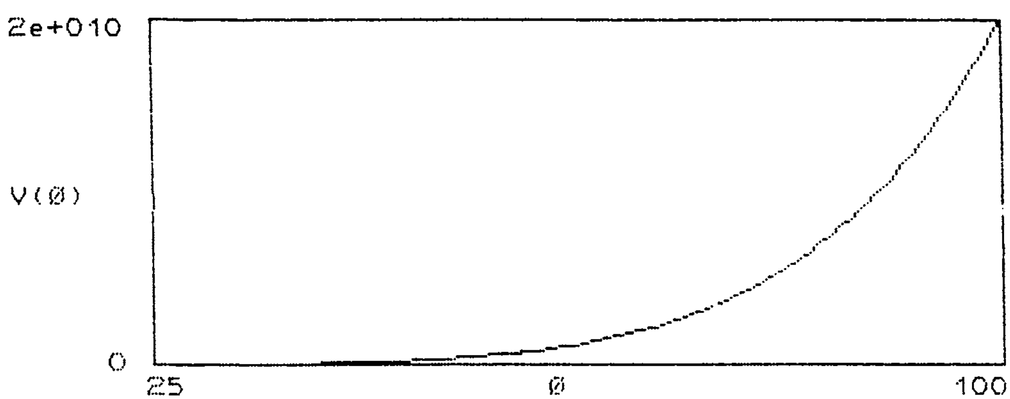
$$\eta := \frac{\lambda}{\epsilon}$$

$$M(\varnothing) := \mathrm{root}\left[M^2 + \lambda \cdot \frac{M}{8 \cdot \pi} + m^2 - \lambda \cdot \frac{\varnothing^2}{2} - \epsilon \cdot \frac{\varnothing^4}{24} - \epsilon \cdot \frac{M^2}{128 \cdot \pi^2}, M\right]$$

$$G(\varnothing) := \frac{-M(\varnothing)}{4 \cdot \pi}$$

$$F(\varnothing) := m^2 \cdot \frac{\varnothing^4}{24 \cdot \eta} - \lambda \cdot \frac{\varnothing^4}{12} - 29 \cdot \epsilon \cdot \frac{\varnothing^6}{6!}$$

$$(\varnothing) := \frac{-M(\varnothing)^3}{6 \cdot \pi} + \frac{\lambda}{8} \cdot \left[\varnothing^2 - \left[2 \cdot \frac{m^2}{\lambda}\right]^2\right] - \eta \cdot m^2 \cdot \frac{\varnothing^4}{24} - \lambda \cdot \frac{G(\varnothing)^2}{8} - \eta \cdot \frac{G(\varnothing)^3}{48} - F(\varnothing)$$



Effective Potential Vs Ø

Fig. 18

$$\emptyset := 0 \,..50$$

$$\lambda := .5$$

$$m := 1$$

$$M := 3 + 2i$$

$$\epsilon := .5$$

$$\eta := \frac{\lambda}{\epsilon}$$

$$M(\emptyset) := \operatorname{root}\left[M^2 + \lambda\cdot\frac{M}{8\cdot\pi} + m^2 - \lambda\cdot\frac{\emptyset^2}{2} - \epsilon\cdot\frac{\emptyset^4}{24} - \epsilon\cdot\frac{M^2}{128\cdot\pi^2}, M\right]$$

$$T := 100000$$

$$y(\emptyset) := \frac{M(\emptyset)}{T}$$

$$G(\emptyset) := \frac{-M(\emptyset)}{4\cdot\pi} + \frac{1}{\pi}\cdot\ln(1 - \exp(-y(\emptyset)))$$

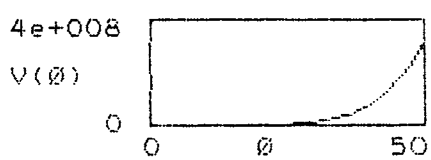
$$F(\emptyset) := m^2\cdot\frac{\emptyset^4}{24\cdot\eta} - \lambda\cdot\frac{\emptyset^4}{12} - 29\cdot\epsilon\cdot\frac{\emptyset^6}{6!}$$

$$tr(\emptyset) := -m^2 + \lambda\cdot\frac{\emptyset^2}{2}$$

$$mass(\emptyset) := tr(\emptyset)^2 + \lambda\cdot\frac{G(\emptyset)}{2} + \epsilon\cdot\frac{G(\emptyset)^2}{8}$$

$$V(\emptyset) := \frac{\lambda}{8}\cdot\left[\emptyset^2 - 2\cdot\frac{m^2}{\lambda}\right]^2 - \frac{\epsilon}{24}\cdot m^2\cdot\emptyset^4 - \frac{\lambda}{8}\cdot G(\emptyset)^2 - \frac{\epsilon}{48}\cdot G(\emptyset)^3 - F(\emptyset)$$

Fig. 19.a



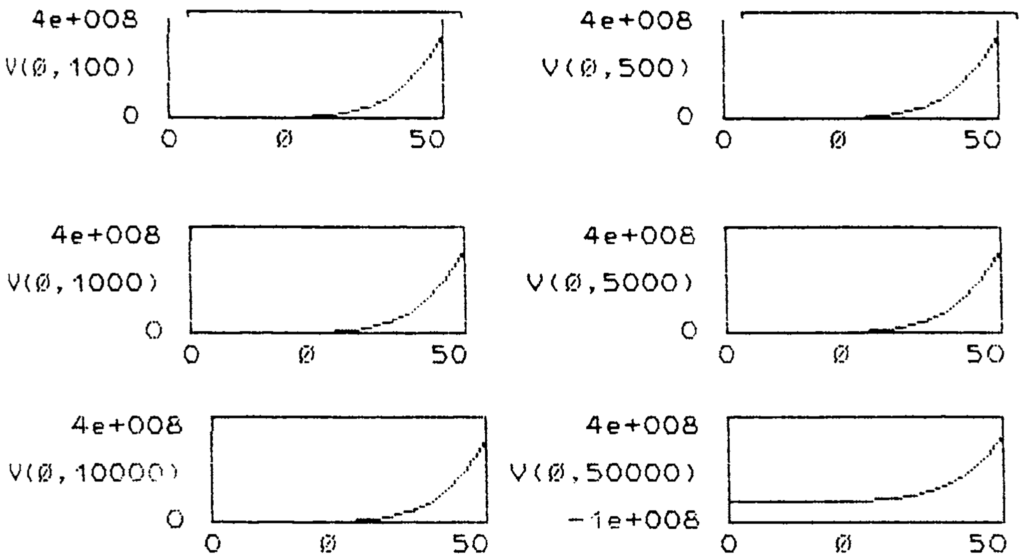


Fig. 19.b

Effective Potential Graph at Finite Temperature

Chapter 3

CJT Formalism at Finite Chemical potential

3.1 Introduction

Recently there has been considerable interest in the effects of finite charge on a quantum field theory. The phase structure ,effective potential at fixed charge and Bose-Einstein condensation have been recently analyzed [60–68]. The analysis is important because the nature of the phase transitions will be affected by the presence of of various conserved charges. Ipso facto the value of the critical temperature of phase transition at which the scalar boson effective mass vanishes will be considerably altered. In this chapter we extend the finite temperature CJT formalism [chapter 2] for scalar fields, to charged scalar field so as to include the effects of conserved charge, through the introduction of chemical potential μ [61,62,63,67,69].

Charges can be bosonic or non-bosonic. In the latter case scalar bosons do not carry charge. The only way in which the chemical potential μ can contribute to the effective mass is through the fermion loops. Effect on critical temperature is very small because of the smallness of the Yukawa couplings [67,68]. It has been shown that for bosonic charges, the chemical potential has striking effects on the critical temperature and the phase structure. Analysis of the problem has been carried out both by using perturbative as well as non-perturbative methods [60,64,65]. For simplicity we consider only a charged scalar field with a chemical potential. Extension to several component fields is straight forward.

An earlier analysis of the phase structure of the theory does not offer an expression for one loop effective potential [61,63]. In ref [64] one loop effective potential is obtained and the consequences are studied. It is hoped that CJT formalism will yield a much more satisfactory picture of the phase structure of the theory.

