BOREL-TYPE SUMMABILITY METHODS

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

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Chapter-I

INTRODUCTION

Summability transformations help us to generalize the concept of limit of a sequence or series, and thus provide us a method to assign limits even to sequences which are divergent. These transformations or methods can be classified into two types.

(i) Sequence to sequence transformations

(ii) Sequence to function transformations

Sequence to sequence transformations are accomplished using infinite matrices. Consider an infinite matrix $C = (c_{nk})$ and a sequence $\{s_n\}$, $n = 0,1,2,\ldots$. Form the new sequence $\{t_n\}$ defined by

$$t_n = \sum_{k=0}^{\infty} c_{nk} s_k$$

We shall assume that the series converges for every n. $\{t_n\}$ is called the C-transform of the given sequence $\{s_n\}$. If $\{t_n\}$ converges to t, then t is called the C-limit of $\{s_n\}$ and we write $s_n \longrightarrow t(C)$.

A transformation C is called a regular summability transformation if it preserves limit in the case of convergent sequences. That is

$$s_n \longrightarrow s \implies t_n \longrightarrow s(C)$$

Silverman-Toeplitz theorem gives necessary and sufficient conditions for a matrix to represent a regular method and thus help us to construct regular transformations. This theorem can be stated as: The necessary and sufficient conditions that the matrix $C = (c_{nk})$ represents a regular transformation are:

(i) $\sum_{k=o}^{\infty} |c_{nk}| < M$, for some M and for all $n=0,1,2,\ldots$.

(ii) $\lim_{n \to \infty} c_{nk} = 0$, for each $k = 0,1,2,\ldots$.

(iii) $\lim_{n \to \infty} \sum_{k=o}^{\infty} c_{nk} = 1$

As an example for the sequence to sequence transformation consider the (C,1) mean (Cesaro mean of order 1). The transformed sequence $\{t_n\}$ of a given sequence $\{s_n\}$ is defined by

$$t_n = \frac{s_o + s_1 + \ldots + s_{n-1}}{n}$$

If $\{s_n\}$ = 1,0,1,0,...., then $\{s_n\}$ is not convergent where as $s_n \longrightarrow \frac{1}{2}(C,1)$. The matrix of (C,1) is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots \end{bmatrix}$$

As an example for sequence to function transformation, consider the Abel method, defined by

$$s_n \longrightarrow s(A),$$

if,

$$\lim_{r \to 1-} (1-r) \sum_{n=o}^{\infty} s_n r^n = s$$

If $\{s_n\}$ denotes the sequence of partial sums of the series $\sum_{o}^{\infty} a_n$, then we have the relation

$$\frac{\Sigma a_n r^n}{1-r} = \sum_{o}^{\infty} s_n r^n$$

Hence for a series $\sum_{o}^{\infty} a_n$, Abel method is defined as

$$\Sigma a_n = s(A),$$

if

$$\lim_{r \to 1-} \sum_{n=o}^{\infty} a_n r^n = s$$

In particular the series 1-2+3-4+ ... is summable Abel with limit $\frac{1}{4}$. The partial sums of the above series determine the sequence 1, -1, 2,-2, ... which is not (C,1) summable.

In the field of summability theory, Tauberian theorems occupy an important position. These theorems provide results of the following type. If a sequence $\{s_n\}$ is summable by a method C and also s_n satisfy some condition (called Tauberian condition) then $\{s_n\}$ is convergent. The first theorem of this character was proved by A. Tauber in 1897 and he proved the following result ([16], Theorem 85).

" If $\Sigma\, a_n$ is summable (Abel) to s and $a_n = o(\frac{1}{n})$, then Σa_n converges to s". This theorem has been generalized by showing that the result is true even when $a_n = O(\frac{1}{n})$ ([16], Theorem 90). A similar theorem for Borel-summability can be stated as follows:

" If Σa_n is Borel summable to s and $a_n = O(\frac{1}{\sqrt{n}})$, then Σa_n converges to s". ([16], Theorem, 156).

Definitions, properties and theorems concerning a large number of regular transformations can be obtained from the book "Divergent Series" by G.H. Hardy ([16]).

Another variant of Tauberian theorem prove results of the following character. " If $\rho \geqslant -\frac{1}{2}$, $a_n = o(n^\rho)$ and Σa_n is summable Borel to s, then Σa_n is summable $(C, 2\rho+1)$ to s". ([16], Theorem 147). Borel method is a particular case of Borel-type method (B, α, β). (Results on (B, α, β) method form the material of this thesis). The above theorem was generalized to (B, α, β) method by Borwein, D [5]. This was again improved by Kwee, B [25] by showing that the result is true with $a_n = O(n^\rho)$. A few papers dealing results of this nature are[7], [8], [10], [26].

The study of Gibbs phenomenon and Lebesgue constants for different summability transformations had been undertaken by many researchers ([18],[19],[27],[28],[29],[30]). Summability transformations help us to extend the domain of convergence of a series of functions. The domain of convergence of a series of Legendre polynomials for different summability transformations had been investigated by many authors ([11], [21], [22], [32]). Tauberian constants for many summability methods had been determined. The problem generally considered in this area can be stated in the following way. Let $\{s_n\}$ be the sequence of partial sums of Σa_n and assume a_n satisfies some tauberian condition. Let $T(x) = \sum_{n=0}^{\infty} c_n(x) s_n$ be a sequence to function summability transformation. Then the results give estimates of

$$\lim_{n\to\infty} \sup_{x_n\to\infty} | T(x_n)-s_n|$$

when neither $\lim T(x)$ nor $\lim s_n$ is assumed to exist. [1], [20], [31], [33] deal results of this nature.

A brief introduction to Summability transformations touching all aspects of the theory is almost an impossible task. Hence in the above introduction, I have restricted the concepts to those which are relevant to the topics discussed in this thesis.

Brief summary of the results in the thesis

Before summarising the results, we first define the Borel-type transformation (B,α,β) which is a generalization of the classical Borel transform. After Borwein,D([5]) we may define (B,α,β) summability as follows:

Let $\{s_n\}$, n=0,1,2,... be a sequence of real or complex numbers. Suppose that $\alpha > 0$, β is real and N, a non-negative integer such that $\alpha N+\beta > 0$. The sequence $\{s_n\}$ is said to be (B,α,β) summable to s, if

$$\underset{x \to \infty}{\text{Limit}} \quad \alpha e^{-x} \sum_{n=N}^{\infty} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} s_n = s$$

The (B,α,β) method is regular and reduces to the classical Borel method when $\alpha=\beta=1$. Being a generalization of the Borel transform (B,α,β) method is also called Borel-type summability method.

In chapter 2, the study of Gibbs phenomenon with regard to (B,α,β) summability is undertaken. It is shown that the Borel-type summability completely preserves the Gibbs phenomenon for Fourier series.

In chapter 3, the Lebesgue constants for (B,α,β) method is calculated. It is shown that the Lebesgue constants $L_B(x)$ for (B,α,β) method is given by

$$L_B(x) = \frac{2}{\pi^2} \log \left(\frac{2x}{\pi^2}\right) - \frac{2}{\pi^2} C - \frac{2}{\pi^2} \int_0^{\pi} \psi\left(\frac{t}{\pi}\right) \sin t \, dt + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$$

where C is the Euler-Mascheroni constant and

$$\Psi(t/\pi) = \Gamma'(t/\pi)/\Gamma(t/\pi)$$

In chapter 4, the domain of summability of Legendre polynomials by the (B,α,β) transform is obtained. It is shown that the sequence $\{s_k(z,w)\}$ of partial sums of the series of Legendre polynomials $\sum_{n=0}^{\infty} (2n+1)\, P_n(z)\, Q_n(w)$ is summable (B,α,β) to $(w-z)^{-1}$ in the region

$$\left\{ z : \operatorname{Re}\left\{\frac{\xi(\phi)}{T(\psi)}\right\}^{1/\alpha} < \lambda,\ \left|\frac{\xi(\phi)}{T(\psi)}\right| < M,\ 0 \leqslant \phi \leqslant \pi,\ 0 \leqslant \psi \right\}$$

where M is a positive constant and $0 < \lambda < 1$.