3.2 Formalism

We consider the O(2) invariant Hamiltonian density ($\mathcal{H}$) for a charged scalar field Φ_a (a=1,2) which is given by

$$\mathcal{H} = \frac{1}{2}\pi_a\pi_a + \frac{1}{2}(\nabla\Phi_a).(\nabla\Phi_a) + \frac{1}{2}m^2\Phi_a\Phi_a + \frac{\lambda}{4!}(\Phi_a\Phi_a)^2 \tag{3.1}$$

The conserved charge Q is given by

$$Q = \int d^3x(\Phi_1\pi_2 - \Phi_2\pi_1) \tag{3.2}$$

In the above equations π represents the canonically conjugate momentum. Introduction of the chemical potential μ is done by computing the net back ground charge in the grand canonical ensemble. The grand canonical partition function is defined to be

$$Z(\beta,\mu) \equiv tr[\exp[-\beta(\mathcal{H}' - \mu Q)]] \tag{3.3}$$

The μ dependent Hamiltonian density is

$$\mathcal{H}(\mu) = \mathcal{H} - \mu(\phi_1\pi_2 - \phi_2\pi_1) \tag{3.4}$$

CJT formalism considers this partition function as the generating functional for finite temperature Green's functions of non local composite fields. $\mathcal{H}'$ includes the effect of sources J and K, and the path integral representation takes the form (suppressing an irrelevant normalization constant)

$$Z_{\beta,\mu}(J,K) = \int \mathcal{D}\Phi_1\mathcal{D}\Phi_2 \exp\left[-I(\Phi,\mu,J,K)\right] \tag{3.5}$$

$$\begin{aligned} I(\Phi,J,K) = I(\Phi,\mu) + \int d^4x \left[\Phi_1(x)J_1(x) + \Phi_2(x)J_2(x)\right. \\ +\frac{1}{2}\left[\int d^4(x)d^4(y)\left[\Phi_1(x)K_1(x,y)\Phi_1(y)\right] + \right. \\ \left.\int d^4(x)d^4(y)\left[\Phi_2(x)K_2(x,y)\Phi_2(y)\right]\right] \end{aligned} \tag{3.6}$$

$$\begin{aligned} I(\Phi,\mu) = \int d^4(x)d^4(y)\left[\Phi_1(x)D_{0,\mu}^{-1}(x-y)\Phi_1(y)\right] + \\ \int d^4(x)d^4(y)\left[\Phi_2(x)D_{0,\mu}^{-1}(x-y)\Phi_2(y)\right] \\ +\mu[\Phi_1\pi_2 - \pi_2\Phi_1] + I_{int}(\Phi) \end{aligned} \tag{3.7}$$

$$I_{int}(\Phi) = \int d^4x \mathcal{L}_{int}(x) \tag{3.8}$$

$$D_{0,\mu}^{-1}(x-y) \equiv -[\Box + m^2 - \mu^2]\delta^4(x-y) \tag{3.9}$$

Using Hamilton's equations we get [61,65]

$$\mu(\Phi_1\pi_2 - \Phi_2\pi_1) = \mu(\Phi_1\dot{\Phi}_2 - \Phi_2\dot{\Phi}_1) \tag{3.10}$$

Presence of the chemical potential μ modifies the classical potential and the μ dependent Lagrangian can be written as

$$\mathcal{L}(\mu) = \pi_a\dot{\Phi}_a - \mathcal{H}(\mu) = \frac{1}{2}\partial_\mu\Phi_a\partial^\mu\Phi_a - V(\Phi) + \mu(\Phi_1\partial_0\Phi_2 + \Phi_2\partial_0\Phi_1) \tag{3.11}$$

$$V(\Phi) = \frac{1}{2}(m^2 - \mu^2)\Phi_a\Phi_a + \frac{\lambda}{4!}(\Phi_a\Phi_a)^2 \tag{3.12}$$

3.3 Effective Potential

Quantum corrections to the classical potential defined above can be obtained by means of the effective potential evaluated in the CJT approach. The expression for the effective action is obtained as a double Legendre transform of ln Z, and eliminating J and K in favor of ϕ and G [chapter 1,Section 2].The functional operator which is the propagator of the interacting theory in the presence of chemical potential is (a,b=1,2)

$$D^{-1}_{ab}(\phi; x, y) = -[\Box + m^2 - \mu^2 + \frac{\lambda}{6}(\phi^2(x))]\delta_{a,b}\delta^4(x-y) \\ -\frac{\lambda}{3}\phi_a(x)\phi_b(x)\delta^4(x-y) \qquad (3.13)$$

The vertices of the shifted theory give the interaction part of the Lagrangian

$$\mathcal{L}_{int}(\phi, \Phi) = -\frac{\lambda}{6}\phi_a(x)\Phi_a(x)\Phi^2(x) + \frac{\lambda}{4!}\Phi^4(x) \qquad (3.14)$$

where $\Phi^2 = \Phi_a\Phi_a$

Presence of the chemical potential will not alter the diagrams to be counted which are obtained by Hartee-Fock approximation [Chapter 2 Fig 2]. The series expansion for the effective action is

$$\Gamma_{\beta,\mu}(\phi, G) = I_{class}(\phi, \mu) + \frac{1}{2} tr \ln D_0 G^{-1} + \frac{1}{2}(D^{-1}G - 1) + \\ \frac{\lambda}{4!}\int d^4x\, [G_{aa}(x,x)G_{bb}(x,x) + 2G_{ab}(x,x)G_{ba}(x,x)] \qquad (3.15)$$

The integral sign also includes the summation over discrete frequencies. By stationarizing this effective action with respect to the propagator G, the gap equation can be written as

$$G^{-1}_{ab}(x,y) = D^{-1}_{ab}[\phi; x, y] + \frac{\lambda}{6}\left[\delta_{ab}G_{cc}(x,x) + 2G_{ab}(x,x)\right]\delta^4(x-y) \quad (3.16)$$

For translation invariant field theories with constant classical background field the effective potential can be derived from the effective action [chapter 1]. The definition of the fourier transformed propagator used in chapter 2 [eqns. 2.7 and 2.8] can be applied here also taking care of the fact that as a consequence of the introduction of the chemical potential the mode frequency will have two values $(k^2+M^2)^{\frac{1}{2}} \stackrel{+}{-} \mu$. The tree level mass matrices are given by

$$m^2_{ab} = (m^2 - \mu^2 + \frac{\lambda}{6}\phi^2)\delta_{ab} + \frac{\lambda}{3}\phi_a\phi_b \quad (3.17)$$

$$D_{ab}(k) = \int \frac{d^4k}{2\pi^4} D_{ab}(x-y)\exp(ik(x-y))$$
$$= \frac{1}{\delta_{ab}k^2 + m^2_{ab}} \quad (3.18)$$

The ansatz for the full propagator G is fixed in terms of an effective mass matrix M_{ab}

$$\begin{aligned} G_{ab}(k) &= \int \frac{d^4k}{2\pi^4} G_{ab}(x-y)\exp(ik(x-y) \\ &= \frac{1}{\delta_{ab}k^2 + M^2_{ab}} \end{aligned} \tag{3.19}$$

To simplify the analysis we assume that the effective mass associated with both the component fields is $M(\phi)$. The shift associated with one of the fields is assumed to vanish ($\phi_2 = 0, \phi_1 = 1$) [60]. The μ dependent effective potential is then given by

$$\begin{aligned} V+\beta,\mu(\phi, M) &= \frac{1}{2}(m_B^2 - \mu^2)\phi^2 + \frac{\lambda_B}{4!}\phi^4 + \\ \int \frac{d^\nu k}{2\pi^\nu} &\ln(k^2 + M^2(\phi)) - \frac{1}{2}[M^2(\phi) - m_B^2 - \frac{\lambda_B}{2}\phi^2]G(x,x) \\ &+ \frac{\lambda_B}{8}G(x,x)G(x,x) \end{aligned} \tag{3.20}$$