In chapter 5, we prove a result on Tauberian constants for (B,α,β) transform of a series Σu_k with the condition $\limsup |\sqrt{k}\, u_k| = L < \infty$. It is shown that if $m \longrightarrow \infty$, $t \longrightarrow \infty$ such that

$$\limsup \frac{m-t}{\sqrt{t}} = Q < \infty, \text{ then}$$

$$\limsup |B(t)-s_m| < A.L, \text{ where}$$

$$A = \sqrt{\frac{2\alpha}{\pi}}\, Q \int_0^Q e^{-\frac{\alpha}{2} z^2} dz + \sqrt{\frac{2}{\alpha\pi}}\, e^{-\frac{\alpha}{2} Q^2}$$

Chapter-II

GIBBS PHENOMENON FOR (B,α,β) SUMMABILITY

Preliminaries

In this section we define the Borel-type summability transformations. We also list some of the basic inequalities and estimates satisfied by the Borel-type transform which will be used in the sequel. The proof of these inequalities and estimates can be found in [5]. These are generalization of the corresponding results for the Borel-transform (cf [16], theorem, 137).

Definition of the Borel-type transforms (B,α,β)

Let $\{s_n\}$, $n=0,1,2,...$ be a sequence of real or complex numbers. Suppose that $\alpha > 0$, β is real and N a non-negative integer such that $\alpha N+\beta > 0$. The sequence $\{s_n\}$ is said to be (B,α,β) summable to the sum s, if

$$\operatorname*{Limit}_{x \to \infty} \alpha e^{-x} \sum_{n=N}^{\infty} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} s_n = s.$$

The (B,α,β) method is regular and reduces to the classical Borel method when $\alpha=\beta=1$.

Lemma 2.1. (cf [5])

$$\alpha e^{-x} \sum_{n=N}^{\infty} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} = 1+o(1) \quad (x \longrightarrow \infty).$$

Lemma 2.2. (cf [5])

Let $x > 0$, $0 < \delta < \frac{1}{\alpha}$, $\frac{1}{2} < \xi < \frac{2}{3}$, $\gamma = \frac{1}{3}(\alpha\delta)^2$

$0 < \eta < 2\xi - 1$ and let

$$u_n = u_n(x) = \alpha e^{-x} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}, \quad n = N, N+1, \ldots$$

Then,

(i) $\displaystyle\sum_{n=N}^{\infty} u_n \longrightarrow 1$ as $x \longrightarrow \infty$

(ii) $u_n \leqslant u_{n+1}$ when $n \leqslant \frac{x}{\alpha} - \frac{\beta}{\alpha} - 1$, and

$u_{n+1} \leqslant u_n$ when $n \geqslant \frac{x}{\alpha} - \frac{\beta}{\alpha} + \frac{1}{\alpha}$

(iii) $\displaystyle\sum_{|n-\frac{x}{\alpha}| > \delta x} u_n = O(e^{-\gamma x}) \quad (x \longrightarrow \infty)$

(iv) $\displaystyle\sum_{|n-\frac{x}{\alpha}| > x^{\xi}} u_n = O(e^{-x^{\eta}}) \quad (x \longrightarrow \infty)$

(v) $$u_n = \frac{\alpha}{\sqrt{2\pi x}}\, e^{\frac{-\alpha^2 h^2}{2x}}\left[1+ O\left(\frac{|h|+1}{x}\right) + O\left(\frac{|h|^3}{x^2}\right)\right]$$

$$= \frac{\alpha}{\sqrt{2\pi x}}\, e^{\frac{-\alpha^2 h^2}{2x}}\left[1+ O(x^{3\xi-2})\right] \text{ where } h = n - \frac{x}{\alpha}$$

(vi) If $\theta > 0$ fixed, then

$$\sum_{|n-\frac{x}{\alpha}|>\theta x^{\xi}} u_n = O(e^{-x^{\eta}}) \quad (x \longrightarrow \infty)$$

(vii) $$\sum_{|n-\frac{x}{\alpha}| > \lambda\sqrt{x}} u_n < \epsilon \text{ , for } x > x_o(\epsilon),\ \lambda > \lambda_o(\epsilon)$$

Theorem 2.1. (cf [13])

The (B,α,β) Summability completely preserves the Gibbs phenomenon for Fourier series.

Proof:

Consider the function $f(t)$ defined on $[0,2\pi)$ by

$$f(t) = \tfrac{1}{2}(\pi-t), \quad (0<t<2\pi)$$
$$= 0 \text{ if } t = 0$$

and extended outside by periodicity.

The Fourier series of f is given by

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

Clearly f is odd and has a jump π at 0. The sequence $\{s_n(t)\}$ of partial sums of the above Fourier series is given by

$$s_n(t) = \frac{-t}{2} + \int_0^t \frac{\sin (n+\frac{1}{2})u}{2 \sin \frac{u}{2}} du$$

Let $B_x(t)$ denote the (B,α,β) transform of $\{s_n(t)\}$. Then,

$$B_x(t) = \alpha e^{-x} \sum_{n=N}^{\infty} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} s_n(t)$$

$$= \alpha e^{-x} \sum_{n=N}^{\infty} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \left(-\frac{t}{2} + \int_0^t \frac{\sin(n+\frac{1}{2})u}{2\sin \frac{u}{2}} du \right)$$

$$= \frac{-t}{2} \alpha e^{-x} \sum_{n=N}^{\infty} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} + \alpha e^{-x} \sum_{n=N}^{\infty} \left(\int_0^t \frac{\sin(n+\frac{1}{2})u}{2 \sin \frac{u}{2}} du\right) \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}$$

$$= \Sigma_1 + \Sigma_2 \tag{2.1}$$

say. Applying Lemma 2.1, it follows that

$$\Sigma_1 = \frac{-t}{2}\,[1+o(1)] \quad (x \longrightarrow \infty) \tag{2.2}$$

By changing the order of integration and summation in Σ_2, which is permissible because of uniform convergence, we obtain.

$$\Sigma_2 = \alpha e^{-x} \int_0^t \left[\sum_{n=N}^{\infty} \frac{(\sin n+\frac{1}{2})u}{2\sin\frac{u}{2}} \cdot \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \right] du$$

$$= \alpha e^{-x} \int_0^t \mathrm{Im} \left[e^{i\frac{u}{2}} \sum_{n=N}^{\infty} e^{inu} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \right] \frac{du}{2\sin\frac{u}{2}}$$

(Im means imaginary part)

$$= \alpha e^{-x} \int_0^t \mathrm{Im} \left[e^{iu(\frac{1}{2}+\frac{1-\beta}{\alpha})} \sum_{n=N}^{\infty} \frac{(xe^{i\frac{u}{\alpha}})^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \right] \frac{du}{2\sin\frac{u}{2}}$$

Now by using Lemma 2.1, we have

$$\Sigma_2 = \alpha e^{-x} \int_0^t \mathrm{Im}\,[e^{iu(\frac{1}{2}+\frac{1-\beta}{\alpha})} \frac{1}{\alpha} e^{xe^{i\frac{u}{\alpha}}} \{1+o(1)\}] \frac{du}{2\sin\frac{u}{2}}$$

$$= \int_0^t \mathrm{Im}\,[e^{iu(\frac{1}{2}+\frac{1-\beta}{\alpha})} e^{-x(1-e^{i\frac{u}{\alpha}})}] \{1+o(1)\} \frac{du}{2\sin\frac{u}{2}} \tag{2.3}$$

Now,

$$\mathrm{Im}[e^{iu(\frac{1}{2}+\frac{1-\beta}{\alpha})} e^{-x(1-e^{i\frac{u}{\alpha}})}]$$

$$= \operatorname{Im}\left[e^{iu(\frac{1}{2}+\frac{1-\beta}{\alpha})-x(1-\cos\frac{u}{\alpha}-i\sin\frac{u}{\alpha})}\right]$$

$$= e^{-x(1-\cos\frac{u}{\alpha})}\ \operatorname{Im}\left[e^{i(x\sin\frac{u}{\alpha}+(\frac{1}{2}+\frac{1-\beta}{\alpha})u)}\right]$$

$$= e^{-2x\sin^2\frac{u}{2\alpha}}\ \sin[x\sin\frac{u}{\alpha}+(\frac{1}{2}+\frac{1-\beta}{\alpha})u]$$