3.4 Effective Mass

Stationary requirements with respect to ϕ and M^2 lead to

$$0 = \frac{\delta V}{\delta\phi} = \phi\left[m_B^2 - \mu^2 + \frac{\lambda_B}{6}\phi^2 + \frac{\lambda_B}{2}G(x,x)\right] \tag{3.21}$$

$$0 = \frac{\partial V}{\partial M^2} = -\frac{1}{2}\left[m^2 - m_B^2 - \frac{\lambda_B}{2}\phi^2 - \frac{\lambda_B}{2}G(x,x)\right]\frac{\partial G(x,x)}{\partial M^2} \qquad (3.22)$$

$$M^2(\phi) = m_B^2 + \frac{\lambda_B}{2}\phi^2 + \frac{\lambda_B}{2}G(x,x) \qquad (3.23)$$

It has been shown in chapter 2 [eqn. 2.10] that non zero turning point is absent in Φ^4 theory at zero temperature for negative λ. In the presence of chemical potential

$$V_\mu(\phi) = \phi\left[M^2(\phi) - \mu^2 - \frac{\lambda_B}{3}\phi^2\right] \qquad (3.24)$$

Here $m^2 - \mu^2$ is not positive-definite, and hence, SSB is possible for negative,positive and zero values of the coupling constant λ. Thus the presence of μ^2 considerably alters the nature of the phase transition. At zero temperature, considering the classical potential we note the following. Normally SSB take place only for $M^2 < 0$. finite temperature correction increases this mass parameter, and at the critical temperature T_c, $m^2 = 0$, and symmetry is restored. The presence of μ^2 decreases the magnitude of m^2 and critical temperature increases. SSB can take place for positive m^2 also when $m^2 < \mu^2$.

In terms of the effective mass the static effective potential is

$$V_{eff} = V_{class} + \frac{1}{2}\int\frac{d^\nu k}{2\pi^\nu}\ln(k^2 + M^2(\phi)) - \frac{\lambda_B}{4}G(x,x)G(x,x) \qquad (3.25)$$

3.5 Renormalization

Performing the summation in the time coordinate considering the effect of μ we can write [60,69].

$$G(x,x) = \int \frac{d^\nu k}{2\pi^\nu} \frac{1}{E} + \frac{2}{\beta} \int \frac{d^\nu k}{2\pi^\nu} \frac{1}{E}$$

$$x \left[\frac{1}{\exp(\beta(E+\mu)-1} + \frac{1}{\exp(\beta(E-\mu)-1} \right] \tag{3.26}$$

$$V_\mu^1 = \int \frac{d^\nu k}{2\pi^\nu} E + \frac{1}{\beta} \int \frac{d^\nu k}{2\pi^\nu}$$

$$\left[\ln\left(1 - e^{\beta(E+\mu)}\right) + \ln\left(1 - e^{\beta(E-\mu)}\right) \right] \tag{3.27}$$

V_μ^1 gives the familiar one loop effective potential at finite chemical potential with the mass term replaced by the effective mass $M(\phi)$.

We find that the introduction of the chemical potential does not introduce any new divergences in the theory. The renormalization procedure employed in ref[7] can be used here also to remove the divergences. Introduction of μ alters the finite part of the propagator G(x,x) [24]. To avoid confusion we denote the renormalisation scale by η instead of μ. In (3+1) dimensions

$$G(M(\phi)) = \frac{M^2(\phi)}{16\pi^2} \ln\left(\frac{M^2(\phi)}{\eta^2}\right)$$
$$+\frac{2}{\beta}\int\frac{d^3k}{2\pi^3}\frac{1}{E}\left[\frac{1}{\exp(\beta(E+\mu)-1}+\frac{1}{\exp(\beta(E-\mu)-1}\right] \tag{3.28}$$

At T=0 the first term alone will survive. The finite cut-off independent effective mass is obtained as

$$M^2(\phi) = \overset{+}{-} m_r^2 + \frac{\lambda_r}{2}\phi^2 + \frac{\lambda_r}{2}G(M(\phi)) \tag{3.29}$$

The finite expression for effective mass is written in the same form as that of the theory with $\mu = 0$. But the two expressions are not identical because of the difference in the definition of G(x,x). Similarity in form permits exactly similar calculations. The final expression for the effective potential is

$$V_{\beta,\mu}(\phi, M(\phi)) = \frac{M^4}{2\lambda_r} - \frac{1}{2}M^2G(M) + \frac{M^4}{64\pi^2}\left[\ln\frac{M^2}{\eta^2} - \frac{1}{2}\right]$$
$$+\frac{1}{\beta}\int\frac{d^3k}{2\pi^3}\ln(1-\exp(\beta(E+\mu))) + \ln(1-\exp(\beta(E-\mu))) -$$
$$\frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{12}\phi^4 \tag{3.30}$$

The unrenormalized λ term does not get cancelled even with the introduction of the chemical potential. The integral appearing in the definition of G(M(ϕ)) is difficult to be evaluated analytically. We evaluated the integral numerically [Table 1]

3.6 (2+1) Dimension

Expression for the effective potential in (2+1) dimensions can be easily obtained by using the results of chapter 2 [eqn. 2.9] by putting $\xi = 0$. After appropriate modification in the definition of $G(M(\phi))$ we get

$$G(M(\phi)) = -\frac{M(\phi)}{4\pi} + \frac{2}{\beta}\int \frac{d^2k}{2\pi^2}\frac{1}{E} \times \left[\frac{1}{\exp(\beta(E+\mu)-1} + \frac{1}{\exp(\beta(E-\mu)-1}\right] \tag{3.31}$$

Integrals appearing in the above equation have been evaluated earlier [60].

$$\frac{2}{\beta}\int\frac{d^2k}{2\pi^2}\frac{1}{E}\times$$
$$\left[\frac{1}{\exp(\beta(E+\mu)-1}+\frac{1}{\exp(\beta(E-\mu)-1}\right]=G(M)=$$
$$-\frac{M}{4\pi}+\frac{1}{2\pi}\left[\frac{1}{\beta^2}\ln\beta^2(M^2-\mu^2)-M+\frac{1}{12}(m^2+\mu^2)\right] \tag{3.32}$$

$$\frac{1}{\beta}\int\frac{d^2k}{2\pi^2}\ln(1-\exp(\beta(E+\mu))+\ln(1-\exp(\beta(E-\mu))$$
$$=-\frac{1}{2\pi}\left[\frac{M^2-\mu^2}{2\beta^2}\ln\beta^2(\omega^2-\mu^2)+\frac{3\mu^2-M^2}{2\beta^2}\right.$$
$$\left.-\frac{M^3}{3\beta}+\frac{M(M^3+3\mu^2)}{36}-\frac{1}{288}(M+\mu)^4+(M-\mu)^4+\frac{2\zeta(3)}{\beta^4}\right) \tag{3.33}$$

Where ζ represents Riemann's zeta function. Using the above equations n a finite expression for effective potential is obtained.

Charge density is defined as the negative derivative of the free energy with respect to the chemical potential [60,65,69]:

$$\rho=-\frac{\partial G(x,x)}{\partial\mu}+\mu\phi^2 \tag{3.34}$$

It has been shown earlier that [60]

$$\frac{\partial}{\partial\mu}G(x,x) = -\frac{\mu}{2\pi}\left[\frac{2}{\beta} - \frac{\ln\beta^2}{\beta}(M^2-\mu^2) + \frac{1}{36}(3M^2-\mu^2)\beta\right] \tag{3.35}$$

In the above equation terms are taken up to first order in β.

$$\rho = -\frac{\mu}{2\pi}\left[\frac{2}{\beta} - \frac{\ln\beta^2}{\beta}(M^2-\mu^2) + \frac{1}{36}(3M^2-\mu^2)\beta\right] + \mu\phi^2 \tag{3.36}$$

3.7 Conclusions

Extending the CJT formalism at finite temperature to include finite chemical potential,the effective potential for a charged scalar field is evaluated. Our study shows that the inclusion of μ alters the phase structure of the theory. In (2+1) dimensions the effective potential could be evaluated in a closed form. Charge density also is calculated.

Numerical evaluation of the effective mass yields a graph (fig. 19) similar to that of the $\mu = 0$ model. This is only natural since when variation with respect to G is performed, all dependence on μ disappears. Graph (fig. 20–23) representing variation of V'(ϕ) for different values of λ, shows the possibility of a minimum value away from $\phi = 0$. This is an expected result and has been shown analytically [61,63]. The integral appearing in $G(M(\phi))$in (3+1) dimensions is evaluated numerically (Table–1). Numerical evaluation of the effective potential at various temperature can be obtained using these values of integrals.