Thus from (2.3) we obtain

$$\Sigma_2 = \int_0^t e^{-2x\sin^2\frac{u}{2\alpha}} \sin[x\sin\frac{u}{\alpha}+u(\frac{1}{2}+\frac{1-\beta}{\alpha})](1+o(1))\ \frac{du}{2\sin\frac{u}{2}}$$

$$= \int_0^t e^{-2x\sin^2\frac{u}{2\alpha}} \sin[x\sin\frac{u}{\alpha}+(\frac{1}{2}+\frac{1-\beta}{\alpha})u]\ \frac{du}{2\sin\frac{u}{2}}$$

$$= \int_0^{\frac{t}{2\alpha}} e^{-2x\sin^2 w} \sin[x\sin 2w + w(\alpha+2-2\beta)]\frac{\alpha\, dw}{\sin(\alpha w)} \tag{2.4}$$

Consequently for small values t_x of t, we obtain from (2.1), (2.2) and (2.4),

$$B_x(t_x)+\frac{t_x}{2} = \int_0^{\frac{t_x}{2\alpha}} e^{-2xw^2} \sin[2wx+w(\alpha+2-2\beta)]\ \frac{dw}{w}$$

$$= \int_0^{\frac{t_x}{2\alpha}} e^{-2xw^2} \sin[w(2x+\alpha+2-2\beta)]\ \frac{dw}{w}$$

The substitution $z = (2x+2-2\beta+\alpha)w$, yield

$$\int_0^{(\frac{x+1-\beta}{\alpha}+\frac{1}{2})t_x} e^{-2xz^2/(2x+2-2\beta+\alpha)^2} \frac{\sin z}{z} dz$$

Hence for any positive number T, if $x \to \infty$, $t_x \to 0$ in such a way that $xt_x \to \alpha T$, then

$$B_x(t_x) = \int_0^T \frac{\sin z}{z} dz. \tag{2.5}$$

Thus equation (2.5) shows that (B,α,β) transform preserves Gibbs phenomenon for Fourier series. When $\alpha=\beta=1$, we obtain the result on Gibbs phenomenon for Borel summability proved by Lorch [30].

Chapter III

LEBESGUE CONSTANTS FOR (B,α,β) SUMMABILITY

In this chapter the Lebesgue constants for Borel-type method of summability is determined. These constants are defined as follows. Let $D_n(t)$ the Dirichlet's kernel, namely $\frac{\sin(2n+1)t}{\sin t}$ and $B_x(t)$ the (B,α,β) transform of the sequence $\{D_n(t)\}$. In finding the Lebesgue constants, we estimate the value of the following integral for large values of x,

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} |B_x(t)|dt \qquad (3.1)$$

The following theorem generalizes the corresponding result for Borel summability [28].

Theorem 3.1. (c.f.[13])

If $L_B(x)$ denote the Lebesgue constants for (B,α,β) summability, then

$$L_B(x) = \frac{2}{\pi^2}\log\left(\frac{2x}{\pi^2}\right) - \frac{2}{\pi^2} C - \frac{2}{\pi^2}\int_0^{\pi}\psi\left(\frac{t}{\pi}\right)\sin t\, dt + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \longrightarrow \infty)$$

where,

$$C = \int_0^1 \frac{1-e^{-y}}{y}\,dy - \int_1^\infty \frac{dy}{ye^y}$$ is the Euler-Mascheroni constant and $\psi(\frac{t}{\pi}) = \frac{\Gamma'(\frac{t}{\pi})}{\Gamma(\frac{t}{\pi})}$

Proof:

Let $B_x(t)$ denote the (B,α,β) transforms of the sequence of functions $\left\{ \frac{\sin(2n+1)t}{\sin t} \right\}$

Then

$$B_x(t) = \alpha e^{-x} \sum_{n=N}^{\infty} \frac{\sin(2n+1)t}{\sin t} \cdot \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}$$

$$= \frac{\alpha e^{-x}}{\sin t} \text{ Im } \left[\sum_{n=N}^{\infty} e^{i(2n+1)t} \cdot \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \right]$$

$$= \frac{\alpha e^{-x}}{\sin t} \text{ Im } \left[e^{it-(\beta-1)\frac{i2t}{\alpha}} \sum_{n=N}^{\infty} \frac{(xe^{\frac{i2t}{\alpha}})^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \right]$$

$$= \frac{\alpha e^{-x}}{\sin t} \text{ Im}\left[e^{i\{t-\frac{2(\beta-1)t}{\alpha}\}} \cdot \frac{1}{\alpha} e^{x e^{\frac{i2t}{\alpha}}} \right]$$

$$= \frac{e^{-x}}{\sin t} \text{ Im } \left[e^{i\{t+2(\frac{1-\beta}{\alpha})t + x(\sin\frac{2t}{\alpha})\}} \cdot e^{x\cos\frac{2t}{\alpha}} \right]$$

$$= \frac{e^{-2x\sin^2\frac{t}{\alpha}}}{\sin t} \cdot \sin\left[x\sin\frac{2t}{\alpha} + 2t\left(\frac{1}{2} + \frac{1-\beta}{\alpha}\right)\right]$$

It follows from (3.1) that the Lebesgue constants $L_B(x)$ for (B,α,β) method is given by

$$L_B(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-2x \sin^2 \frac{t}{\alpha}} \left|\sin\left[x \sin \frac{2t}{\alpha} + 2t\left(\frac{1}{2} + \frac{1-\beta}{\alpha}\right)\right]\right| \frac{dt}{\sin t} \tag{3.2}$$

The evaluation of the integral is divided into many stages and hence the following lemmas.

Lemma 3.2.

Let,

$$L_{B_1}(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-2x \sin^2 \frac{t}{\alpha}} \left|\sin\left[x \sin\frac{2t}{\alpha} + 2t\left(\frac{1}{2} + \frac{1-\beta}{\alpha}\right)\right]\right| \frac{dt}{t}$$

Then $L_B(x) = L_{B_1}(x) + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$

Proof:

Using the following inequalities

$$0 < t - \sin t < \frac{t^3}{3!} \quad \text{for } t \geqslant 0$$
$$1 < \frac{t}{\sin t} < \frac{\pi}{2} \quad \text{for } 0 \leqslant t \leqslant \frac{\pi}{2} \tag{3.3}$$

it follows that

$$0 \leqslant L_B(x) - L_{B_1}(x)$$
$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left[\frac{1}{\sin t} - \frac{1}{t}\right] e^{-2x \sin^2 \frac{t}{\alpha}} \left|\sin\left[x \sin \frac{2t}{\alpha} + 2t\left(\frac{1}{2} + \frac{1-\beta}{\alpha}\right)\right]\right| dt$$

$$\leqslant \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{t-\sin t}{t \sin t} e^{-2x \sin^2 \frac{t}{\alpha}} dt$$

$$\leqslant \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{t^3}{3!} \cdot \frac{1}{t} \cdot \frac{\pi}{2t} e^{-2x \sin^2 \frac{t}{\alpha}} dt$$

$$= \frac{1}{6} \int_0^{\frac{\pi}{2}} t\, e^{-2x \sin^2 \frac{t}{\alpha}} dt$$

Substitution of $y = \frac{t}{\alpha}$ reduces the integral to

$$\frac{\alpha^2}{6} \int_0^{\frac{\pi}{2\alpha}} y\, e^{-2x \sin^2 y} dy$$

$$\leqslant \frac{\alpha^2}{6} \int_0^{M\pi} y\, e^{-2x\sin^2 y} dy \quad \text{where } M = \left[\frac{1}{2\alpha} \right] + 1$$

$$= \frac{\alpha^2}{6} \sum_{n=1}^{M} \int_{(n-1)\pi}^{n\pi} y\, e^{-2x \sin^2 y} dy$$

Also,

$$\int_{(n-1)\pi}^{n\pi} y\, e^{-2x \sin^2 y} dy < n\pi \int_{(n-1)\pi}^{n\pi} e^{-2x \sin^2 y} dy$$

$$= n\pi \int_0^{\pi} e^{-2x \sin^2 y} dy$$

$$= 2n\pi \int_0^{\frac{\pi}{2}} e^{-2x \sin^2 y} dy$$

$$\leqslant 2n\pi \int_0^{\frac{\pi}{2}} e^{-8x \frac{y^2}{\pi^2}} dy \quad \text{(using 3.3)}$$

$$< \quad 2n\pi \int_0^\infty e^{-8x \frac{y^2}{\pi^2}} \, dy$$

The substitution $z = \sqrt{8x}\,\frac{y}{\pi}$ shows that

$$\int_0^\infty e^{-8x \frac{y^2}{\pi^2}} \, dy = O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$$

Since M is finite, it followsthat

$$L_B(x) - L_{B_1}(x) = O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$$

Thus

$$L_B(x) = L_{B_1}(x) + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$$

Lemma 3.3.

Let

$$L_{B_2}(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-2x \frac{t^2}{\alpha^2}} \left|\sin[x \sin \frac{2t}{\alpha} + 2t(\tfrac{1}{2} + \tfrac{1-\beta}{\alpha})]\right| \frac{dt}{t}$$

Then

$$L_B(x) = L_{B_2}(x) + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$$

Proof: We have

$$\begin{aligned} 0 &\leqslant L_{B_1}(x) - L_{B_2}(x) \\ &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left[e^{-2x \sin^2 \frac{t}{\alpha}} - e^{-2x \frac{t^2}{\alpha^2}}\right] \\ &\qquad \left|\sin[x \sin \frac{2t}{\alpha} + 2t(\tfrac{1}{2} - \tfrac{1-\beta}{\alpha})]\right| \frac{dt}{t} \end{aligned}$$

$$= \frac{2}{\pi} \int_{0}^{\frac{\pi}{2\alpha}} [e^{-2x \sin^2 y} - e^{-2x y^2}] \, | \sin[x \sin 2y + 2\alpha y(\frac{1}{2} + \frac{1-\beta}{\alpha})] \, | \frac{dy}{y}$$

Consequently,

$$0 \leqslant L_{B_1}(x) - L_{B_2}(x) \leqslant \frac{2}{\pi} \int_{0}^{\frac{\pi}{2\alpha}} \frac{e^{2x(y^2-\sin^2 y)}-1}{e^{2xy^2} . y} \, dy \qquad (3.4)$$

For approximating the integral on the right hand side we consider the following cases for different values of α.

<u>Case-1</u>

Let $\alpha \geqslant \frac{1}{2}$. This implies $\frac{\pi}{2\alpha} \leqslant \pi$.

From equation (3.4) it follows that,

$$L_{B_1}(x)-L_{B_2}(x) \leqslant \frac{2}{\pi} \int_{0}^{\pi} \frac{e^{2x(y^2-\sin^2 y)}-1}{e^{2xy^2} . y} \, dy$$

$$= \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \frac{e^{2x(y^2-\sin^2 y)}-1}{e^{2xy^2} . y} \, dy + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{e^{2x(y^2-\sin^2 y)}-1}{e^{2xy^2} . y} \, dy$$

$$= I_1 + I_2,$$

say. To estimate I_1 and I_2 we make use of the following inequalities,

$$\sin y \leqslant y, \quad \text{for } y \geqslant 0$$
$$e^y - 1 < ye^y \text{for } y \geqslant 0 \qquad (3.5)$$

Using the above inequality in I_1, we see that

$$I_1 \leqslant \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{2x(y^2-\sin^2 y)\ e^{2x(y^2-\sin^2 y)}}{ye^{2xy^2}} .dy$$

$$= \frac{4x}{\pi} \int_0^{\frac{\pi}{2}} (y+\sin y)\ (y-\sin y)\ e^{-2x\ \sin^2 y} \ . \frac{dy}{y}$$

$$\leqslant \frac{4x}{\pi} \int_0^{\frac{\pi}{2}} 2y.\ \frac{y^3}{3!}\ e^{\frac{-8x}{\pi^2} y^2} \ . \frac{dy}{y} \text{ (Using (3.3))}$$

$$= \frac{4x}{3\pi} \int_0^{\frac{\pi}{2}} y^3\ e^{-\frac{8x}{\pi^2} y^2} dy$$

The substitution $z = \frac{8x}{\pi^2} y^2$ shows that

$$I_1 \leqslant \frac{\pi^3}{96x} \int_0^{2x} ze^{-z}\ dz$$

Since the intepal $\int_0^{\infty} ze^{-z}\ dz$ converges, it is bounded.

It follows that

$$I_1 = O(\tfrac{1}{x}) \qquad (x \to \infty)$$

Also,

$$I_2 = \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{e^{2x(y^2-\sin^2 y)}-1}{e^{2xy^2}.y} dy$$

$$\leqslant \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} e^{-2x \sin^2 y} \cdot \frac{dy}{y}$$

$$\leqslant \frac{4}{\pi^2} \int_{\frac{\pi}{2}}^{\pi} e^{-2x \sin^2 y} dy$$

$$= \frac{4}{\pi^2} \int_{0}^{\frac{\pi}{2}} e^{-2x \sin^2 y} dy$$

$$\leqslant \frac{4}{\pi^2} \int_{0}^{\frac{\pi}{2}} e^{-\frac{8x}{\pi^2} y^2} dy \qquad \text{(using (3.3))}$$

$$\leqslant \frac{4}{\pi^2} \int_{0}^{\infty} e^{-\frac{8x}{\pi^2} y^2} dy$$

The substitution $z = \frac{\sqrt{8x}}{\pi} y$ shows that

$$I_2 \leqslant \frac{4}{\pi\sqrt{8x}} \int_{0}^{\infty} e^{-z^2} dz = \frac{1}{\sqrt{2\pi x}}$$

Thus we see that

$$I_2 = O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty).$$

It follows that for $\alpha \geqslant \frac{1}{2}$

$$L_{B_1}(x) - L_{B_2}(x) = O\left(\frac{1}{\sqrt{x}}\right) \qquad (x \to \infty).$$

<u>Case 2</u>:

Let $0 < \alpha < \frac{1}{2}$. Set $M = \left[\frac{1}{2\alpha}\right] + 1$.

Consequently from (3.4) we see that

$$L_{B_1}(x)-L_{B_2}(x) \leqslant \frac{2}{\pi} \int_0^{M\pi} \frac{e^{2x(y^2-\sin^2 y)}-1}{ye^{2xy^2}}\, dy$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{e^{2x(y^2-\sin^2 y)}-1}{y\, e^{2xy^2}}\, dy + \frac{2}{\pi} \int_{\pi}^{M\pi} \frac{e^{2x(y^2-\sin^2 y)}-1}{ye^{2xy^2}}\, dy$$

$$= I_3 + I_4,$$

say, As in case-1, we find that

$$I_3 = O\left(\frac{1}{\sqrt{x}}\right) \qquad (x \to \infty).$$

Further,

$$I_4 = \frac{2}{\pi} \sum_{r=2}^{M} \int_{(r-1)\pi}^{r\pi} \frac{e^{2x(y^2-\sin^2 y)}-1}{ye^{2xy^2}}\, dy .$$

The following calculation shows that each integral on the right hand side is $O\left(\frac{1}{\sqrt{x}}\right)$.

To this end note that

$$\int_{(r-1)\pi}^{r\pi} \frac{e^{2x(y^2-\sin^2 y)}-1}{e^{2xy^2}.y}\,dy < \frac{1}{(r-1)\pi}\int_{(r-1)\pi}^{r\pi} e^{-2x\sin^2 y}\,dy$$

$$= \frac{2}{(r-1)\pi}\int_{0}^{\frac{\pi}{2}} e^{-2x\sin^2 y}\,dy$$

$$\leqslant \frac{2}{(r-1)\pi}\int_{0}^{\frac{\pi}{2}} e^{\frac{-8x}{\pi^2}y^2}\,dy$$

$$= O\left(\frac{1}{\sqrt{x}}\right) \qquad (x \to \infty)$$

Since M is finite, it follows that

$$I_4 = O\left(\frac{1}{\sqrt{x}}\right) \qquad (x \to \infty)$$

Thus we see that for all values of α

$$L_{B_1}(x) - L_{B_2}(x) = O\left(\frac{1}{\sqrt{x}}\right) \qquad (x \to \infty)$$

This result together with Lemma 3.2 shows that

$$L_B(x) = L_{B_2}(x) + O\left(\frac{1}{\sqrt{x}}\right) \qquad (x \to \infty)$$

Lemma 3.4.