Our computational ability permits only an upper limit two orders of magnitude higher than the temperature value considered. It seems that more sophisticated computational procedures will not alter the values considerably.

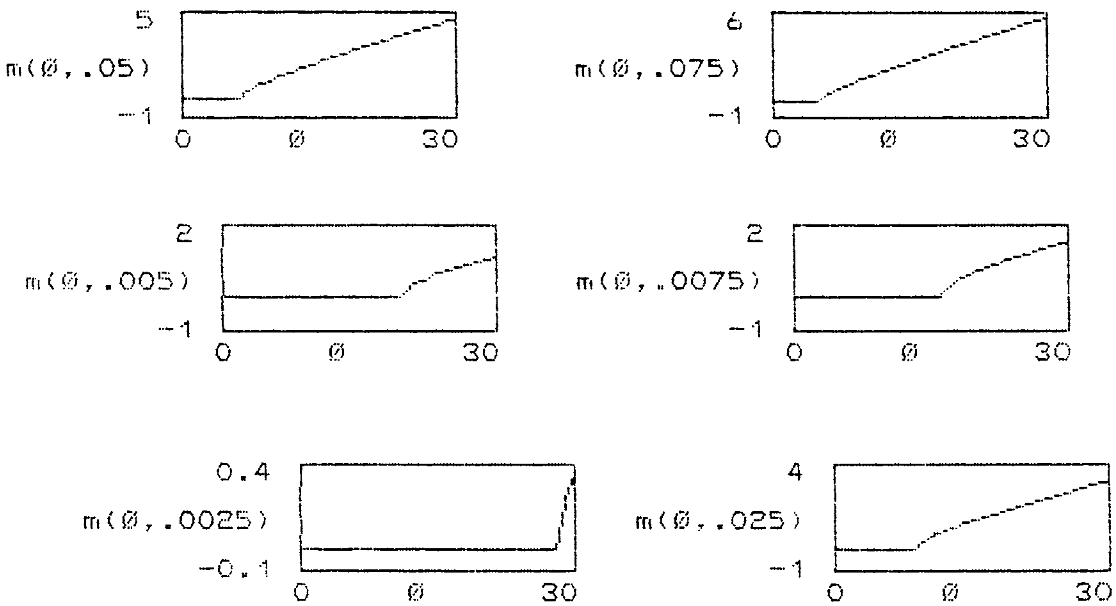
$$\emptyset := 0, .5 \,..30$$

$$\lambda := .05$$

$$n := 1$$

$$m := 3 + 2i$$

$$:= \frac{m^2}{16 \cdot \pi^2} \cdot \ln\left[\frac{m^2}{n^2}\right] \qquad m(\emptyset,\lambda) := \mathrm{root}\left[m^2 + n^2 - \frac{\lambda}{2} \cdot \emptyset^2 - \frac{\lambda}{2} \cdot G(m), m\right]$$



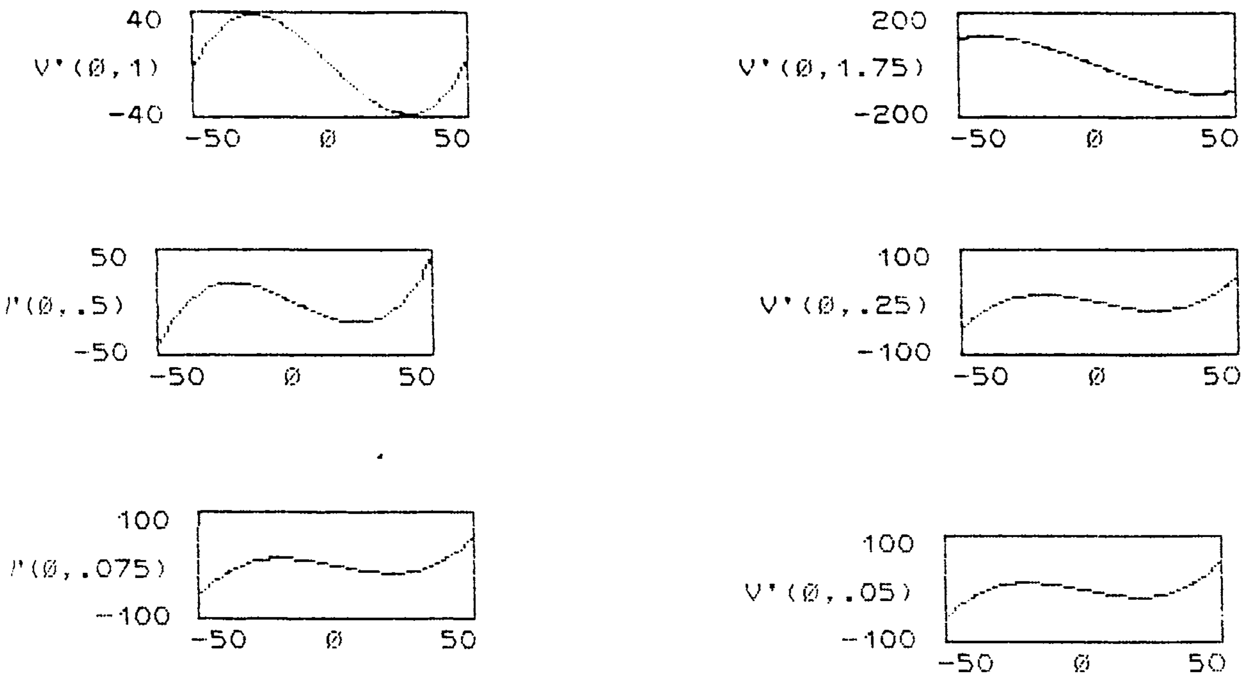
Graph for effective mass with chemical potential

Fig. 19.c

$:= -50 \,..\, 50$ $\qquad \lambda := .005 \qquad n := 1 \qquad m := 3 + 2i$

$$) := \frac{m^2}{16 \cdot \pi^2} \cdot \ln\left[\frac{m^2}{n^2}\right] \qquad m(\emptyset) := \mathrm{root}\left[m^2 + n^2 - \frac{\lambda}{2} \emptyset^2 - \frac{\lambda}{2} \cdot G(m), m\right]$$

$$,\mu) := m(\emptyset)^2 \cdot \emptyset - \mu^2 \cdot \emptyset - \frac{\lambda}{3} \cdot \emptyset^3$$