Let

$$L_{B_3}(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-2x\frac{t^2}{\alpha^2}} \left| \sin[2x\frac{t}{\alpha} + 2t(\frac{1}{2} + \frac{1-\beta}{\alpha})] \right| \frac{dt}{t}$$

Then

$$L_B(x) = L_{B_3}(x) + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$$

Proof: Now,

$$|L_{B_2}(x) - L_{B_3}(x)| \leqslant \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-2x\frac{t^2}{\alpha^2}} \Big| \sin[x \sin\frac{2t}{\alpha} + 2t(\frac{1}{2} + \frac{1-\beta}{\alpha})] - \sin[x \frac{2t}{\alpha} + 2t(\frac{1}{2} + \frac{1-\beta}{\alpha}] \Big| \frac{dt}{t}$$

Applying the trigonometric formula

$$\text{SinC} - \text{SinD} = 2 \sin\frac{C-D}{2} \cos\frac{C+D}{2}$$

it follows that

$$|L_{B_2}(x) - L_{B_3}(x)| \leqslant \frac{4}{\pi} \int_0^{\frac{\pi}{2}} e^{-2x\frac{t^2}{\alpha^2}} \left| \sin[\frac{x}{2}(\frac{2t}{\alpha} - \sin\frac{2t}{\alpha})] \right| \frac{dt}{t}$$

$$\leqslant \frac{4x}{\pi} \int_0^{\frac{\pi}{2}} e^{-2x\frac{t^2}{\alpha^2}} \left(\frac{2t}{\alpha} - \sin\frac{2t}{\alpha} \right). \frac{dt}{t}$$

$$\leqslant \frac{4x}{\pi} \int_0^{\frac{\pi}{2}} e^{-2x\frac{t^2}{\alpha^2}} \cdot \frac{8t^3}{6\alpha^3} \frac{dt}{2t} \quad \text{(using (3.3))}$$

$$= \frac{8x}{3\pi\ \alpha^3} \int_0^{\frac{\pi}{2}} t^2\ e^{-2x\frac{t^2}{\alpha^2}}\ dt$$

$$= \frac{8x}{3\pi} \int_0^{\frac{\pi}{2\alpha}} y^2\ e^{-2xy^2}\ dy$$

$$\leqslant \frac{8x}{3\pi} \int_0^{\infty} y^2\ e^{-2xy^2}\ dy$$

$$= \frac{8x}{3\pi}\left[y\ \frac{e^{-2xy^2}}{-4x} \Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-2xy^2}}{4x}\ dy \right]$$

$$= \frac{2}{3\pi} \int_0^{\infty} e^{-2xy^2}\ dy \quad = \frac{2}{3\pi\sqrt{2x}} \int_0^{\infty} e^{-z^2}\ dz$$

Thus $L_{B_2}(x) - L_{B_3}(x) = 0\ (\frac{1}{\sqrt{x}}) \quad (x \to \infty)$

Together with Lemma 3.3, it follows that

$$L_B(x) = L_{B_3}(x) + O(\frac{1}{\sqrt{x}}) \quad (x \to \infty)$$

Lemma 3.5.

If $f_x(t) = \frac{1}{t}\ (1 - e^{-2x\frac{t^2}{\alpha^2}})$

then
$$\int_0^{\frac{\pi}{2}} |f_x'(t)|dt = O(\sqrt{x}) \qquad (x \to \infty)$$

Proof:

Differentiating we obtain,

$$f_x'(t) = \frac{1}{t^2}\left[\left(4x\frac{t^2}{\alpha^2}+1\right) e^{-2x\frac{t^2}{\alpha^2}} - 1\right]$$

set $y = 2x\frac{t^2}{\alpha^2}$ and $g(y) = t^2 f_x'(t)$ (3.6)

consequently,

$$g(y) = (2y+1)\, e^{-y} - 1$$

and

$$g(y) = 0 \iff 2y+1 = e^y$$

This clearly shows that, there exists a unique positive value for y say $2\delta^2$ such that

$$g(2\delta^2) = 0, \text{ and}$$

$$\begin{aligned} g(y) &< 0 \text{ if } y > 2\delta^2 \\ g(y) &> 0 \text{ if } y < 2\delta^2 \end{aligned} \qquad (3.7)$$

(3.6) and (3.7) together shows that

$$f_x'(t) > 0 \quad \text{if } 0 < t < \frac{\alpha\delta}{\sqrt{x}}$$

and $f_x'(t) < 0$ if $t > \frac{\alpha\delta}{\sqrt{x}}$

consequently

$$\int_0^{\frac{\pi}{2}} |f_x'(t)|dt = \int_0^{\frac{\alpha\delta}{\sqrt{x}}} f_x'(t)dt - \int_{\frac{\alpha\delta}{\sqrt{x}}}^{\frac{\pi}{2}} f_x'(t)dt.$$

$$= f_x\left(\frac{\alpha\delta}{\sqrt{x}}\right) - f_x(o) - f_x\left(\frac{\pi}{2}\right) + f_x\left(\frac{\alpha\delta}{\sqrt{x}}\right)$$

$$= \frac{2\sqrt{x}}{\alpha\delta}(1-e^{-2\delta^2}) - \frac{2}{\pi}\left(1-e^{\frac{-\pi^2 x}{2\alpha^2}}\right)$$

$$= O(\sqrt{x}) \qquad (x \to \infty)$$

Lemma 3.6.

If

$$L(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left|\sin[2x\frac{t}{\alpha} + 2t(\frac{1}{2} + \frac{1-\beta}{\alpha})]\right| \frac{dt}{t}$$

then,

$$L(x) = \frac{4}{\pi^2}\log x - \frac{4}{\pi^2}\log\alpha - \frac{2}{\pi^2}\int_0^{\pi}\psi(\tfrac{t}{\pi})\sin t\, dt + O(\tfrac{1}{x}) \quad (x \to \infty)$$

where $\psi(\frac{t}{\pi}) = \Gamma'(\frac{t}{\pi}) / \Gamma(\frac{t}{\pi})$

Proof:

$$L(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left|\sin\left[\left\{2\left(\frac{x+1-\beta}{\alpha}\right) + 1\right\} t\right]\right| \frac{dt}{t} \tag{3.8}$$

To estimate this integral we make use of the following result proved in [29]

If $$L_1(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{|\sin(2x+1)t|}{t}\, dt$$

then,

$$L_1(x) = \frac{4}{\pi^2}\log x - \frac{2}{\pi^2}\int_0^{\pi} \psi\left(\frac{t}{\pi}\right)\sin t\, dt + O\left(\frac{1}{x}\right) \quad (x \to \infty)$$

It follows that

$$L(x) = \frac{4}{\pi^2}\log\left(\frac{x+1-\beta}{\alpha}\right) - \frac{2}{\pi^2}\int_0^{\pi} \psi\left(\frac{t}{\pi}\right)\sin t\, dt + O\left(\frac{1}{\frac{x+1-\beta}{\alpha}}\right) \quad (x \to \infty)$$

$$= \frac{4}{\pi^2}\log x - \frac{4}{\pi^2}\log\alpha - \frac{2}{\pi^2}\int_0^{\pi} \psi\left(\frac{t}{\pi}\right)\sin t\, dt + O\left(\frac{1}{x}\right) \quad (x \to \infty)$$

Lemma 3.7. If

$$d(x) = L(x) - L_{B_3}(x)$$

then,

$$d(x) = \frac{2}{\pi^2}\log x + \frac{2}{\pi^2}\log\frac{\pi^2}{2\alpha^2} + \frac{2}{\pi^2}C + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$$

where C is the Euler-Mascheroni constant given by

$$C = \int_0^1 \frac{1-e^{-y}}{y}\,dy - \int_1^\infty \frac{dy}{ye^y}$$

Proof:

Using the expressions of $L(x)$ and $L_{B_3}(x)$ from (3.8) and Lemma 3.4, we obtain,

$$d(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left(\frac{1-e^{-2x\frac{t^2}{\alpha^2}}}{t}\right) \left| \sin\left[\left\{\frac{2(x+1-\beta)}{\alpha}+1\right\} t\right]\right| dt$$

$$= \frac{4}{\pi^2} \int_0^{\frac{\pi}{2}} \left(\frac{1-e^{-2x\frac{t^2}{\alpha^2}}}{t}\right) dt + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$$

Set $y = 2x\frac{t^2}{\alpha^2}$. Then,

$$d(x) = \frac{2}{\pi^2} \int_0^{\frac{\pi^2}{2\alpha^2}x} \frac{1-e^{-y}}{y}\,dy + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$$

$$= \frac{2}{\pi^2} \int_0^1 \frac{1-e^{-y}}{y}\,dy + \frac{2}{\pi^2} \int_1^{\frac{\pi^2}{2\alpha^2}x} \frac{1-e^{-y}}{y}\,dy + O\left(\frac{1}{\sqrt{x}}\right)$$

$$= \frac{2}{\pi^2}\left[\int_0^1 \frac{1-e^{-y}}{y}\,dy - \int_1^{\frac{\pi^2}{2\alpha^2}x} \frac{dy}{ye^y}\right] + \frac{2}{\pi^2} \log\left(\frac{\pi^2}{2\alpha^2}x\right)$$

$$+ O\left(\frac{1}{\sqrt{x}}\right)$$

$$= \frac{2}{\pi^2} \log x + \frac{2}{\pi^2} \log\left(\frac{\pi^2}{2\alpha^2}\right) + \frac{2}{\pi^2} \left[\int_0^1 \frac{1-e^{-y}}{y} dy - \int_1^\infty \frac{dy}{ye^y} \right]$$

$$+ \frac{2}{\pi^2} \int_{\frac{\pi^2}{2\alpha^2}x}^\infty \frac{dy}{ye^y} + O\left(\frac{1}{\sqrt{x}}\right)$$

The integral $\int_{\frac{\pi^2}{2\alpha^2}x}^\infty \frac{dy}{ye^y}$ vanishes exponentially as $x \to \infty$

and hence can be absorbed in $O\left(\frac{1}{\sqrt{x}}\right)$. Also it is known that the value of the integral

$$\int_0^1 \frac{1-e^{-y}}{y} dy - \int_1^\infty \frac{dy}{ye^y}$$

is equal to Euler-Mascherioni constant C.

Thus we see that

$$d(x) = L(x) - L_{B_3}(x)$$

$$= \frac{2}{\pi^2} \log x + \frac{2}{\pi^2} \log\left(\frac{\pi^2}{2\alpha^2}\right) + \frac{2}{\pi^2}C + O\left(\frac{1}{\sqrt{x}}\right)$$

$$(x \to \infty)$$

Proof of theorem.

Lemma 3.4, Lemma 3.6 and Lemma 3.7 together show that

$$L_B(x) = L(x) - d(x) + O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$$

$$= \frac{2}{\pi^2}\log\left(\frac{2x}{\pi^2}\right) - \frac{2}{\pi^2} C - \frac{2}{\pi^2}\int_0^{\pi} \psi\left(\frac{t}{\pi}\right)\sin t \, dt$$

$$+ O\left(\frac{1}{\sqrt{x}}\right) \quad (x \to \infty)$$

Chapter IV

(B,α,β) SUMMABILITY OF LEGENDRE SERIES

In this chapter the domain of summability of a series of Legendre polynomials by (B,α,β) method is obtained. The corresponding result for the Borel exponential method [22] follows as a particular case. The series to be considered is the expansion of $(t-z)^{-1}$ in terms of Legendre polynomials. To this end consider the Legendre polynomials of the first and second kind of n^{th} degree denoted by $P_n(z)$ and $Q_n(w)$ respectively. The Laplace integral representation of $P_n(z)$ and $Q_n(w)$ are given by (c.f [40]).

$$P_n = P_n(z) = \frac{1}{\pi} \int_0^{\pi} \xi^n \, d\phi \tag{4.1}$$

and

$$Q_n = Q_n(w) = \int_0^{\infty} \tau^{-n-1} \, d \tag{4.2}$$

where,

$$\xi = \xi(\phi) = z + \sqrt{(z^2-1)} \cos\phi$$

$$\tau = \tau(\psi) = w + \sqrt{(w^2-1)} \cosh\psi$$

The banach of $\sqrt{(z^2-1)}$ is so chosen that $z + \sqrt{(z^2-1)}$ lies in the exterior of the unit circle.

Write,

$$s_k = s_k(z,w) = \sum_{n=o}^{k} (2n+1)\, P_n(z)\, Q_n(w)$$

and

$$d_n = d_n(z,w) = P_{n+1}(z)\, Q_n(w) - P_n(z)\, Q_{n+1}(w) \qquad (4.3)$$

We have by Christoffel formula ([40], page 321)

$$\frac{1}{w-z} = s_n + (n+1) \frac{1}{w-z}\, d_n \qquad (4.4)$$

By Hein's theorem ([40], page 322) the sequence $\{s_n(z,w)\}$ converges to $(w-z)^{-1}$ in the interior of the ellipse with foci $\pm$ 1 and passing through w. The following theorem asserts that the sequence $\{s_n(z,w)\}$ is summable by (B,α,β) method to $(w-z)^{-1}$ in a wider region.

Theorem 4.1. (c.f [14])

The sequence $\{s_k\ (z,w)\}$ of partial sums of the series of Legendre polynomials $\sum_{n=o}^{\infty} (2n+1)\, P_n(z)\, Q_n(w)$ is summable (B,α,β) to $(w-z)^{-1}$ in the region

$$\left\{ z:\ \mathrm{Re}\left\{\frac{\S(\phi)}{\Upsilon(\psi)}\right\}^{\frac{1}{\alpha}} < \lambda\ ,\ \left|\frac{\S(\phi)}{\Upsilon(\psi)}\right| < M\ ,\ 0 \leqslant \phi \leqslant \pi,\ 0 \leqslant \psi \right\}$$

where M is a positive number and $0 < \lambda < 1$.

Proof:

Let $T(x) = \sum_{k=N}^{\infty} u_k(x)s_k$, where

$$u_k(x) = \alpha e^{-x} \frac{x^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}$$

Then, using (3.4) we see that

$$T(x) = \sum_{k=N}^{\infty} u_k(x) \left[\frac{1}{w-z} - (k+1) \frac{1}{w-z} d_k \right]$$

$$= \frac{1}{w-z} \sum_{k=N}^{\infty} u_k(x) - \frac{1}{w-z} \sum_{k=N}^{\infty} (k+1) u_k(x) d_k$$

$$= \Sigma_1 - \Sigma_2$$

say. Applying Lemma 2.1, it follows that

$$\Sigma_1 = \frac{1}{w-z} [1+ o(1)] \quad (x \to \infty).$$

Hence

$$\lim_{x \to \infty} T(x) = (w-z)^{-1}$$

if and only if

$$\sum_{k=N}^{\infty} (k+1) u_k(x) d_k = o(1) \qquad (x \to \infty) \tag{4.5}$$

We now investigate the region where (4.5) is satisfied. Using the equations (4.1), (4.2), (4.3) and (4.4) we have

$$\sum_{k=N}^{\infty}(k+1)\ u_k(x)d_k = \frac{1}{\pi}\sum_{k=N}^{\infty}(k+1)u_k(x)\int_0^{\infty}\int_0^{\pi}(\frac{\xi}{T})^k\ (\frac{\xi}{T}-\frac{1}{T^2})d\phi\ d\psi \tag{4.6}$$

Change of order of integration and summation in (4.6) is permissible if

$$|\frac{\xi}{T}| \leqslant M < \infty$$

with this assumption we obtain from (4.6)

$$\sum_{k=N}^{\infty}(k+1)u_k(x)d_k = \frac{1}{\pi}\int_0^{\infty}\int_0^{\pi}(\frac{\xi}{T}-\frac{1}{T^2})\ [\sum_{k=N}^{\infty}(k+1)u_k(x)(\frac{\xi}{T})^k]d\phi\ d\psi$$

$$= o(1)\quad (x\to\infty)$$

provided

$$\sum_{k=N}^{\infty}(k+1)u_k(x)\ (\frac{\xi}{T})^k = o(1)\qquad (x\to\infty)$$

Now

$$\sum_{k=N}^{\infty}(k+1)u_k(x)\ (\frac{\xi}{T})^k = \alpha e^{-x}\sum_{k=N}^{\infty}(k+1)\ \frac{x^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}\ \frac{[(\frac{\xi}{T})^{\frac{1}{\alpha}}]^{\alpha k+\beta-1}}{(\frac{\xi}{T})^{\frac{\beta-1}{\alpha}}}$$

$$= \alpha\ e^{-x}\ (\tfrac{\xi}{T})^{\frac{1-\beta}{\alpha}} \sum_{k=N}^{\infty} (k+1)\ \frac{[x(\frac{\xi}{T})^{\frac{1}{\alpha}}]^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} \tag{4.7}$$