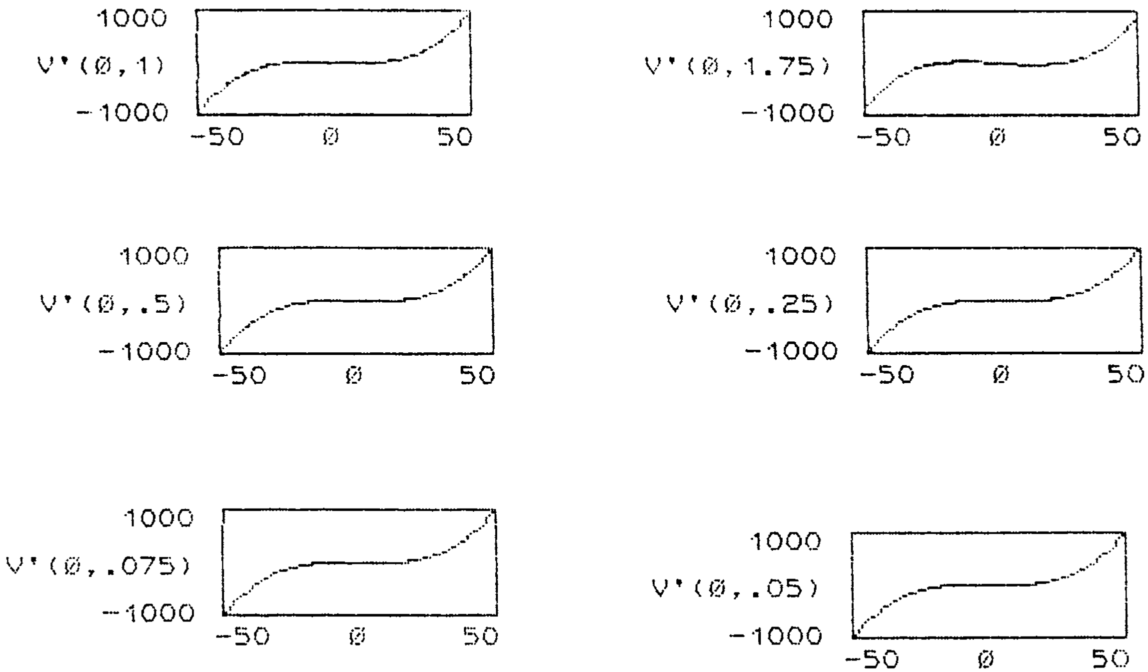
Derivative of effective potential

Fig 20

$:= -50 \,..\, 50$ $\quad \lambda := .05 \quad \mu := 1 \quad m := 3 + 2i$

$$G(m) := \frac{m^2}{16\cdot\pi^2}\cdot\ln\left[\frac{m^2}{\mu^2}\right] \qquad m(\varnothing) := \mathrm{root}\left[m^2 + \mu^2 - \frac{\lambda}{2}\cdot\varnothing^2 - \frac{\lambda}{2}\cdot G(m), m\right]$$

$$V'(\varnothing,\mu) := m(\varnothing)^2\cdot\varnothing - \mu^2\cdot\varnothing - \frac{\lambda}{3}\cdot\varnothing^3$$



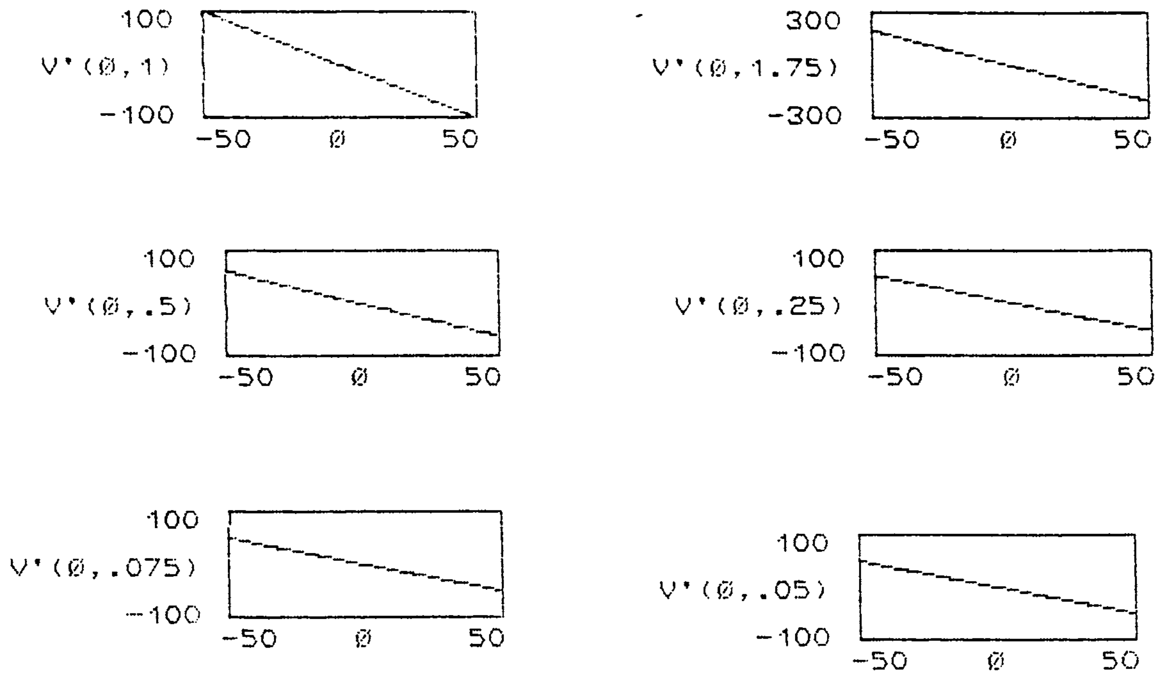
Derivative of effective potential

Fig 21

$= -50 \,..\, 50 \qquad \lambda := 0 \qquad n := 1 \qquad m := 3 + 2i$

$$G(m) := \frac{m^2}{16 \cdot \pi^2} \cdot \ln\left[\frac{m^2}{n^2}\right] \qquad m(\emptyset) := \mathrm{root}\left[m^2 + n^2 - \frac{\lambda}{2} \cdot \emptyset^2 - \frac{\lambda}{2} \cdot G(m), m\right]$$

$$V'(\emptyset, \mu) := m(\emptyset)^2 \cdot \emptyset - \mu^2 \cdot \emptyset - \frac{\lambda}{3} \cdot \emptyset^3$$



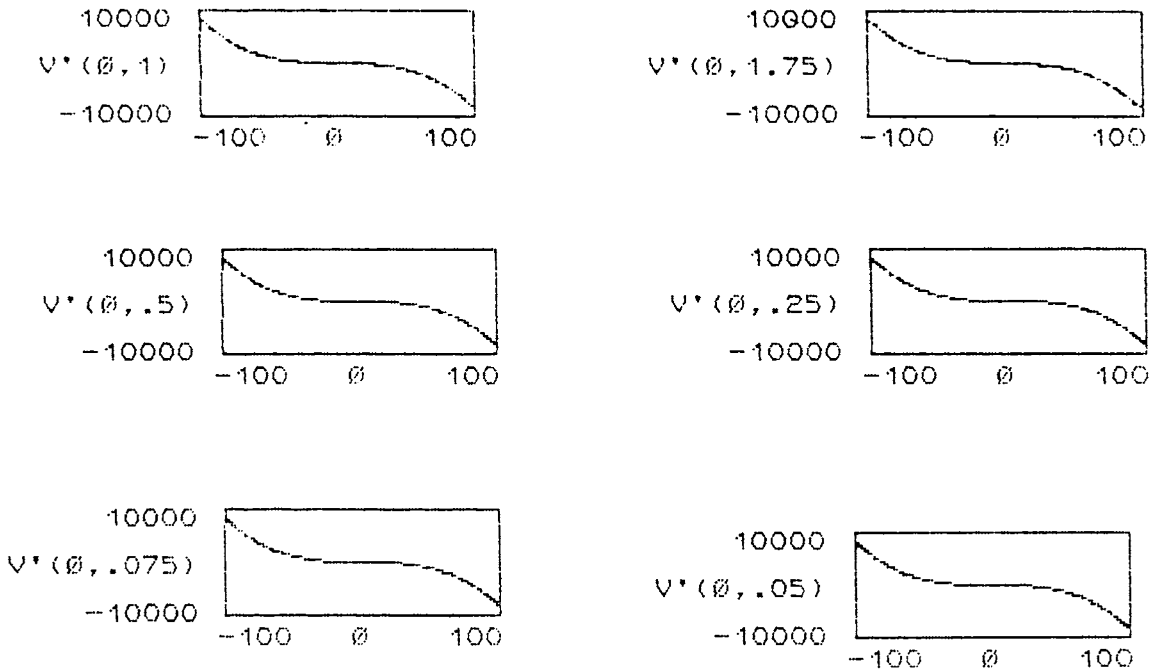
Derivative of effective potential

Fig 22

$00 \,..\, 100$ $\quad \lambda := -.05 \quad n := 1 \quad m := 3 + 2i$

$$:= \frac{m^2}{16 \cdot \pi^2} \cdot \ln\left[\frac{m^2}{n^2}\right] \qquad m(\emptyset) := \mathrm{root}\left[m^2 + n^2 - \frac{\lambda}{2} \cdot \emptyset^2 - \frac{\lambda}{2} \cdot G(m), m\right]$$

$$V'(\emptyset, \mu) := m(\emptyset)^2 \cdot \emptyset - \mu^2 \cdot \emptyset - \frac{\lambda}{3} \cdot \emptyset^3$$