Write $I(x) = \sum_{k=N}^{\infty} \frac{[x(\frac{\xi}{T})^{\frac{1}{\alpha}}]^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}$

Then $I'(x) = \sum_{k=N}^{\infty} (\alpha k+\beta-1)\ \frac{[x(\frac{\xi}{T})^{\frac{1}{\alpha}}]^{\alpha k+\beta-2}}{\Gamma(\alpha k+\beta)}\ (\frac{\xi}{T})^{\frac{1}{\alpha}}$

$$= \frac{\alpha}{x} \sum_{k=N}^{\infty} \frac{k[x(\frac{\xi}{T})^{\frac{1}{\alpha}}]^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} + \frac{\beta-1}{x} I(x)$$

consequently right side of equation (4.7) reduces to

$$\alpha e^{-x}(\tfrac{\xi}{T})^{\frac{1-\beta}{\alpha}}\ [\{I'(x) - \frac{\beta-1}{x} I(x)\}\ \frac{x}{\alpha} + I(x)] \tag{4.8}$$

For large values of x, Lemma 2.1 shows that

$$I(x) = \frac{1}{\alpha}\ e^{x(\frac{\xi}{T})^{\frac{1}{\alpha}}}$$

$$I'(x) = \frac{1}{\alpha}\ e^{x(\frac{\xi}{T})^{\frac{1}{\alpha}}}\ (\tfrac{\xi}{T})^{\frac{1}{\alpha}}$$

Consequently from (4.7) and (4.8) we obtain

$$\sum_{k=N}^{\infty}(k+1)u_k(x)d_k = \alpha e^{-x}\left(\frac{\varsigma}{\gamma}\right)^{\frac{1-\beta}{\alpha}}\left[\frac{x}{\alpha}\, I'(x) + \frac{\alpha+1-\beta}{\alpha}\, I(x)\right]$$

$$= \alpha e^{-x}\left(\frac{\varsigma}{\gamma}\right)^{\frac{1-\beta}{\alpha}}\left[\ \frac{x}{\alpha}\cdot\frac{1}{\alpha}\ e^{x\left(\frac{\varsigma}{\gamma}\right)^{\frac{1}{\alpha}}}\left(\frac{\varsigma}{\gamma}\right)^{\frac{1}{\alpha}} + \frac{\alpha+1-\beta}{\alpha}\cdot\frac{1}{\alpha}\ e^{x\left(\frac{\varsigma}{\gamma}\right)^{\frac{1}{\alpha}}}\ \right]$$

$$= \frac{1}{\alpha}\ e^{-x}\ e^{x\left(\frac{\varsigma}{\gamma}\right)^{\frac{1}{\alpha}}}\left[\left(\frac{\varsigma}{\gamma}\right)^{\frac{1-\beta}{\alpha}}\left[(\alpha+1-\beta)+\ x\left(\frac{\varsigma}{\gamma}\right)^{\frac{1}{\alpha}}\right]\right] \qquad (4.9)$$

Let $\left(\frac{\varsigma}{\gamma}\right)^{\frac{1}{\alpha}} = a+ib$. Then right hand side of (4.9) is equal to

$$= e^{-x(1-a)}\ e^{ixb}\left[\ \frac{\alpha+1-\beta+x(a+ib)}{\alpha(a+ib)^{\frac{\beta-1}{\alpha}}}\ \right]$$

$$= o(1) \qquad (x \to \infty)$$

if $1-a > 0$. That is real part of

$$\left(\frac{\varsigma}{\gamma}\right)^{\frac{1}{\alpha}} < \lambda < 1.$$

Chapter-V

TAUBERIAN CONSTANTS FOR BOREL-TYPE SUMMABILITY

In this chapter we obtain Tauberian constants for Borel-type summability. The corresponding result for Borel-summability proved in [1] follows as a particular case of the following theorem

Theorem 5.1. (c.f. [15])

Let Σu_k satisfy the tauberian condition

$$\limsup |\sqrt{k}\, u_k| = L < \infty \tag{5.1}$$

Let $m \to \infty$, $t \to \infty$ such that

$$\limsup \frac{m-t}{\sqrt{t}} = Q < \infty \tag{5.2}$$

Then

$$\limsup |B(t) - s_m| < AL$$

where

$$A = \sqrt{\frac{2\alpha}{\pi}}\; Q \int_0^Q e^{-\frac{\alpha}{2}z^2}\, dz + \sqrt{\frac{2}{\alpha\pi}}\; e^{-\frac{\alpha}{2}Q^2}$$

Lemma 5.2. (c.f[1]).

It is easily seen that

(i) If $0 < k < \mu$, then

$$\sum_{j=k+1}^{\mu} j^{-\frac{1}{2}} = \int_k^{\mu} x^{-\frac{1}{2}} dx - \epsilon_k = 2(\sqrt{\mu} - \sqrt{k}) - \epsilon_k$$

where $\epsilon_k \longrightarrow 0$ as $k \longrightarrow \infty$.

(ii) $|\sqrt{m} - \sqrt{\mu}|\, m^{\frac{1}{2}} < |m-\mu| \qquad (m, \mu > 0)$

Proof of the theorem:

Let $j^* = \max(j, N)$, $\{s_k\}$ the sequence of partial sums of $\sum_0^{\infty} u_j$ and $B(t)$ the (B, α, β) transform of $\{s_k\}$. Then

$$B(t) = \alpha\, e^{-t} \sum_{k=N}^{\infty} \frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} \left(\sum_{j=o}^{k} u_j \right)$$

$$= \sum_{j=o}^{\infty} \left[\alpha e^{-t} \sum_{k=j^*}^{\infty} \frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} \right] u_j$$

Consequently

$$B(t)-s_m = \sum_{j=o}^{m} [-\{1-\alpha e^{-t} \sum_{j^*}^{\infty} \frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}\}] u_j$$

$$+ \sum_{j=m+1}^{\infty} [\alpha e^{-t} \sum_{j^*}^{\infty} \frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}] u_j$$

$$= \sum_{j=o}^{m} [-\alpha e^{-t} \sum_{N}^{j^*-1} \frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}] u_j$$

$$+ \sum_{j=m+1}^{\infty} [\alpha e^{-t} \sum_{j^*}^{\infty} \frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}] u_j$$

Let $c_\mu = - \alpha \mu^{-\frac{1}{2}} e^{-t} \sum_{N}^{\mu-1} \frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}$ if $N \leqslant \mu \leqslant m$

$$= \alpha \mu^{-\frac{1}{2}} \sum_{\mu}^{\infty} \frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} \quad \text{if } \mu > m$$

Let $x_k = \sqrt{k}\, u_k$ so that

$$\limsup |x_k| = L < \infty$$

and

$$B(t)-s_m = \Sigma\; c_\mu\; x_\mu$$

For fixed μ, $\underset{t\to\infty}{\mathrm{Lt}}\; c_\mu = 0.$

To determine the Tauberian constant A in the theorem we follow the method used in [1] where it is shown that

$$A = \lim\sup \Sigma\; |c_\mu| \tag{5.3}$$

Use of Lemma 5.2 shows that

$$\sum_{N}^{\infty} |c_\mu| = \alpha e^{-t} \sum_{\mu=N}^{m-1} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} \left(\sum_{\mu+1}^{m} j^{-\frac{1}{2}} \right)$$

$$+ \alpha e^{-t} \sum_{\mu=m+1}^{\infty} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} \left(\sum_{m+1}^{\mu} j^{-\frac{1}{2}} \right)$$

$$= 2\alpha e^{-t} \sum_{\mu=N}^{m-1} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} \left(\sqrt{m} - \sqrt{\mu} \right)$$

$$+ 2\alpha e^{-t} \sum_{\mu=m+1}^{\infty} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} \left(\sqrt{\mu} - \sqrt{m} \right)$$

$$- \alpha e^{-t} \sum_{\mu=N}^{m-1} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} \epsilon_\mu - \epsilon_m\, \alpha e^{-t} \sum_{\mu=m+1}^{\infty} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)}$$

Since ϵ_μ and ϵ_m are null sequences and (B,α,β) transform is regular it follows that

$$\sum_N^\infty |c_\mu| = 2\alpha e^{-t} \sum_N^\infty \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} \left| \sqrt{m} - \sqrt{\mu} \right| + o(1) \ (m\to\infty).$$

For each $\delta > 0$, define

$$B_{\frac{1}{2}} = \left\{ \mu / \ |\mu - t| > \delta\sqrt{t} \right\}$$

Then

$$\sum_{\mu=N}^\infty |c_\mu| = 2\alpha e^{-t} \sum_{\mu\in B_{\frac{1}{2}}} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} \left| \sqrt{m} - \sqrt{\mu} \right|$$

$$+ 2\alpha e^{-t} \sum_{\mu\notin B_{\frac{1}{2}}} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} \left| \sqrt{m} - \sqrt{\mu} \right|$$

$$= \Sigma_1 + \Sigma_2, \text{ say.}$$

Use of Lemma 5.2 (ii) and Cauchy-Schwartz inequality on Σ_1 shows that

$$\Sigma_1 \leqslant \frac{2}{\sqrt{m}} \alpha e^{-t} \sum_{\mu\in B_{\frac{1}{2}}} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} |m-\mu|$$

$$\leqslant \frac{2}{\sqrt{m}} \left[\alpha e^{-t} \sum_{\mu \in B_{\frac{1}{2}}} (m-\mu)^2 \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)}\right]^{1/2} \left[\alpha e^{-t} \sum_{\mu \in B_{\frac{1}{2}}} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)}\right]^{1/2}$$

An application of Lemma 2.2 shows that $\Sigma_1 \rightarrow 0$.

Now we shall consider the behaviour of the second sum

$$\Sigma_2 = 2\alpha e^{-t} \sum_{\mu \notin B_{\frac{1}{2}}} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} \left| \sqrt{m} - \sqrt{\mu} \right|$$

$$= 2\alpha e^{-t} \sum_{|\mu-t| \leqslant \delta\sqrt{t}} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} |\sqrt{m}-\sqrt{\mu}|$$

Condition (5.2) in the theorem implies that

$$\frac{m-t}{t} \rightarrow 0 \text{ and hence } \frac{m}{t} \rightarrow 1.$$

Also $|\mu-t| \leqslant \delta\sqrt{t}$ implies

$$\left| \frac{\mu}{m} - \frac{t}{m} \right| < \frac{\delta\sqrt{t}}{m}$$

It follows that if $\mu \notin B_{\frac{1}{2}}$, $\frac{\mu}{m} \rightarrow 1$.

Consequently

$$| \sqrt{m} - \sqrt{\mu} | = \frac{|m-\mu|}{\sqrt{m}+\sqrt{\mu}} \simeq \frac{|m-\mu|}{2\sqrt{m}}$$

Thus we see that

$$\Sigma_2 = \frac{\alpha e^{-t}}{\sqrt{m}} \sum_{\mu \notin B_{\frac{1}{2}}} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} |m-\mu|$$

$$= \frac{\alpha e^{-t}}{\sqrt{m}} \sum_{\mu=N}^{\infty} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} |m-\mu| + o(1)$$

Let

$$\phi(t) = \frac{\alpha e^{-t}}{\sqrt{m}} \sum_{\mu=N}^{\infty} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} |m-\mu|$$

$$= \frac{\alpha e^{-t}}{\sqrt{m}} \sum_{\mu=N}^{m} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} (m-\mu) + \frac{\alpha e^{-t}}{\sqrt{m}} \sum_{m+1}^{\infty} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} (\mu-m)$$

$$= \Sigma_3 + \Sigma_4, \text{ say.}$$

Now,

$$\Sigma_3 = \frac{\alpha e^{-t}}{\sqrt{m}} \sum_{\mu=N}^{m} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} [(m-t)-(\mu-t)]$$

$$= \Sigma_{31} - \Sigma_{32}, \text{ say}$$

Then,

$$\Sigma_{31} = \frac{m-t}{\sqrt{m}} \alpha e^{-t} \sum_{\mu=N}^{m} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)}$$

Let $\frac{1}{2} < \xi < \frac{2}{3}$. Use of Lemma 2.2 shows that

$$\Sigma_{31} = \frac{m-t}{\sqrt{m}} \sum_{t-t^{\xi}}^{m} \sqrt{\frac{\alpha}{2\pi t}}\, e^{\frac{-\alpha}{2t}(\mu-t)^2} + o(1)$$

$$= \frac{m-t}{\sqrt{m}} \sqrt{\frac{\alpha}{2\pi t}} \int_{t-t^{\xi}}^{m} e^{\frac{-\alpha}{2t}(x-t)^2}\, dx + o(1)$$

The substitution $z = \frac{x-t}{\sqrt{t}}$ and the assumption (5.2) in the theorem show that

$$\Sigma_{31} = \sqrt{\frac{\alpha}{2\pi}}\, Q \int_{-\infty}^{Q} e^{\frac{-\alpha}{2} z^2} dz + o(1)$$

$$= o(1) + \sqrt{\frac{\alpha}{2\pi}}\, Q \left[\int_{-\infty}^{o} e^{\frac{-\alpha}{2}z^2} dz + \int_{o}^{Q} e^{-\frac{\alpha}{2}z^2} dz \right]$$

$$= o(1) + \sqrt{\frac{\alpha}{2\pi}}\, Q \left[\sqrt{\frac{\pi}{2\alpha}} + \int_{o}^{Q} e^{\frac{-\alpha}{2}z^2} dz \right]$$

$$= o(1) + \frac{Q}{2} + \sqrt{\frac{\alpha}{2\pi}}\, Q \int_{o}^{Q} e^{\frac{-\alpha}{2}z^2} dz.$$

A similar treatment of Σ_{32} yield

$$\Sigma_{32} = \frac{\alpha e^{-t}}{\sqrt{m}} \sum_{\mu=N}^{\infty} \frac{t^{\alpha\mu+\beta-1}}{\Gamma(\alpha\mu+\beta)} (\mu-t)$$

$$= \frac{1}{\sqrt{m}} \sqrt{\frac{\alpha}{2\pi t}} \int_{t-t^{\delta}}^{m} (x-t)\, e^{-\frac{\alpha}{2}(x-t)^2} dx + o(1)$$

The substitution $z = \frac{x-t}{\sqrt{t}}$ together with the fact $\lim \frac{m}{t} = 1$ shows that

$$\Sigma_{32} = \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{Q} z\, e^{-\frac{\alpha}{2} z^2} dz + o(1)$$

$$= o(1) + \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{-Q} z\, e^{-\frac{\alpha}{2}z^2} dz$$

$$= o(1) - \sqrt{\frac{1}{2\alpha\pi}}\, e^{-\frac{\alpha}{2}Q^2}$$

Thus we see that

$$\Sigma_3 = \Sigma_{31} - \Sigma_{32}$$

$$= \frac{Q}{2} + \sqrt{\frac{\alpha}{2\pi}}\, Q \int_{0}^{Q} e^{-\frac{\alpha}{2}z^2} dz + \sqrt{\frac{1}{2\alpha\pi}}\; e^{-\frac{\alpha}{2}Q^2}$$

A similar calculation with Σ_4 will show that

$$\Sigma_4 = \Sigma_{41} - \Sigma_{42}, \text{ where}$$

$$\Sigma_{41} = \sqrt{\frac{\alpha}{2\pi}} \int_Q^\infty e^{-\frac{\alpha}{2}z^2} . z \, dz = \frac{1}{\sqrt{2\alpha\pi}} e^{-\frac{\alpha}{2}Q^2}$$

$$\Sigma_{42} = \sqrt{\frac{\alpha}{2\pi}} Q \int_Q^\infty e^{-\frac{\alpha}{2}z^2} dz = \frac{Q}{2} - \sqrt{\frac{\alpha}{2\pi}} Q \int_o^Q e^{-\frac{\alpha}{2}z^2} dz$$

Thus we see that

$$\sum_{\mu=N}^\infty |c_\mu| < \Sigma_3 + \Sigma_4$$

$$= \sqrt{\frac{2\alpha}{\pi}} Q \int_o^Q e^{-\frac{\alpha}{2}z^2} dz + \sqrt{\frac{2}{\alpha\pi}} e^{-\frac{\alpha}{2}Q^2} \qquad (5.4)$$

(5.3) and (5.4) together completes the proof of the theorem.

When $\alpha=\beta=1$, the value reduces to

$$\sqrt{\frac{2}{\pi}} \left[Q \int_o^Q e^{-\frac{1}{2}z^2} dz + e^{-\frac{Q^2}{2}} \right]$$

which is proved in [1].

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