Derivative of effective potential

Fig 23

$$\emptyset := -10, -8 \,..\, 10 \qquad := .05 \qquad n := 1 \quad m := 3 + 2i$$

$$G(m) := \frac{-m}{4 \cdot \pi} \qquad m(\emptyset) := \mathrm{root}\left[m^2 + n^2 - \frac{\lambda}{2} \cdot \emptyset^2 - \frac{\lambda}{2} \cdot G(m), m\right]$$

$$\beta := 10^{-5} \qquad \mu := .05 \qquad e(k) := \left[k^2 + m(\emptyset)^2\right]^{.5}$$

$$\emptyset) := \frac{2}{\beta} \cdot \int_0^{10^7} \frac{k^2}{4 \cdot \pi} \cdot \frac{1}{e(k)} \cdot \left[\frac{1}{\exp(\beta \cdot (e(k) + \mu) - 1)}\right] + \left[\frac{1}{\exp(\beta \cdot (e(k) - \mu) - 1)}\right] dk$$

Ø	I(Ø)
-10	$4.32682 \cdot 10^{14}$
-8	$4.32682 \cdot 10^{14}$
-6	$4.32682 \cdot 10^{14}$
-4	$4.32682 \cdot 10^{14}$
-2	$4.32682 \cdot 10^{14}$
0	$4.32682 \cdot 10^{14}$
2	$4.32682 \cdot 10^{14}$
4	$4.32682 \cdot 10^{14}$
6	$4.32682 \cdot 10^{14}$
8	$4.32682 \cdot 10^{14}$
10	$4.32682 \cdot 10^{14}$

le shows that the value of the egral is independent of the iation of Ø value.

Representative Numerical Values of Integrals

Table 1

$$\emptyset := -8, -6 \,..\, 8 \qquad \lambda := 0 \qquad n := 1 \quad m := 3 + 2i$$

$$G(m) := \frac{-m}{4 \cdot \pi} \qquad m(\emptyset) := \mathrm{root}\left[m^2 + n^2 - \frac{\lambda}{2} \cdot \emptyset^2 - \frac{\lambda}{2} \cdot G(m), m\right]$$

$$\beta := 10^{-5} \qquad \mu := .05 \qquad e(k) := \left[k^2 + m(\emptyset)^2\right]^{.5}$$

$$I(\emptyset) := \frac{2}{\beta} \cdot \int_0^{10^7} \frac{k^2}{4 \cdot \pi} \cdot \frac{1}{e(k)} \cdot \left[\frac{1}{\exp(\beta \cdot (e(k) + \mu) - 1)}\right] + \left[\frac{1}{\exp(\beta \cdot (e(k) - \mu) - 1)}\right] dk$$

$\emptyset$
-8
-6
-4
-2
0
2
4
6
8

$I(\emptyset)$
$4.32682 \cdot 10^{14}$
$4.32682 \cdot 10^{14}$
$4.32682 \cdot 10^{14}$
$4.32682 \cdot 10^{14}$
$4.32682 \cdot 10^{14}$
$4.32682 \cdot 10^{14}$
$4.32682 \cdot 10^{14}$
$4.32682 \cdot 10^{14}$
$4.32682 \cdot 10^{14}$

le shows that the value of the
egral is independent of the
iation of Ø value.

Representative Numerical Values of Integrals

Table 1 A

$$\emptyset := -8,-6\,..8 \qquad \lambda := -.05 \qquad n := 1 \qquad m := 3 + 2i$$

$$G(m) := \frac{-m}{4\cdot\pi} \qquad m(\emptyset) := \mathrm{root}\left[m^2 + n^2 - \frac{\lambda}{2}\cdot\emptyset^2 - \frac{\lambda}{2}\cdot G(m), m\right]$$

$$\beta := 10^{-5} \qquad \mu := .05 \qquad e(k) := \left[k^2 + m(\emptyset)^2\right]^{.5}$$

$$I(\emptyset) := \frac{2}{\beta}\cdot\int_0^{10^7} \frac{k^2}{4\cdot\pi}\cdot\frac{1}{e(k)}\cdot\left[\frac{1}{\exp(\beta\cdot(e(k)+\mu) - 1)}\right] + \left[\frac{1}{\exp(\beta\cdot(e(k)-\mu) - 1)}\right] dk$$

Ø	I(Ø)
-8	$4.32682\cdot 10^{14} - 5.23509i\cdot 10^{4}$
-6	$4.32682\cdot 10^{14} - 4.47521i\cdot 10^{4}$
-4	$4.32682\cdot 10^{14}$
-2	$4.32682\cdot 10^{14}$
0	$4.32682\cdot 10^{14}$
2	$4.32682\cdot 10^{14}$
4	$4.32682\cdot 10^{14}$
6	$4.32682\cdot 10^{14} - 4.47521i\cdot 10^{4}$
8	$4.32682\cdot 10^{14} - 5.23509i\cdot 10^{4}$

le shows that the real values of the
egral is independent of the
iation of Ø value.

Representative Numerical Values of Integrals

Table 1 B

$\emptyset := -8, -6 \,..\, 8 \qquad \lambda := .05 \qquad n := 1 \quad m := 3 + 2i$

$$G(m) := \frac{-m}{4 \cdot \pi} \qquad m(\emptyset) := \mathrm{root}\left[m^2 + n^2 - \frac{\lambda}{2}\cdot\emptyset^2 - \frac{\lambda}{2}\cdot G(m), m\right]$$

$$\beta := 10^{-5} \qquad \mu := 1 \qquad e(k) := \left[k^2 + m(\emptyset)^2\right]^{.5}$$

$$(\emptyset) := \frac{2}{\beta}\int_0^{10^7} \frac{k^2}{4\,\pi}\cdot\frac{1}{e(k)}\cdot\left[\frac{1}{\exp(\beta\cdot(e(k) + \mu) - 1)}\right] + \left[\frac{1}{\exp(\beta\cdot(e(k) - \mu) - 1)}\right] dl$$

$\emptyset$	$I(\emptyset)$
-8	$4.32678 \cdot 10^{14}$
-6	$4.32678 \cdot 10^{14}$
-4	$4.32678 \cdot 10^{14}$
-2	$4.32678 \cdot 10^{14}$
0	$4.32678 \cdot 10^{14}$
2	$4.32678 \cdot 10^{14}$
4	$4.32678 \cdot 10^{14}$
6	$4.32678 \cdot 10^{14}$
8	$4.32678 \cdot 10^{14}$

ble shows that the real values of the tegral is independent of the riation of Ø value.

Representative Numerical Values of Integrals

Table 1 C

Chapter 4

Schrődinger Picture Formalism

4.1 Introduction

Variational methods in functional Schrődinger picture have been shown to be very useful in the study of detailed structures of the quantum fields, both for the bosonic and fermionic field theories [39,42,47]. Recently various interesting applications of this formalism has been presented. In (2+1) dimensional Thirring model [52]. Gaussian approximation provides better information than the large -N approximation. it has been used to derive (2+1) dimensional effective potential in Liouville model [46]. Quantum field theoretic analysis of inflation dynamics in a (2+1) dimensional universe has been worked out using this method [71]. Taking into consideration the recent interest of the formalism in (2+1) dimensional theories we propose to apply the method to the most general renormalizable scalar theory in (2+1) dimensions that is a Φ^6 model.We find that the effective potential expression that emerges using functional Schrödinger picture is same as that derived using CJT formalism in chapter 2.

4.2 Effective Action

We propose to apply the formalism to a Φ^6 model and since Φ^6 coupling effects show up only at the three loop level we write the expression up to three loops [chapter 1,eqn. 1.41].

$$\Gamma = \int dt \Big[\int_x \hat{\Pi}\dot{\hat{\phi}} - \frac{1}{2}(\nabla\hat{\phi})^2 - V(\hat{\phi}) + \int_{x,y} \Sigma\dot{G} - 2\int_{x,y,z} \Sigma G \Sigma - \int_x \frac{1}{8}G^{-1}(x,x,t) - \frac{1}{2}\nabla_x^2 G(x,y,t)|_{x=y} + \frac{1}{2}V^{(2)}(\hat{\phi})G(x,x,t)\Big] - \frac{1}{8}V^{(4)}(\hat{\phi})\int_x G(x,x,t)^2 - \frac{1}{16}V^{(6)}(\hat{\phi})\int_x G^3(x,x,t) \tag{4.1}$$

In this chapter we follow the standard practice in the functional Schrödinger picture representing expectation values as $\hat{\phi}$ equivalent to shift ϕ used in preceding chapters. Identifying the first term as the classical action and performing variations we obtain

$$
\begin{aligned}
\frac{\delta\Gamma}{\delta\hat{\phi}(x,t)} = 0 \longrightarrow \dot{\hat{\Pi}}(x,t) &= \nabla_x^2\hat{\phi}(x,t) - V^{(1)}(\hat{\phi}) \\
&-\frac{1}{2}V^{(3)}(\hat{\phi})G(x,x,t) - \frac{1}{8}V^{(5)}(\hat{\phi})G^2(x,x,t) \qquad (4.2) \\
\frac{\delta\Gamma}{\delta\hat{\Pi}(x,t)} = 0 \longrightarrow \dot{\Sigma}(x,y,t) &+ 2\int_z \Sigma(x,z,t)\Sigma(x,y,t) \\
= \frac{1}{8}G^{-2}(x,y,t) + \Big[\frac{1}{2}\nabla_x^2 &- \frac{1}{2}V^{(2)}(\hat{\phi}) - \frac{1}{4}V^{(4)}(\hat{\phi})G(x,x,t) \\
&-\frac{1}{4}V^{(6)}(\hat{\phi})G^2(x,x,t)\Big]\delta^r(x-y) \qquad (4.3) \\
\frac{\delta\Gamma}{\delta\Sigma(x,y,t)} = 0 \longrightarrow \dot{G}(x,y,t) &= 2\Big[\int_z G(x,z,t)\Sigma(x,y,t) \\
&+\sigma(x,z,t)G(z,y,t)\Big] \qquad (4.4)
\end{aligned}
$$

4.3 Static Effective Potential

It has been shown that renormalization of time-dependent equations can be achieved along the same lines of the renormalization of the static effective potential. We therefore evaluate the static effective potential for Φ^6 model at zero temperature. This can be achieved by taking $\hat{\phi}$ to be 'x' independent and putting $\Sigma = 0$. The classical potential for the model is given in chapter 2 [eqn.2.1]. The effective potential is given by given in chapter 2. By evaluating the derivatives we get

$$
\begin{aligned}
V_{eff}(\hat{\phi}, G) = \frac{1}{2}m^2\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 + \frac{\xi}{6!}\hat{\phi}^6 \\
+ \left(\frac{1}{2}m^2 + \frac{\lambda}{4}\hat{\phi}^2 + \frac{\xi}{48}\hat{\phi}^4\right) G(x,x) \\
+ \left(\frac{\lambda}{8} + \frac{\xi}{16}\hat{\phi}^2\right) G^2(x,x) + \frac{\xi}{48}G^3(x,x) \\
+\frac{1}{8}trG^{-1}(x,x) - \frac{1}{2}\nabla_x^2 G(x,x)
\end{aligned}
\tag{4.5}
$$

By performing variation with respect to G the gap equation is obtained.

$$
\begin{aligned}
\frac{1}{4}G^{-2}(x,y) = \Bigg[-\delta_x^2 + m^2 + \frac{\lambda}{2}\hat{\phi}^2 + \frac{\xi}{24}\hat{\phi}^4 \\
+\frac{\lambda}{2}G(x,y) + \frac{\xi}{4}\hat{\phi}^2 G^2(x,y) + \frac{\xi}{48}G^3(x,y)\Bigg]\,\delta(x-y)
\end{aligned}
\tag{4.6}
$$

Assuming translation invariance the Fourier transform of a function is defined as

$$f(x) \equiv \int \frac{d^\nu k}{2\pi^\nu} e^{ikx} f(k) \tag{4.7}$$

$$G(x,x) = \int \frac{d^\nu k}{2\pi^\nu} \frac{1}{2} \left[k^2 + m^2 + \frac{\lambda}{2}\hat{\phi}^2 + \frac{\xi}{24}\hat{\phi}^4 \right] \times$$
$$\left. \frac{\lambda}{2} G(x,x) + \frac{\xi}{4}\hat{\phi}^2 G(x,x) + \frac{\xi}{48} G^2(x,x) \right]^{-\frac{1}{2}} \tag{4.8}$$

$$G(x,x) = \int \frac{d^\nu k}{2\pi^\nu} \frac{1}{2(k^2 + M^2)} \tag{4.9}$$

where an anzatz is fixed for G in terms of an effective mass [chapter 2,eqn.2.6]. The effective mass M is treated here as a variational parameter which is $\hat{\phi}$ dependent.The static effective potential can be written as

$$V_{eff}(\phi, M) = \frac{1}{2} \int \frac{d^\nu k}{2\pi^\nu} (k^2 + M^2) +$$
$$\left(\frac{1}{2} m_B^2 \hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 + \frac{\xi_B}{6!}\hat{\phi}^6 \right)$$
$$+ \frac{1}{2} \left[M^2 - m_B^2 - \frac{\lambda_B}{2}\hat{\phi}^2 - \frac{\xi_B}{24}\hat{\phi}^4 \right] G(x,x) +$$
$$+ \left[\frac{\lambda}{8} + \frac{\xi}{16}\hat{\phi}^2 \right] G^2(x,x) + \frac{\xi}{48} G^3(x,x) \tag{4.10}$$

Eqn (4.10) shows that the effective potential expression obtained here is the same as the one obtained using Gaussian effective potential approach [17]. This is only natural since when time dependence is not taken into account definitions of effective action in both the approaches coincide. From the earlier analysis [chapter 2] it also becomes clear that the formalism is equivalent to CJT approach at zero temperature . Both the equations differ by a $\xi\phi^2$ term. This term does not contribute when daisy and super daisy diagrams are considered through Hartee-fock approximation. At ϕ^4 level both the approaches are exactly identical.

Considering the first and second terms alone of eqn(4.10) it can be seen that one loop effective potential result is contained in the expression with the mass term replaced by the effective mass. Identity with the Gaussian effective potential results become more transparent if we make the following correspondence in notation.

$$G(x,x) \longrightarrow I_0 = \int \frac{d^\nu k}{2\pi^\nu} \frac{1}{2(k^2+m^2)} \tag{4.11}$$

$$\frac{1}{2} \int \frac{d^\nu k}{2\pi^\nu} (k^2+m^2) \longrightarrow I_1 \tag{4.12}$$

$$M \longrightarrow \Omega \tag{4.13}$$

4.4 Finite temperature calculation for Φ^6 model

The Hamiltonian for a Φ^6 model is given by [33,34]

$$H = \int d^3x \left(\frac{1}{2}\Pi^2(x) - \frac{1}{2}\phi(x)\nabla^2\phi(x) + \frac{1}{2}m^2\phi^2(x) + \frac{\lambda}{4!}\phi^4(x) + \frac{\xi}{6!}\phi^6(x)\right) \tag{4.14}$$

To evaluate the expectation values the following variational procedure is used. Gaussian density matrix for a free theory is obtained as an extension of density matrix described in chapter 1[eqn. (1.52). This density matrix is chosen as a test function. for the interacting theory this test function is expressed in terms of variational parameters and variation is performed with respect to them. For a free field theory density matrix (test function) is written as [33,70]

$$\rho(\phi_1\phi_2) = det^{\frac{1}{2}}\left[\omega\tanh(\frac{\beta\omega}{2})\right]\exp\left[-\frac{1}{2}\int d^3x\int d^3y\right.$$

$$[\phi_1(x)(\omega\coth(\beta\omega))(x,y)\phi_1(y)+\phi_2(x)(\omega\coth(\beta\omega))(x,y)\phi_2(y)$$
$$-2\phi_1(x)(\omega\csc h(\beta\omega)(x,y)\phi_2(y)]] \qquad (4.15)$$

In terms of the variational parameters Λ, E, Ω and b the density matrix for the interacting theory is written as

$$\rho(\phi_1,\phi_2)=N\exp\left[-\frac{1}{2}\int d^3x\int d^3y\,[\phi_1(x)\Lambda(x,y)\phi_1(y)\right.$$
$$\left.+\phi_2(x)\Lambda(x,y)\phi_2(y)-\phi_1(x)E(x,y)\phi_2(y)]\right] \qquad (4.16)$$

where

$$N=det^{1/2}\left[\Omega\tanh\left(\frac{b\Omega}{2}\right)\right] \qquad (4.17)$$

$$\Lambda(x,y)=\Omega(x,x)\coth b\Omega(x,y) \qquad (4.18)$$

$$E(x,y)=2\Omega(x,x)\csc hb\Omega(x,y) \qquad (4.19)$$

Average values are obtained for any observable $\hat{O}$ by the relation

$$\langle \hat{O} \rangle = \int \mathcal{D}\phi \hat{O}(\phi)\rho(\phi,\phi) \tag{4.20}$$

Thus

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{2}\Omega^{-1}\coth(\frac{\beta\Omega}{2})(x,y) \tag{4.21}$$

$$\begin{aligned} \langle \Pi(x)\Pi(y) \rangle &= \int \mathcal{D}\phi \frac{\partial^2 \rho(\phi,\phi')}{\partial\phi\partial\phi'}|_{\phi=\phi'} \\ &= \frac{1}{2}\Omega\coth(\frac{b\Omega}{2})(x,y) \end{aligned} \tag{4.22}$$

By evaluating the entropy $S = -\langle \rho \ln \rho \rangle$ and $\langle H \rangle$, we can calculate the Helmholtz free energy F:

$$F = U - TS \tag{4.23}$$

$$\beta F = tr\rho \ln \rho + \beta tr\rho H = \langle \ln \rho \rangle + \beta \langle H \rangle \tag{4.24}$$

The effective potential is obtained from the free energy by

$$V_{eff} \equiv \frac{F}{\int d^2x} \tag{4.25}$$

The evaluation of the entropy function and the expectation value of the Hamiltonian for a Φ^4 theory has already been given [33].Additional terms appearing in the expectation value of Φ^6 theory are a ϕ^2 dependent part and a ϕ^2 independent part. We consider only the ϕ^2 independent part because in the CJT formalism we used the Hartee Fock approximation which eliminates the ϕ^2 dependent term.

$$
\begin{aligned}
\langle H\rangle = \int d^2x \left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\xi}{6!}\phi^6\right) \\
+ \int d^2x \left[\frac{1}{4}\Omega + \frac{1}{4}\left(-\nabla^2 + m^2 + \frac{\lambda}{2}\phi^2\right)\Omega^{-1}\right] \coth\frac{b\Omega}{2}(x,x) \\
+\frac{\lambda}{32}\left(\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x)\left(\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x) + \\
+\frac{\xi}{144}\left(\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x)\left(\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x) \\
\left(\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x)
\end{aligned}
\tag{4.26}
$$

Taking the already computed value of entropy the expression for free energy can be written [33]

$$\beta F = tr \ln\left(2\sinh\frac{b\Omega}{2}\right) + \beta\int d^2x\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\xi}{6!}\phi^6\right)$$
$$+\int d^2x\left[\frac{\beta}{4}\Omega + \frac{\beta}{4}\left(-\nabla^2 + m^2 + \frac{\lambda}{2}\phi^2\right)\Omega^{-1} - \frac{1}{2}b\Omega\right]\coth\frac{b\Omega}{2}(x,x)$$
$$+\frac{\beta\lambda}{32}\left(\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x)\left(\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x) +$$
$$+\frac{\beta\xi}{144}\left(\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x)\left(\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x)$$
$$\times\left(\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x) \quad (4.27)$$

Taking into account the translation invariance of the theory and using the fourier transform expression for Effective potential is obtained.

$$\beta V_{eff} = \int\frac{d^2k}{2\pi^2}\left(2\sinh\frac{b\Omega}{2}\right) + \beta\int\frac{d^2k}{2\pi^2}\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\xi}{6!}\phi^6\right)$$
$$+\int\frac{d^2k}{2\pi^2}\left[\frac{\beta}{4}\Omega + \frac{\beta}{4}\left(-\nabla^2 + m^2 + \frac{\lambda}{2}\phi^2\right)\Omega^{-1} - \frac{1}{2}b\Omega\right]\coth\frac{b\Omega}{2}(x,x)$$
$$+\frac{\beta\lambda}{32}\left(\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x)\left(\int\frac{d^2k'}{2\pi^2}\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x) +$$
$$+\frac{\beta\xi}{144}\left(\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x)\left(\int\frac{d^2k'}{2\pi^2}\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x)$$
$$\times\left(\int\frac{d^2k''}{2\pi^2}\Omega^{-1}\coth\frac{b\Omega}{2}\right)(x,x) \quad (4.28)$$

The above equations can be varied in different ways. But we have to impose the condition that the correct free theory expression is obtained by putting $\lambda = 0$. This is true if we choose Ω and bΩ as variational parameters. First of all we vary with respect to Ω keeping bΩ fixed.This produces the following gap equation.

$$\Omega^2 = k^2 + m^2 + \frac{1}{2}\lambda\phi^2 + \frac{1}{24}\xi\phi^4 + \frac{\lambda}{4}\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\left(\frac{b\Omega}{2}\right) + \frac{\xi}{8}\left[\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\left(\frac{b\Omega}{2}\right)\int\frac{d^2k'}{2\pi^2}\Omega^{-1}\coth\left(\frac{b\Omega}{2}\right)\right] \tag{4.29}$$

Varying with respect to bΩ keeping Ω constant we get b=β where [eqn.(4.29)] is used. Thus one of the variational parameters introduced is identified as the inverse temperature. Put

$$M^2 = m^2 + \frac{1}{2}\lambda\phi^2 + \frac{1}{24}\xi\phi^4 + \frac{\lambda}{4}\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\left(\frac{b\Omega}{2}\right) \frac{\xi}{8}\left[\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\left(\frac{b\Omega}{2}\right)\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\left(\frac{b\Omega}{2}\right)\right] \tag{4.30}$$

The expression for effective mass becomes

$$M^2 = k^2 + m^2 \tag{4.31}$$

$$\ln\left(2\sinh\frac{b\Omega}{2}\right) = \ln(1 - e^{b\Omega}) \tag{4.32}$$

The expression for finite temperature effective potential is

$$\begin{aligned} V_{eff} = & \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!} + \frac{\xi}{6!} + \frac{1}{2}\int\frac{d^2k}{2\pi^2}(k^2 + M^2)^{1/2} \\ & + \frac{1}{\beta}\int\frac{d^2k}{2\pi^2}\ln(1 - e^{b\Omega}) - \frac{1}{4}(M^2 - m^2 - \frac{\lambda}{2}\phi^2) \\ & - \frac{1}{24}\xi\phi^4\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\left(\frac{b\Omega}{2}\right) + \\ & \frac{\lambda}{32}\left(\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\frac{b\Omega}{2}\right)\left(\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\frac{b\Omega}{2}\right) + \\ & + \frac{\xi}{144}\left(\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\frac{b\Omega}{2}\right)\left(\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\frac{b\Omega}{2}\right) \\ & \left(\int\frac{d^2k}{2\pi^2}\Omega^{-1}\coth\frac{b\Omega}{2}\right) \end{aligned} \tag{4.33}$$

In order to compare with the effective mass expression obtained for Φ^6 model we make the following identification.

$$G(x,x) \longrightarrow \int \frac{d^2k}{2\pi^2} \Omega^{-1} \coth\left(\frac{b\Omega}{2}\right) \tag{4.34}$$

By comparing with the finite temperature effective potential expression for Φ^6 model we can see that both equations are identical.

The preceding procedure can also be used to establish the equivalence between CJT formalism and Schrodinger picture formalism for the charged scalar field with a finite chemical potential [chapter 3]

4.5 Conclusions

In this chapter we have established the equivalence between self-consistent composite operator formalism and functional Schrödinger picture formalism both at zero temperature and finite temperature. The equations are exactly identical. As far as Φ^6 model is concerned use of Hartee-Fock approximation eliminates diagrams depending on ϕ and hence the expressions are not exactly identical. This raises certain doubts about the validity of Hartee-Fock approximation in Φ^6 model. But when the purpose is to include daisy and superdaisy diagrams in the summation and when we consider only scalar models the approximation is valid.

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